Cluster Algebras and q-Painlevé Equations

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Web-seminar on Painlevé equations and related topics

This talk is based on the collaborative research:

[1] T. Masuda, N. Okubo & TT: Birational Weyl group actions and q-Painlevé equations via mutation combinatorics in cluster algebras, arXiv:2303.06704.

[2] T. Masuda, Y. Mizuno, N. Okubo, Y. Terashima & TT: ——, Part II: τ-function formalism (a tentative title), in preparation.

§1 Background and Results

a discrete Painlevé equation	$\fallingdotseq$	birational action of a translation part of an affine Weyl group

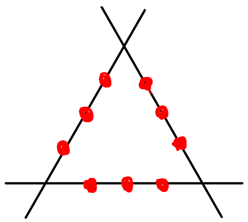
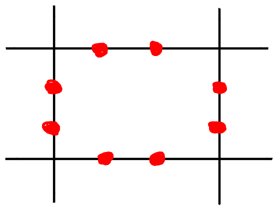
Weyl group action arises in an algebro-geometric setup

Cf.) Dolgachev–Ortland: *Point sets in projective spaces and theta functions.* (1988)

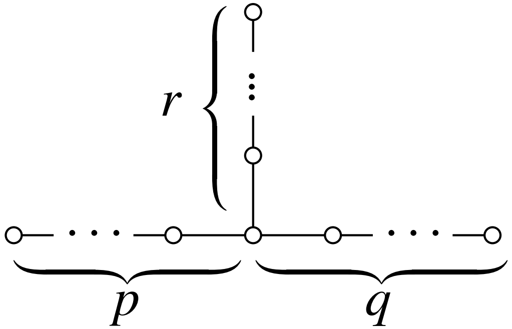
- 9 point blow-up of $\mathbb{P}^2 \curvearrowleft W(E_8^{(1)})$

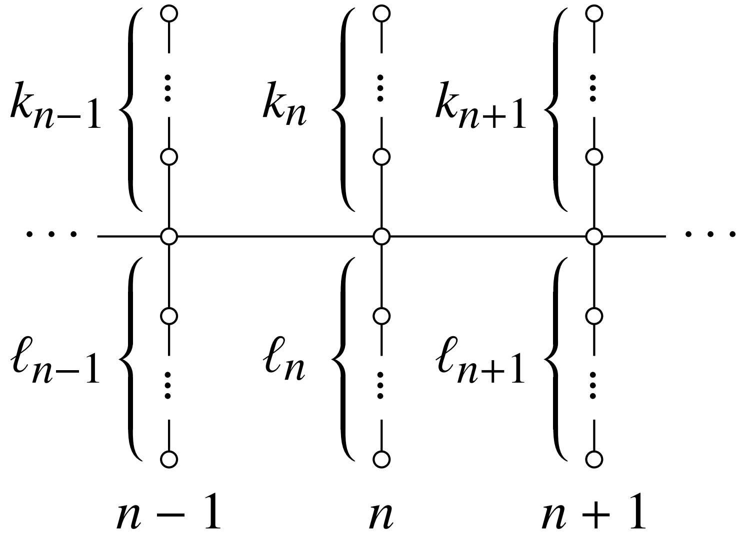
$\rightsquigarrow$ Classification of 2-D Painlevé systems with the elliptic-difference Painlevé equation of type $E_8^{(1)}$ at the top (Sakai's list).

Ref.) Sakai: Rational surfaces associated with affine root systems and geometry of the Painlevé equations. Comm. Math. Phys. **220** (2001)

 $\curvearrowleft W(E_6^{(1)})$  $\curvearrowleft W(D_5^{(1)})$ **q-difference**

- It can be $\left\{ \begin{array}{l} \text{not affine} \\ \text{higher dimensional} \end{array} \right\}$ $\rightsquigarrow$ Various Dynkin diagrams

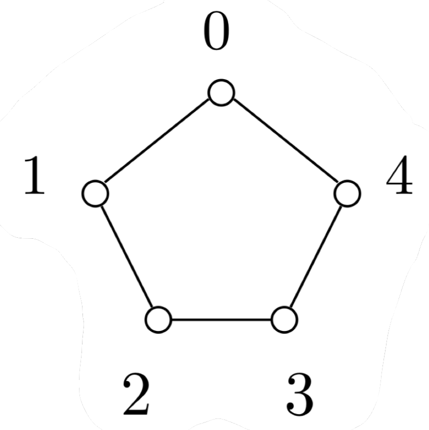




Ref.) Tsuda: Tropical Weyl group action via point configurations and τ-functions of the q-Painlevé equations. Lett. Math. Phys. **77** (2006);
Tsuda–Takenawa: Tropical representation of Weyl groups associated with certain rational varieties. Adv. Math. **221** (2009)

Thanks to the geometric background (= the space of initial values), these Weyl group actions admit τ-function formalism.

E.g.) $A_4^{(1)}$ type (q-$P_{\rm V}$)



$$\mathbb{C}(a_0, a_1, \ldots, a_4; f, g) \curvearrowleft W(A_4^{(1)}) = \langle s_i \ (0 \le i \le 4) \rangle$$

$$s_i(a_j) = a_j a_i^{-c_{ij}}, \quad s_0(f) = \frac{f}{a_0} \cdot \frac{a_2 a_3 f + a_0 g + a_0 a_3}{a_2 a_3 f + a_0^2 g + a_0 a_3} \quad \text{etc.}$$

$$\mathbb{C}(a_0, a_1, \ldots, a_4; \tau_1, \tau_2, \ldots, \tau_7) \curvearrowleft W(A_4^{(1)}) \qquad (f, g) = \left(\frac{\tau_1 \tau_2}{\tau_3 \tau_7}, \frac{\tau_4 \tau_5}{\tau_6 \tau_7} \right)$$

$$s_1(\tau_{\{1,2\}}) = \tau_{\{2,1\}}, \quad s_4(\tau_{\{4,5\}}) = \tau_{\{5,4\}},$$

$$s_2(\tau_1) = \frac{a_0 a_1}{a_3^2} \frac{a_0 \tau_4 \tau_5 + a_2 a_3 \tau_6 \tau_7}{\tau_3}, \quad s_2(\tau_3) = \frac{a_0 a_1}{a_2 a_3^2} \frac{a_0 \tau_4 \tau_5 + a_3 \tau_6 \tau_7}{\tau_1},$$

$$s_3(\tau_4) = \frac{a_2}{a_0^2 a_3 a_4} \frac{a_2 a_3 \tau_1 \tau_2 + a_0 \tau_3 \tau_7}{\tau_6}, \quad s_3(\tau_6) = \frac{a_2 a_3}{a_0^2 a_4} \frac{a_2 \tau_1 \tau_2 + a_0 \tau_3 \tau_7}{\tau_4},$$

$$s_0(\tau_1) = \frac{a_4}{a_0^2 a_1 a_2} \frac{a_2 a_3 \tau_1 \tau_2 \tau_6 + a_0 \tau_3 \tau_4 \tau_5 + a_0 a_3 \tau_3 \tau_6 \tau_7}{\tau_4 \tau_7},$$

$$s_0(\tau_4) = \frac{a_0 a_4}{a_1 a_2} \frac{a_2 a_3 \tau_1 \tau_2 \tau_6 + a_0 \tau_3 \tau_4 \tau_5 + a_3 \tau_3 \tau_6 \tau_7}{\tau_1 \tau_7},$$

$$s_0(\tau_7) = \frac{a_4}{a_0 a_1 a_2} \frac{a_2 a_3 \tau_1 \tau_2 \tau_6 + a_0^2 \tau_3 \tau_4 \tau_5 + a_0 a_3 \tau_3 \tau_6 \tau_7}{\tau_1 \tau_4},$$

Laurent phenomenon and positivity, **just like cluster algebras!**

Prior research (q-Painlevé from cluster algebras)

- Okubo and Hone–Inoue derived some individual equations.
  d-KP, KdV **Y-system, T-system**

Ref.) Okubo: Discrete integrable systems and cluster algebras. RIMS Kokyuroku Bessatsu **B41** (2013);
Okubo: Bilinear equations and q-discrete Painlevé equations satisfied by variables and coefficients in cluster algebras. J. Phys. A **48** (2015);
Hone–Inoue: Discrete Painlevé equations from Y-systems. J. Phys. A **47** (2014)

- Bershtein–Gavrylenko–Marshakov derived all the 2-D q-Painlevé systems in Sakai's list at the level of affine Weyl group symmetries.

Ref.) Bershtein–Gavrylenko–Marshakov: Cluster integrable systems, q-Painlevé equations and their quantization. J. High Energy Phys. **2018** (2018)

- Mizuno gave a geometric interpretation and complement to [BGM]'s result through toric surfaces.

Ref.) Mizuno: q-Painlevé equations on cluster Poisson varieties via toric geometry. Selecta Math. **30** (2024)

Summary of results

[1] We present a systematic way to derive a birational representation of Weyl groups from cluster algebras, by means of a graph-combinatorial approach. Our framework covers a broad class of Dynkin diagrams and in affine case leads to (higher-order) q-Painlevé equations. The key ingredient is a mutation sequence, called the **reflection associated with a cycle graph**. Symplectic structure is also discussed.

[2] To establish **τ-function formalism** for the above representation of Weyl groups, another new idea was needed; i.e. a **non-normalized cluster algebra** with two series of variables. Relations in Weyl groups are proven using the 'separation formula' and 'synchronicity theorem', which are fundamental tools in the theory of cluster algebras.

§2 Cluster Mutations

Ref.) Fomin–Zelevinsly: Cluster algebras. IV. Coefficients. Compos. Math. **143** (2007)

$Q = (V, E)$: a quiver (a directed graph)

$V = V(Q) = \{1, 2, \dots, N\}$: the vertex set of Q

$E = E(Q) \subset V \times V$: the edge set of Q

Assume that Q has no loops $i \to i$ nor 2-cycles $i \to j \to i$.

$\mathbb{Q}(y_1, y_2, \dots, y_N)$: a field of rational functions of $\boldsymbol{y} = (y_1, y_2, \dots, y_N)$

Mutation $\mu_k : (\boldsymbol{y}, Q) \mapsto (\boldsymbol{y}', Q')$ **in direction** $k \in V$

Mutated quiver $\boxed{Q' = \mu_k(Q)}$ is defined by the procedure:

(1) **Add** a new edge $i \to j$ for each subgraph $i \to k \to j$

(2) **Reverse** the orientation of all edges including vertex k

(3) **Erase** the 2-cycles appeared.

Mutated y-variables $\boxed{\boldsymbol{y}' = \mu_k(\boldsymbol{y})}$ are defined by the birational transformations:

$$y_i' = \begin{cases} {y_k}^{-1} & (i = k) \\ y_i\, {y_k}^{[b_{ki}]_+}(1 + y_k)^{-b_{ki}} & (i \neq k) \end{cases}$$

where $B = (b_{ij})_{i,j=1}^N$ is the (signed) adjacency matrix of Q and $[a]_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.

Alternatively, the mutation μ_k can be described as

$$\mu_k(B) = {}^{\mathrm{T}}A_k B A_k,$$

$$\mu_k(y^{\boldsymbol{m}}) = y^{A_k \boldsymbol{m}}(1 + y_k)^{-(B\boldsymbol{m})_k}$$

by a unimodular matrix

$$A_k \overset{\text{def}}{=} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ [b_{k1}]_+ & \cdots & [b_{k,k-1}]_+ & -1 & [b_{k,k+1}]_+ & \cdots & [b_{k,N}]_+ \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \leftarrow \text{the } k\text{th row}$$

where $y^{\boldsymbol{m}} = \prod_{i=1}^{N} y_i^{m_i}$ is a multi-index notation. Note that ${A_k}^2 = \mathrm{id}$.

In general, it holds that

$$\mu_k{}^2 = \mathrm{id} \quad \text{and} \quad \mu_i \circ \mu_j = \mu_j \circ \mu_i \ \text{ if } \ b_{ij} = 0.$$

A symmetric group $\mathfrak{S}_V$ acts on y-variables as permutations of labels:

$$\sigma(y_i) = y_{\sigma^{-1}(i)}, \quad \sigma \in \mathfrak{S}_V$$

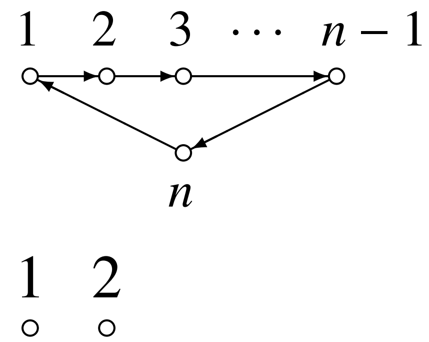
and it holds that $(i, j) \circ \mu_i = \mu_j \circ (i, j)$ holds.

Let G_Q be the set of compositions of mutations and permutations that keep $Q = (V, E)$ invariant. Then G_Q provides a nontrivial group of birational transformations on $\mathbb{Q}(y_1, y_2, \ldots, y_N)$. This is what we are interested in.

§3 Reflections Associated with Cycle Graphs

First we consider an oriented cycle of length n:

$C = (1 \to 2 \to \cdots \to n \to 1) =$

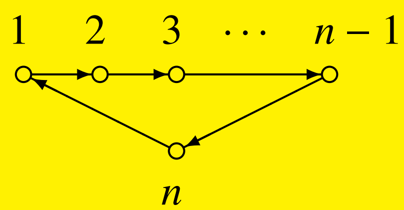


$(n \geq 3)$

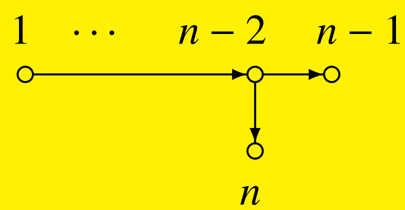
$(n = 2)$

Apply $M = \mu_{n-1} \circ \mu_{n-2} \circ \cdots\cdots \circ \mu_2 \circ \mu_1$ to C:

$C =$



$\overset{M}{\longleftrightarrow}$



$= M(C)$

Two vertices $n-1, n$ are symmetric in $M(C)$.

$$R_C \overset{\text{def}}{=} M^{-1} \circ (n-1, n) \circ M$$

Reflection associated with a cycle graph C

The trident graph $M(C)$ is invariant under a permutation $(n-1, n)$, thus

$$R_C(C) = M^{-1} \circ (n-1, n) \circ M(C) = M^{-1} \circ M(C) = C.$$

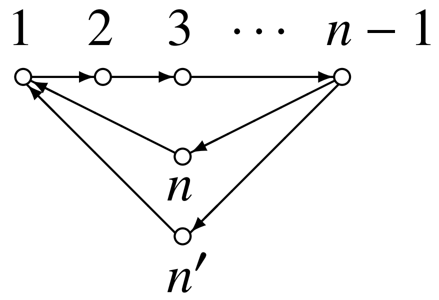
It is immediate from $\mu_k{}^2 = (n-1, n)^2 = \mathrm{id}$ that $R_C{}^2 = \mathrm{id}.$

- Note that the mutation sequence R_C itself has already appeared in several areas of mathematics, including higher Teichmüller theory:

Ref.) Goncharov–Shen: Donaldson-Thomas transformations of moduli spaces of G-local systems. Adv. Math. **327** (2018);
Inoue–Ishibashi–Oya: Cluster realization of Weyl groups and higher Teichmüller theory. Selecta Math. **27** (2019);
Inoue–Lam–Pylyavskyy: On the cluster nature and quantization of geometric R-matrices. Publ. RIMS **55** (2019);
Bucher: Maximal green sequences for cluster algebras associated to the n-torus. arXiv:1412.3713

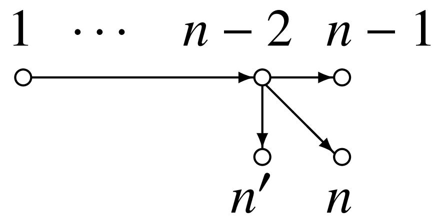
Next we consider a quiver Q obtained from C by adding a copy n' of vertex n.

$Q =$



Applying $M = \mu_{n-1} \circ \mu_{n-2} \circ \cdots\cdots \circ \mu_2 \circ \mu_1$, we have

$M(Q) =$



Three vertices $n-1, n, n'$ are symmetric in $M(Q)$, thus $R_C(Q) = Q$.

Proposition 3.1 $(R_C \circ (n, n'))^3 = \mathrm{id}$

Proof M and (n, n') are commutative. Hence we have $R_C \circ (n, n') =$

$$M^{-1} \circ (n-1, n) \circ M \circ (n, n') = M^{-1} \circ \underbrace{(n-1, n) \circ (n, n')}_{\text{a cyclic permutation of order 3}} \circ M. \quad \blacksquare$$

Characterization of a quiver invariant under the reflections

A subgraph $H \subseteq Q$ is called an **induced subgraph** if $E(H)$ consists of all the edges of Q whose endpoints both belongs to $V(H)$.

Suppose a quiver Q contains a cycle $C = (1 \to 2 \to \cdots \to n \to 1)$ as an induced subgraph.

Theorem 3.2 Q is R_C-invariant $\Longleftrightarrow$ Q satisfies the condition

(W) : For any vertex v outside of C,
$\#\{\text{edges from } v \text{ to } C\} = \#\{\text{edges from } C \text{ to } v\}$

• (W) means that: any vertex v outside of C connects to C with a **'wedge graph'** $\mathrm{w} = (k \to v \to \ell)$, where $\{k, \ell\} \subset V(C)$.

Cf.) Goncharov–Shen: Donaldson-Thomas transformations of moduli spaces of G-local systems. Adv. Math. **327** (2018) $\leftarrow$ Sufficiency of (W) was also proven.

<u>Definition 3.3</u> A cycle subgraph $C \subseteq Q$ is said to be **balanced** if C is an induced subgraph and satisfies the condition (W).

Hereafter, we always assume $C \subseteq Q$ is balanced when we consider a reflection R_C.

$$\therefore \quad R_C(Q) = Q$$

An explicit formula of $R_C \curvearrowright \mathbb{Q}(\boldsymbol{y})$

$C = (1 \to 2 \to \cdots \to n \to 1)$: an n-cycle

$R_C = M^{-1} \circ (n-1, n) \circ M, \quad M = \mu_{n-1} \circ \mu_{n-2} \circ \cdots\cdots \circ \mu_2 \circ \mu_1$

$$F_k(y_1, y_2, \ldots, y_n) \overset{\text{def}}{=} 1 + y_{k+1} + y_{k+1}y_{k+2} + \cdots + y_{k+1}y_{k+2}\cdots y_{k+n-1}$$

The birational action of R_C is described as

$$R_C(y_k) = \frac{F_{k-1}}{y_{k+1}F_{k+1}} \quad (k = 1, 2, \ldots, n)$$

where $y_{i+n} = y_i$.

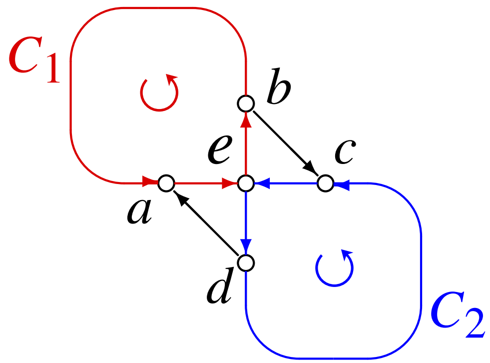
$\therefore \quad \rho^{-1} \circ R_C \circ \rho = R_C, \quad \rho = (1, 2, \ldots, n) \in \mathfrak{S}_n$ **Rotational symmetry**

($\rightsquigarrow \sigma^{-1} \circ R_C \circ \sigma = R_C, \quad \forall \sigma \in \mathfrak{S}_n$ **Permutation symmetry**)

§4 Relations of Reflections

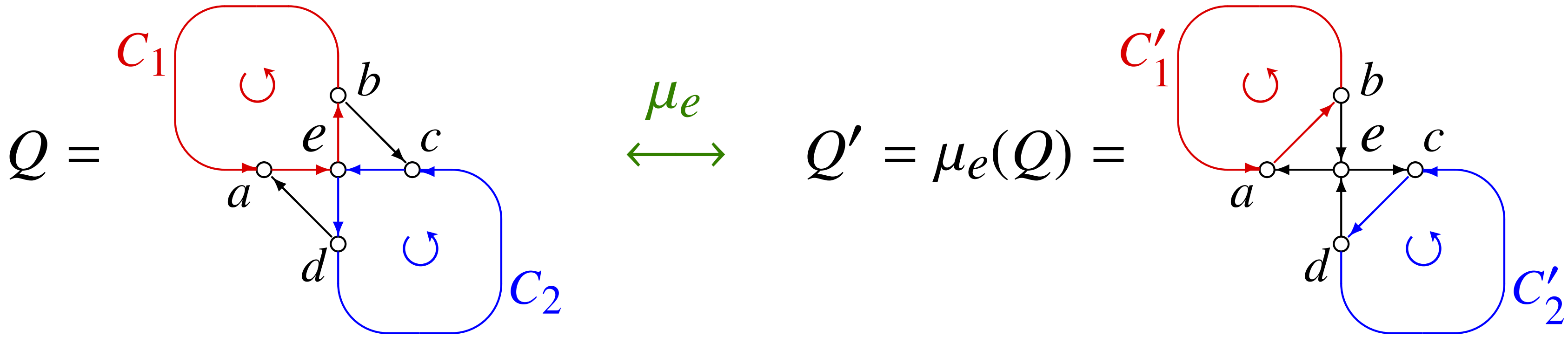
We consider a quiver Q such that two balanced cycles C_1 and C_2 intersect at a vertex e as follows:

$$Q =$$



The existence of two edges $b \to c$ and $d \to a$ guarantees that C_i ($i = 1, 2$) are balanced, thus $R_{C_i}(Q) = Q$.

Apply the mutation μ_e at the crossing vertex e to ‘separate’ C_1 and C_2.



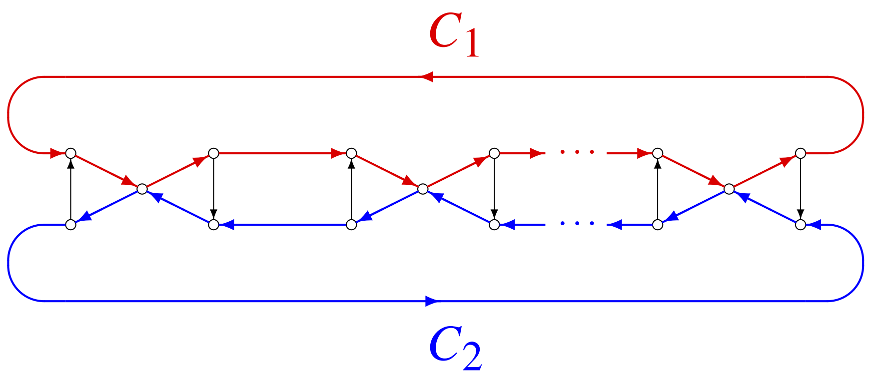
Thanks to the rotational symmetry of reflections, it holds that

$$R_{C_i} = \mu_e \circ R_{C_i'} \circ \mu_e \qquad (i = 1, 2)$$

Because two cycles C_1' and C_2' are not adjacent, the reflections $R_{C_1'}$ and $R_{C_2'}$ mutually commute. Therefore, so do R_{C_1} and R_{C_2}:

$$\begin{aligned} R_{C_1} \circ R_{C_2} &= \mu_e \circ R_{C_1'} \circ R_{C_2'} \circ \mu_e \\ &= \mu_e \circ R_{C_2'} \circ R_{C_1'} \circ \mu_e = R_{C_2} \circ R_{C_1} \end{aligned}$$

No matter how many times the two cycles C_1 and C_2 intersect, if we separate them by mutation at each crossing vertex, we can again deduce the commutativity of R_{C_1} and R_{C_2} from the rotational symmetry.

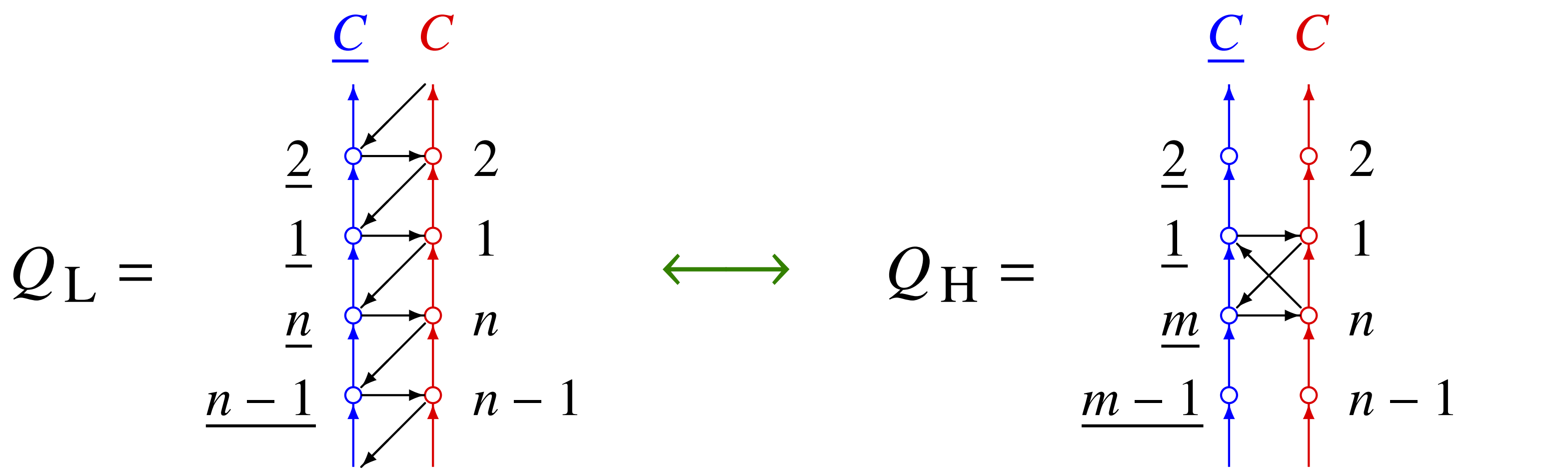
$Q_\times =$ 

In general, for a quiver $Q \supseteq Q_\times$ (as an induced subgraph) we have

Theorem 4.1 Reflections associated with two intersecting cycles are commutative; i.e. $R_{C_1}R_{C_2} = R_{C_2}R_{C_1}$ if $C_1 \cap C_2 \neq \emptyset$.

• Let $Q_{\rm L}$ be a quiver consisting of two n-cycles $\underline{C}$ and C connected with consecutive wedge graphs $\underline{i} \to i \to \underline{i-1}$ $(i \in \mathbb{Z}/n\mathbb{Z})$ like a **ladder**.

• Let $Q_{\rm H}$ be a quiver consisting of m-cycle $\underline{C}$ and n-cycle C connected with a ‘**hinge**’ $1 \to \underline{m} \to n \to \underline{1} \to 1$.



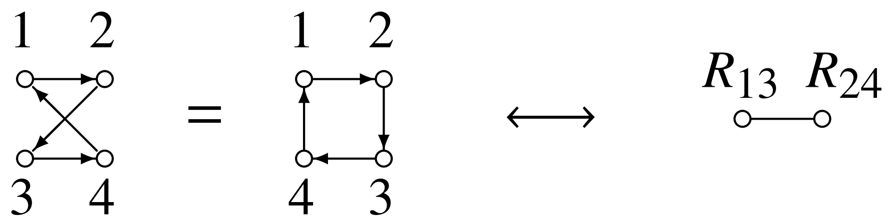
Theorem 4.2 In both cases (i) $Q \supseteq Q_{\rm L}$ and (ii) $Q \supseteq Q_{\rm H}$, the braid relation $R_C R_{\underline{C}} R_C = R_{\underline{C}} R_C R_{\underline{C}}$ $\left(\Leftrightarrow (R_C R_{\underline{C}})^3 = {\rm id}\right)$ holds.

Proof Due to the rotational symmetry, both cases reduce to a much simpler result via mutations: (i) $\rightsquigarrow$ (ii) $\rightsquigarrow$ Prop 3.1(= Cycle & a copy) ■

§5 Examples

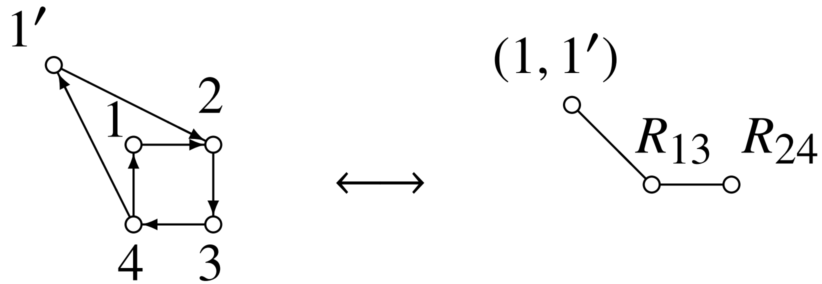
q-P_{VI}

Adjacent 2-cycles $C_{13} = (1 \to 3 \to 1)$ and $C_{24} = (2 \to 4 \to 2)$:

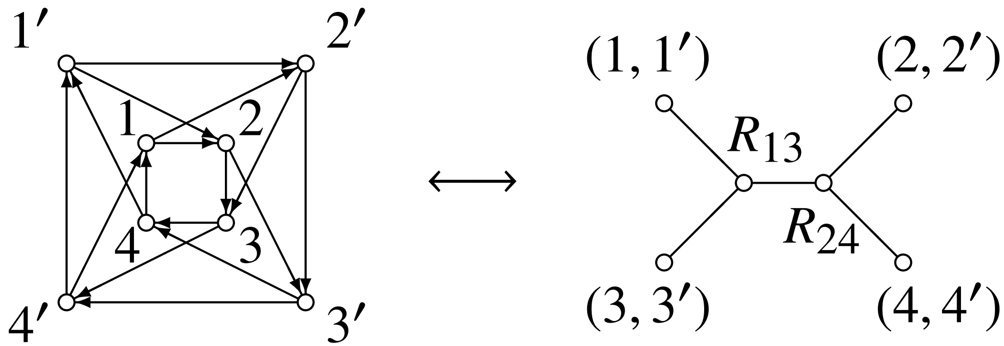
$Q_0 =$ 

A_2-type

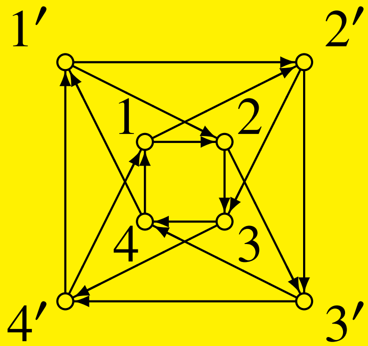
Add a copy i' of vertex i to Q_0:

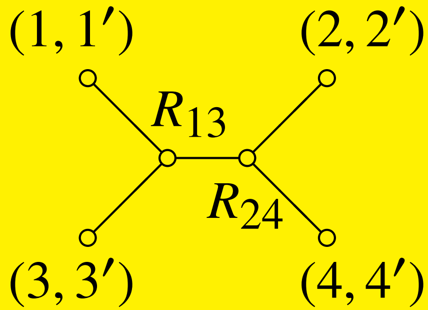
$Q_1 =$ 

A_3-type

$Q =$ 

$D_5^{(1)}$-type

$Q =$ 

$\longleftrightarrow$ 

$D_5^{(1)}$**-type**

Birational action of a translation in the affine Weyl group

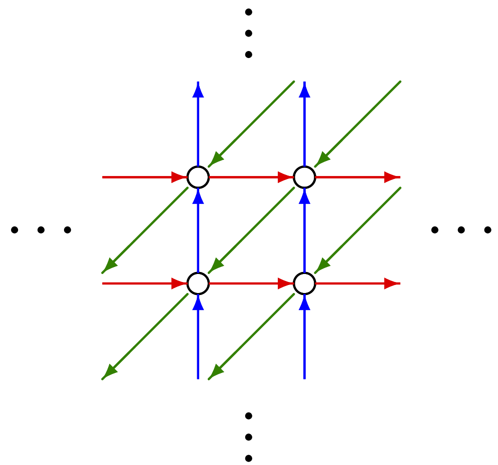
$$W(D_5^{(1)}) = \langle R_{13}, R_{24}, \underbrace{(1,1'), (2,2'), (3,3'), (4,4')}_{\text{transpositions of vertices}} \rangle \subset G_Q$$

coincides with q-$P_{\rm VI}$.

• In a similar manner, we can construct birational realizations of Weyl groups corresponding to various Dynkin diagrams.

Higher dimensional case $m, n \geq 2$ (and one of them > 2)

$$V = \{v_{i,j} \mid i \in \mathbb{Z}/m\mathbb{Z},\ j \in \mathbb{Z}/n\mathbb{Z}\}$$
$$E = \{v_{i,j} \to v_{i+1,j},\ v_{i,j} \to v_{i,j+1},\ v_{i+1,j+1} \to v_{i,j}\}$$

$Q = (V, E) =$ 

The quiver Q contains the following balanced cycles:

- **Vertical** $C_i^v = (v_{i,1} \to v_{i,2} \to \cdots \to v_{i,n} \to v_{i,1}) \quad (i \in \mathbb{Z}/m\mathbb{Z})$
- **Horizontal** $C_j^h = (v_{1,j} \to v_{2,j} \to \cdots \to v_{m,j} \to v_{1,j}) \quad (j \in \mathbb{Z}/n\mathbb{Z})$

If m and n are not relatively prime, there is another type ($g = \gcd(m, n)$):

- **Diagonal** $C_k^d = (v_{k,0} \to v_{k-1,-1} \to v_{k-2,-2} \to \cdots \to v_{k,0}) \quad (k \in \mathbb{Z}/g\mathbb{Z})$

Reflections $s_i^\triangle = R_{C_i^\triangle}$ $(\triangle = v, h, d)$ keeps Q invariant.

- Theorem 4.1 implies that $s_i^\triangle s_j^{\triangle'} = s_j^{\triangle'} s_i^\triangle \ (\forall i, j)$ if $\triangle \neq \triangle'$.
- Theorem 4.2 implies that $(s_i^\triangle s_{i+1}^\triangle)^3 = \mathrm{id}$ for each $\triangle = v, h, d$.

$$\therefore \quad W(A_{m-1}^{(1)}) \times W(A_{n-1}^{(1)}) \times W(A_{g-1}^{(1)}) \simeq \langle s_i^v, s_i^h, s_i^d \rangle \curvearrowright \mathbb{Q}(\{y_{i,j}\})$$

E.g.) Reflections associated with **'vertical cycles'** are described as

$$s_i^v(y_{i,j}) = \frac{F_{i,j-1}}{y_{i,j+1}F_{i,j+1}}, \qquad \frac{s_i^v(y_{i+1,j})}{y_{i+1,j}} = \begin{cases} \dfrac{s_i^v(y_{i-1,j-1})}{y_{i-1,j-1}} = \dfrac{y_{i,j}F_{i,j}}{F_{i,j-1}} & (\text{if } m \geq 3) \\ \dfrac{y_{i,j}y_{i,j+1}F_{i,j+1}}{F_{i,j-1}} & (\text{if } m = 2) \end{cases}$$

where $F_{i,j} = 1 + \sum_{a=1}^{n-1} \prod_{b=1}^{a} y_{i,j+b}$. (**Horizontal** and **diagonal** are similar.)

• The **vertical** and **horizontal** parts, i.e. $W(A^{(1)}_{m-1}) \times W(A^{(1)}_{n-1})$, are equivalent to those realized by Yamada and Kajiwara–Noumi–Yamada.

Ref.) Yamada: A birational representation of Weyl group, combinatorial R-matrix and discrete Toda equation. (in Proceedings of Physics and Combinatorics 2000); Kajiwara–Noumi–Yamada: Discrete dynamical systems with $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry. Lett. Math. Phys. **60** (2002)

• Based on our framework including **diagonal**, Okubo–Suzuki showed that various higher-order q-P_{VI}, such as the q-Garner system, can be derived from the same Q if $m = 2$ and n is even greater than two.

Ref.) Okubo–Suzuki: Generalized q-Painlevé VI systems of type $(A_{2n+1} + A_1 + A_1)^{(1)}$ arising from cluster algebra. IMRN **2022** (2022)

• In fact, the symmetry of Q increases only in the exceptional cases:

$$(m, n) = (2, 4), (4, 2) \Rightarrow D^{(1)}_5, \qquad (m, n) = (3, 3) \Rightarrow E^{(1)}_6$$

§6 Reduction of Dynamical Systems

$B = (b_{ij})_{i,j=1}^{N}$: the adjacency matrix of Q

(= a skew symmetric integer matrix)

$\{y_i, y_j\} \stackrel{\text{def}}{=} b_{ij} y_i y_j$ **Poisson bracket**

Ref.) Gekhtman–Shapiro–Vainshtein: *Cluster Algebras and Poisson Geometry.* (2010)

With a multi-index notation $y^{\boldsymbol{m}} = \prod_{i=1}^{N} y_i^{m_i}$, Leibniz's rule shows

$$\{y^{\boldsymbol{m}}, y^{\boldsymbol{n}}\} = \left({}^{\mathrm{T}}\boldsymbol{m}\, B\, \boldsymbol{n}\right) y^{\boldsymbol{m}+\boldsymbol{n}}.$$

$\therefore$ $y^{\boldsymbol{m}}$ is a Casimir function $\Longleftrightarrow$ $\boldsymbol{m} \in \ker B$

Lemma 6.1 If $\boldsymbol{m} \in \ker B \cap \mathbb{Z}^N$ then

$$\mu_k(y^{\boldsymbol{m}}) = y^{A_k \boldsymbol{m}} \quad \text{and} \quad A_k \boldsymbol{m} \in \ker \mu_k(B) \cap \mathbb{Z}^N$$

Proof The mutation rule $\mu_k(B) = {}^{\mathrm{T}}A_k B A_k$, $\mu_k(y^{\boldsymbol{m}}) = y^{A_k \boldsymbol{m}}(1+y_k)^{-(B\boldsymbol{m})_k}$ and $A_k{}^2 = \mathrm{id}$ lead us to the desired result. ■

Lemma 6.2 **(Normal form of a skew-symmetric integer matrix B)**

$\exists\, U = (u_{ij})_{i,j=1}^{N} \in GL_N(\mathbb{Z})$ s.t.

$$ {}^{\mathrm{T}}UBU = \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & h_2 \\ -h_2 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & h_\ell \\ -h_\ell & 0 \end{pmatrix} \oplus O_{N-2\ell}, $$

where the positive integers $h_1, h_2, \ldots, h_\ell$ satisfying $h_i | h_{i+1}$ are unique.

Ref.) Newman: *Integral Matrices.* (1972)

Write U by arranging column vectors $\boldsymbol{u}_i$ $(1 \le i \le N)$ in a row.

$$ U = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2\ell}, \underbrace{\boldsymbol{u}_{2\ell+1}, \ldots, \boldsymbol{u}_N}_{\ker B}) \in GL_N(\mathbb{Z}). $$

- The first 2ℓ half $\rightsquigarrow$ $f_i = y^{\boldsymbol{u}_{2i-1}},\ g_i = y^{\boldsymbol{u}_{2i}}\ (1 \le i \le \ell)$ **'Canonical coordinates'**
- The latter $N-2\ell$ half ($\ker B$) $\rightsquigarrow$ $\kappa_i = y^{\boldsymbol{u}_{2\ell+i}}\ (1 \le i \le N-2\ell)$ **Casimir**

The new variables $\{f_i, g_i, \kappa_i\}$ are Laurent monomials in the original $\{y_i\}$, and vise versa ($\because$ unimodularity of U).

Lemma 6.2 implies

$$\{f_i, f_j\} = \{g_i, g_j\} = 0, \quad \{f_i, g_j\} = h_i \delta_{ij} f_i g_j$$

'Canonical commutation relations'

A transformation φ of (f_i, g_i) preserves $\{\cdot, \cdot\}$ $\iff$ φ preserves the 2-form

$$\omega = \sum_{i=1}^{\ell} \frac{1}{h_i} \frac{\mathrm{d} f_i \wedge \mathrm{d} g_i}{f_i g_i}$$

In general, it holds that $\{\mu_k(y_i), \mu_k(y_j)\} = \mu_k(b_{ij})\, \mu_k(y_i) \mu_k(y_j)$. Hence, in particular, $\forall \varphi \in G_Q$ preserves the Poisson bracket $\{\cdot, \cdot\}$.

Theorem 6.3 G_Q gives a group of birational canonical transformations on $\mathbb{K}(f_1, \ldots, f_\ell, g_1, \ldots, g_\ell)$ in the sense that $\forall \varphi \in G_Q$ preserves the 2-form ω. Here the coefficient field is $\mathbb{K} = \mathbb{Q}(\kappa_1, \ldots, \kappa_{N-2\ell})$ and the number of canonical coordinates is $2\ell = \mathrm{rank} B$.

$\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_r\}$: a basis of $\ker B$, $\quad r = N - 2\ell$

Assume: Each $\boldsymbol{\eta}_i \in \mathbb{R}^N$ is either of the following two types:

(i) $\boldsymbol{\varepsilon}_C = \sum_{i \in V(C)} \boldsymbol{e}_i$ **cycle type**; (ii) $\boldsymbol{e}_i - \boldsymbol{e}_j$ **transposition type**

where $C \subseteq Q$ is a balanced cycle and $\boldsymbol{e}_i = {}^{\mathrm{T}}(0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0) \in \mathbb{R}^N$.

Introduce the (multiplicative) **root variables** $a_k = y^{\boldsymbol{\eta}_k}$ attached to a basis $\boldsymbol{\eta}_k$ $(1 \leq k \leq r)$ of $\ker B$. Define the corresponding reflections as

$$s_k = \begin{cases} R_C & \text{if } \boldsymbol{\eta}_k \text{ is of cycle type (i)} \\ (i, j) & \text{if } \boldsymbol{\eta}_k \text{ is of transposition type (ii).} \end{cases}$$

Then the transformation law takes the form $s_i(a_j) = a_j {a_i}^{-c_{ij}}$.

(Furthermore, under a certain technical assumption)

Theorem 6.4 $A = (c_{ij})_{i,j=0}^{r-1}$ is a generalized Cartan matrix (GCM) i.e.

(i) $c_{ii} = 2$ (ii) $c_{ij} \in \mathbb{Z}_{\leq 0}$ $(i \neq j)$ (iii) $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$

§7 τ-Function Formalism (Outline)

$Q = (V, E)$, $B = (b_{ij})_{i,j=1}^{N}$: a quiver and its adjacency matrix

	Seed	Mutations
Until now	$(Q, \boldsymbol{y})$	μ_k
From now	$(Q, \boldsymbol{p}, \boldsymbol{x})$	$\mu_k^{(\gamma)}$ $(\gamma \in \mathbb{R})$

- Quiver mutation is the same as before: $Q' = \mu_k^{(\gamma)}(Q) = \mu_k(Q)$
- Variable mutations $\boldsymbol{x}' = \mu_k^{(\gamma)}(\boldsymbol{x})$ and $\boldsymbol{p}' = \mu_k^{(\gamma)}(\boldsymbol{p})$:

$$x_i' = \begin{cases} \dfrac{{p_k}^{\gamma} \prod_{j=1}^{N} {x_j}^{[b_{jk}]_+} + {p_k}^{\gamma-1} \prod_{j=1}^{N} {x_j}^{[-b_{jk}]_+}}{x_k} & (i = k) \\ x_i & (i \neq k) \end{cases}$$

$$p_i' = \begin{cases} {p_k}^{-1} & (i = k) \\ p_i\, {p_k}^{\gamma[b_{ki}]_+ + (1-\gamma)[b_{ik}]_+} & (i \neq k) \end{cases}$$

In general, it holds that

$$\mu_k^{(\gamma)} \circ \mu_k^{(1-\gamma)} = \mathrm{id} \quad \text{and} \quad \mu_i^{(\gamma)} \circ \mu_j^{(\delta)} = \mu_j^{(\delta)} \circ \mu_i^{(\gamma)} \ \text{ if } \ b_{ij} = 0.$$

Note that $\hat{y}_i = p_i \prod_{j=1}^{N} x_j^{b_{ji}}$ obey the same mutation rule as y_i.

Let $C = (1 \to 2 \to \cdots \to n \to 1) \subseteq Q$ be a balanced cycle. For any $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) \in \mathbb{R}^{n-1}$, we define

$$R_C[\boldsymbol{\gamma}] = \mu_1^{(1-\gamma_1)} \circ \cdots \circ \mu_{n-1}^{(1-\gamma_{n-1})} \circ (n-1, n) \circ \mu_{n-1}^{(\gamma_{n-1})} \circ \cdots \circ \mu_1^{(\gamma_1)}$$

Reflection associated with a cycle graph C

Lemma 7.1 **(Rotational symmetry of R_C with parameter change)**

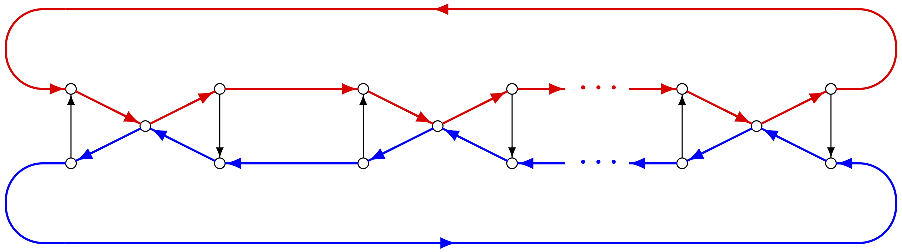
$$R_C[\boldsymbol{\gamma}] = \rho^{-\ell} \circ R_C[\boldsymbol{\gamma}'] \circ \rho^{\ell}, \quad \rho = (1, 2, \ldots, n) \in \mathfrak{S}_n$$

where

$$\gamma'_i = \frac{\Gamma_{i+\ell} - \Gamma_\ell}{\Gamma_{i+\ell+1} - \Gamma_\ell},$$

with $\Gamma_0 = 0$, $\Gamma_i = \prod_{j=i}^{n-1} \gamma_j \ (1 \le i \le n-1)$, $\Gamma_{i+n} = \Gamma_i + 1 \quad (i \in \mathbb{Z})$.

As similar to §4, this rotational symmetry is a crucial key to proving the relations of reflections. For example, we have



Theorem 7.2 Let C_1 and C_2 be two intersecting cycles. It holds that

$$R_{C_1}[\gamma]R_{C_2}[\delta] = R_{C_2}[\delta]R_{C_1}[\gamma]$$

if parameters (γ, δ) satisfy the constraint

$$\sum_{u=v\in C_1\cap C_2} (\Gamma_{u-1}(\Delta_{v+1} - \Delta_v) + \Gamma_u(\Delta_{v-1} - \Delta_{v+1}) + \Gamma_{u+1}(\Delta_v - \Delta_{v-1})) = 0.$$

As before, Γ_i and Δ_i are certain monomials in γ and δ, respectively.

Thank you for your attention!