Formulas for some confluent hypergeometric functions in several variables*

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Themes of the talk:

Part A.:
Branching formula for multivariate Laguerre polynomials

Part B.:
Difference equation for class-one Whittaker functions

Part A.

Branching formula for multivariate Laguerre polynomials

I.A Multivariate Laguerre polynomials

II.A Recurrence relations

III.A Branching formula

IV.A Explicit formulas for Laguerre polynomials

I.A Multivariate Laguerre Polynomials

Macdonald 1988, Lassalle 1991, Baker and Forrester 1997, vD 1997, ...

Orthogonality

— Weight function (Selberg integral, Random Matrices, Calogero-Moser systems)

$$\Delta(x) = e^{-\omega(x_1^2+\cdots+x_n^2)}|x_1\cdots x_n|^{2g_1-1}\prod_{1\leq j<k\leq n}|x_j^2-x_k^2|^{2g_0}$$

— Partitions

$$\Lambda_n := \{\lambda = (\lambda_1,\ldots,\lambda_n) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$$

endowed with Lexicographical Ordering.

—Laguerre polynomials

$$L_\lambda(x) = x_1^{2\lambda_1}\cdots x_n^{2\lambda_n} + \text{lower terms}$$

such that:
–$L_\lambda(x)$ invariant w.r.t. action of (signed) permutation group
–$L_\lambda(x)$, $\lambda \in \Lambda_n$ form orthogonal system w.r.t. $\Delta(x)$ (Gram-Schmidt).

Confluent hypergeometric differential equation

$$DL_\lambda = 4\omega(\lambda_1 + \cdots + \lambda_n)L_\lambda$$

$$D = \sum_{1\le j\le n}\Big(-\frac{\partial^2}{\partial x_j^2} - \frac{2g_1-1}{x_j}\frac{\partial}{\partial x_j} + 2\omega x_j\frac{\partial}{\partial x_j}\Big)$$
$$- 2g_0\sum_{1\le j<k\le n}\left(\frac{1}{x_j+x_k}\Big(\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x_k}\Big) + \frac{1}{x_j-x_k}\Big(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\Big)\right)$$

Relation to Calogero-Moser system

This confluent hypergeometric differential equation is equivalent to the stationary Schrödinger equation with a harmonically confined rational Calogero-Moser potential:

$$
\begin{aligned}
u(x_1, \ldots, x_n) = \sum_{1 \le j \le n} \left(\frac{(g_1 - \frac{1}{2})(g_1 - \frac{3}{2})}{x_j^2} + \omega^2 x_j^2 \right) \\
+ g_0(g_0 - 1) \sum_{1 \le j \ne k \le n} \left(\frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2} \right)
\end{aligned}
$$

Constant term value

$$L_\lambda(0) = (-\omega)^{-|\lambda|} \prod_{1\leq j<k\leq n} \frac{(1+(k-j)g_0)_{\lambda_j-\lambda_k}}{((k-j)g_0)_{\lambda_j-\lambda_k}} \prod_{1\leq j\leq n} ((n-j)g_0+g_1)_{\lambda_j}$$

where $(a)_k = a(a+1)\dots(a+k-1)$ denotes the Pochammer symbol.

Orthogonality relations

$$\int_{\mathbb{R}^n} L_\lambda(x)\overline{L_\mu(x)}\Delta(x)\mathrm{d}x = \frac{(2\pi)^{n/2}\delta_{\lambda,\mu}}{\omega^{g_0 n(n-1)+ng_1+2\lambda_1+\cdots+2\lambda_n}}$$

$$\times \prod_{1\le j\le n} \Gamma((n-j)g_0 + g_1 + \lambda_j)\Gamma(1+(n-j)g_0+\lambda_j)$$

$$\times \prod_{1\le j<k\le n} \frac{\Gamma((k-j+1)g_0+\lambda_j-\lambda_k)\Gamma(1+(k-j-1)g_0+\lambda_j-\lambda_k)}{\Gamma((k-j)g_0+\lambda_j-\lambda_k)\Gamma(1+(k-j)g_0+\lambda_j-\lambda_k)}$$

II.A Recurrence Relations

vD 1997

Elementary symmetric functions

$$E_r(x) = (-\omega)^r \sum_{1\le j_1<\cdots<j_r\le n} x_{j_1}^2 \ldots x_{j_r}^2$$

$(r = 1, \ldots, n)$

Recurrence formula

$$\boxed{E_r(x)L_\lambda(x) = \sum_{\substack{\mu\in\Lambda_n\\ \mu\sim_r\lambda}} C_{\lambda,r}^{\mu,n}(g_0, g_1, \omega)L_\mu(x)}$$

$\mu \sim_r \lambda$ iff $\mu = \lambda + e_{J_+} - e_{J_-}$ with $|J_+| + |J_-| \le r$

(where $e_1, \ldots, e_n$ is the standard unit basis, $J_+, J_- \subset \{1, \ldots, n\}$ with $J_+ \cap J_- = \emptyset$, and $e_J = \sum_{j\in J} e_j$)

Recurrence coefficients

$$C_{\lambda,r}^{\mu,n}(g_0,g_1,\omega) =$$

$$\frac{L_\lambda(0)}{L_\mu(0)} V_{J_+,J_-}^n(\lambda;g_0,g_1,\omega)\, U_{(J_+\cup J_-)^c,r-|J_+|-|J_-|}^n(\lambda;g_0,g_1,\omega)$$

with

$$V_{\epsilon J}^n(\lambda;g_0,g_1,\omega) = \prod_{j\in J_+}((n-j)g_0+g_1+\lambda_j)\prod_{j\in J_-}((n-j)g_0+\lambda_j)$$

$$\times \prod_{\substack{j\in J_+\\ j'\in J_-}}\left(1+\frac{g_0}{(j'-j)g_0+\lambda_j-\lambda_{j'}}\right)\left(1+\frac{g_0}{(j'-j)g_0+\lambda_j-\lambda_{j'}+1}\right)$$

$$\times \prod_{\substack{j\in J_+\\ k\notin J_+\cup J_-}}\left(1+\frac{g_0}{(k-j)g_0+\lambda_j-\lambda_k})\right)\prod_{\substack{j\in J_-\\ k\notin J_+\cup J_-}}\left(1-\frac{g_0}{(k-j)g_0+\lambda_j-\lambda_k}\right)$$

$$U_{K,p}^n(\lambda;g_0,g_1,\omega) = (-1)^p\times$$

$$\sum_{\substack{L_+,L_-\subset K\\ L+\cap L_-=\emptyset\\ |L_+|+|L_-|=p}}\prod_{l\in L_+}((n-l)g_0+g_1+\lambda_l)\prod_{l\in L_-}((n-l)g_0+\lambda_l)$$

$$\times \prod_{l\in L_+,l'\in L_-}\left(1+\frac{g_0}{(l'-l)g_0+\lambda_l-\lambda_{l'}}\right)\left(1+\frac{g_0}{(l'-l)g_0+\lambda_l-\lambda_{l'}+1}\right)$$

$$\times \prod_{\substack{l\in L_+\\ k\in K\setminus(L_+\cup L_-)}}\left(1+\frac{g_0}{(k-l)g_0+\lambda_l-\lambda_k})\right)\prod_{\substack{l\in L_-\\ k\in K\setminus(L_+\cup L_-)}}\left(1-\frac{g_0}{(k-l)g_0+\lambda_l-\lambda_k}\right)$$

Simplest case: $r = 1$ (generalized three-term recurrence)

Baker and Forrester 1997, vD 1997

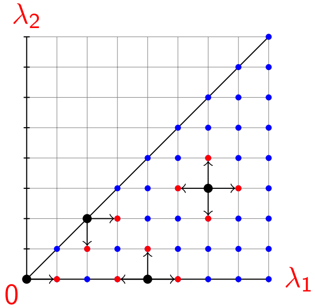
$$-\omega \mathbf{L}_\lambda(x) \sum_{1\le j\le n} x_j^2 =$$

$$\sum_{\substack{1\le j\le n \\ \lambda+e_j\in\Lambda}} V_j(\lambda)(\mathbf{L}_{\lambda+e_j}(x) - \mathbf{L}_\lambda(x)) + \sum_{\substack{1\le j\le n \\ \lambda-e_j\in\Lambda}} V_{-j}(\lambda)(\mathbf{L}_{\lambda-e_j}(x) - \mathbf{L}_\lambda(x))$$

$$\mathbf{L}_\lambda(x) = L_\lambda(x)/L_\lambda(0)$$

$$V_j(\lambda) = ((n-j)g_0 + g_1 + \lambda_j) \prod_{\substack{1\le k\le n \\ k\ne j}} \left(1 + \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k}\right)$$

$$V_{-j}(\lambda) = ((n-j)g_0 + \lambda_j) \prod_{\substack{1\le k\le n \\ k\ne j}} \left(1 - \frac{g_0}{(k-j)g_0 + \lambda_j - \lambda_k}\right)$$


λ_2
λ_1
0

III.A Branching Formula

vD-Emsiz 2014

Notation

—For $\lambda, \mu \in \Lambda_n$ the skew diagram $\lambda \setminus \mu$ is a horizontal strip iff

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n$$

—We write $\mu \preceq \lambda$ iff there exists a $\nu \in \Lambda_n$ such that the skew diagrams λ/ν and ν/μ are horizontal strips.

—$m^n - \lambda$ ($\lambda_1 \leq m$) is the partition such that

$$(m^n - \lambda)_j = m - \lambda_{n+1-j} \qquad (j = 1, \ldots, n).$$

—$\lambda' \in \Lambda_m$ ($m \geq \lambda_1$) denotes the conjugate partition of $\lambda \in \Lambda_n$

Branching formula

$$L_\lambda(x_1,\ldots,x_n,x)=\sum_{\substack{\mu\in\Lambda_n\\ \mu\preceq\lambda}} L_\mu(x_1,\ldots,x_n)L_{\lambda/\mu}(x)$$

Branching polynomials

$$L_{\lambda/\mu}(x)=\sum_{0\leq k\leq d} B^k_{\lambda/\mu}(g_0,g_1,\omega)x^{2k}$$

$$d=|\{1\leq j\leq n \mid \lambda'_j-\mu'_j=1\}|.$$

Coefficients

$$B^k_{\lambda/\mu}(g_0,g_1,\omega)=\omega^{k-m}(-g_0)^{|\lambda|-|\mu|-m}C^{(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(1/g_0,g_1/g_0,g_0\omega)$$

(with $k=0,\ldots,d$ and $m=\lambda_1$).

Proof by degeneration from analogous branching formula for Koornwinder-Macdonald multivariate Askey-Wilson polynomials, which is proved in turn by means of their recurrence relations and Mimachi's reproducing kernel.

IV.A Explicit Formulas for Laguerre Polynomials

vD-Emsiz 2014

The multivariate Laguerre polynomials are given by

$$L_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\mu^{(i)}\in\Lambda_i, i=1,\ldots,n \\ \mu^{(1)}\preceq\cdots\preceq\mu^{(n)}=\lambda}} \prod_{1\le i\le n} L_{\mu^{(i)}/\mu^{(i-1)}}(x_i)$$

where

$$L_{\mu^{(i)}/\mu^{(i-1)}}(x_i) = \begin{cases} \text{branching polynomial} & \text{if } i>1 \\ \text{monic Laguerre polynomial} & \text{if } i=1 \end{cases}$$

$n = 1$

The monic Laguerre polynomial of degree m is given by

$$\boxed{L_m(x) = \sum_{0 \le k \le m} B^k_{m/0}(g_1, \omega) x^{2k}}$$

with

$$\boxed{B^k_{m/0}(g_1, \omega) = \omega^{k-m} C^{0^m, m}_{0^m, m-k}(1/g_0, g_1/g_0, g_0\omega)}$$

Compare with standard ${}_1F_1$ representation:

$$L_m(x) = (-\omega)^{-m} (g_1)_m \; {}_1F_1(-m; g_1; \omega x^2).$$

Part B.

Difference equation for class-one Whittaker functions

I.B Toda system

II.B Whittaker function

III.B Difference equation

Kostant 1979, Olshanetsky-Perelomov 1983, Goodman-Wallach 1986, ...

Quantum hamiltonian

$$\boxed{H_R := \Delta - 2\sum_{\alpha\in S} e^{-\langle\alpha,x\rangle}}$$

Notation

Δ = Laplacian on $\mathbb{R}^n$

$\langle\cdot,\cdot\rangle$ = standard inner product on $\mathbb{R}^n$

S = simple basis for a crystallographic root system $R \subset \mathbb{R}^n$

Assumption R is irreducible and reduced.

Classical series

$$H_A = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - 2\left(e^{-x_1+x_2} + e^{-x_2+x_3} + \cdots + e^{-x_{n-1}+x_n}\right)$$

$$H_B = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - 2\left(e^{-x_1+x_2} + e^{-x_2+x_3} + \cdots + e^{-x_{n-1}+x_n} + e^{-x_n}\right)$$

$$H_C = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - 2\left(e^{-x_1+x_2} + e^{-x_2+x_3} + \cdots + e^{-x_{n-1}+x_n} + e^{-2x_n}\right)$$

$$H_D = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - 2\left(e^{-x_1+x_2} + e^{-x_2+x_3} + \cdots + e^{-x_{n-1}+x_n} + e^{-x_{n-1}-x_n}\right)$$

II.B Whittaker Function

Jacquet 1967, Kostant 1979, Hashizume 1982, Goodman-Wallach 1986, Baudoin-O'Connel 2011, ...

Definition

The class-one Whittaker function $F_\xi(x)$ on R is a solution of the Toda eigenvalue equation

$$H_R F_\xi(x) = \langle \xi, \xi \rangle F_\xi(x)$$

that is smooth and of moderate exponential growth in $x \in \mathbb{R}^n$ and holomorphic in the spectral variable $\xi \in \mathbb{C}^n$.

Normalization conditions

$$F_{w\xi}(x) = F_\xi(x) \quad (\forall w \in W)$$

and

$$\lim_{x\to+\infty} e^{\langle w_0\xi, x\rangle} F_\xi(x) = \prod_{\alpha\in R^+} \eta_\alpha^{-\langle \xi,\alpha^\vee\rangle}\,\Gamma(\langle \xi, \alpha^\vee\rangle)$$

for $\mathrm{Re}(\xi)$ in the positive fundamental chamber.

Notation

W = Weyl group

w_0 = longest element of W

$\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ $\qquad \eta_\alpha = \sqrt{\frac{2}{\langle\alpha,\alpha\rangle}}$

vD-Emsiz 2014

For any dominant weight ω with $\langle \omega, \alpha^\vee \rangle \leq 2$ for all $\alpha \in R$:

$$\sum_{\nu \in P(\omega)} \sum_{\eta \in W_\nu (w_\nu^{-1}\omega)} U_{\nu,\eta}(\xi) V_\nu(\xi) F_{\xi+\nu}(x) = e^{\langle \omega, x \rangle} F_\xi(x)$$

where

$$V_\nu(\xi) = \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha^\vee \rangle > 0}} \frac{\eta_\alpha}{\langle \xi, \alpha^\vee \rangle} \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha^\vee \rangle = 2}} \frac{\eta_\alpha}{1 + \langle \xi, \alpha^\vee \rangle}$$

$$U_{\nu,\eta}(\xi) = \prod_{\substack{\alpha \in R_\nu \\ \langle \eta, \alpha^\vee \rangle > 0}} \frac{\eta_\alpha}{\langle \xi, \alpha^\vee \rangle} \prod_{\substack{\alpha \in R_\nu \\ \langle \eta, \alpha^\vee \rangle = 2}} \frac{-\eta_\alpha}{1 + \langle \xi, \alpha^\vee \rangle}$$

Notation

W_ν = the stabilizer $\{w \in W \mid w\nu = \nu\}$ $\quad R_\nu = \{\alpha \in R \mid \langle \nu, \alpha^\vee \rangle = 0\}$

w_ν = shortest element in W mapping ν to the dominant chamber

$P(\omega)$ = saturated set of weights on the convex hull of $W\omega$

Proof

Based on q-difference equations for the Macdonald polynomials (Macdonald 1988, vD-Emsiz 2011) via the following chain of degenerations:

Macdonald polynomials $\overset{\text{vD-Emsiz 2014}}{\Longrightarrow}$ Heckman-Opdam hypergeometric functions $\overset{\text{Shimeno 2008}}{\Longrightarrow}$ Whittaker functions

Simplest difference equation

When ω is minuscule (i.e. $\langle \omega, \alpha^\vee \rangle \leq 1$ for all $\alpha \in R$) then:

$$\sum_{\nu \in W\omega} F_{\xi+\nu}(x) \prod_{\substack{\alpha \in R \\ \langle \nu, \alpha^\vee \rangle > 0}} \frac{\eta_\alpha}{\langle \xi, \alpha^\vee \rangle} = e^{\langle \omega, x \rangle} F_\xi(x)$$

(as $P(\omega) = W\omega$ in this situation)

Concrete example: difference equations for R of type A

$$\boxed{\sum_{\substack{J\subset\{1,\ldots,n\}\\|J|=r}} F_{\xi+e_J}(x)\prod_{\substack{j\in J\\k\neq J}}(\xi_j-\xi_k)^{-1}=e^{x_1+\cdots+x_r}F_\xi(x)}$$

$r = 1, \ldots, n$

cf. Ruijsenaars 1990 (classical mechanics), Babelon 2003 (via Mellin-Barnes type integrals), Kozlowski 2013 (via quantum inverse scattering method)

Happy 60th Anniversary!

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