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Web-seminar on Painlevé Equations and related topics

q -Painlevé equation and q -middle convolution

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1 Abstract

It is known that q -difference Painlevé equations are obtained as compatibility conditions of two linear q -difference operators (Lax pair). The main target of the talk is the linear q -difference equation which produces the q -Painlevé VI by the connection preserving deformation, which was introduced by Jimbo and Sakai in 1996.

We investigate the symmetry of the linear q -difference equations, especially by the q -middle convolution. It is related to the q -Heun equation and its q -integral transformation.

Some basic matters on the q -middle convolution will be also explained.

2 Painlevé equations (P1, P2, ..., P6)

$$\text{P1} \quad \frac{d^2 \lambda}{dt^2} = 6\lambda^2 + t,$$

$$\text{P2} \quad \frac{d^2 \lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha,$$

⋮

⋮

$$\begin{aligned} \text{P6} \quad \frac{d^2 \lambda}{dt^2} = & \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \left(\frac{d\lambda}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \frac{d\lambda}{dt} \\ & + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \left\{ \alpha + \beta \frac{t}{\lambda^2} + \gamma \frac{(t-1)}{(\lambda-1)^2} + \delta \frac{t(t-1)}{(\lambda-t)^2} \right\}. \end{aligned}$$

Painlevé equations are characterized by the **Painlevé property**: the only movable singularities are poles.

Painlevé equations can be obtained by the compatibility condition of two linear differential operators (Lax pair).

In the case of Painlevé VI (P6), it is related to monodromy preserving deformation of the linear differential equations

$$\frac{dY}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} \right) Y, \quad Y = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix},$$

where A_0, A_1, A_t are 2×2 matrices.

Painlevé variables λ, μ are contained in the coefficients of these matrices.

 λ is realized as the zero of $a_{12}(x)$, $\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t} = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$.

Painlevé VI $\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda},$

$$H = \frac{1}{t(t-1)} [\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) + (\theta_t-1)\lambda(\lambda-1)\}\mu + \kappa_1(\kappa_2+1)(\lambda-t)].$$

3 Discrete Painlevé equations

In 1990s, discrete Painlevé equations were studied from the following viewpoints

- Singularity confinement
- Algebraic entropy = 0: (polynomial growth of degrees of solutions)
- Symmetry (of affine Weyl group)

Sakai (2001): Geometric construction from algebraic surfaces

elliptic difference $E_8^{(1)}$

q -difference $E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_5^{(1)} (= qP6), A_4^{(1)}, E_3^{(1)}, \dots$

additive difference $E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_4^{(1)} (= P6), A_3^{(1)} (= P5), \dots$

Kajiwara, Noumi, Yamada gave a review of discrete Painlevé equations (J. Phys. A, 2017)

In this talk, we mainly discuss q -difference $D_5^{(1)} (= qP6)$, which we denote by $q-P(D_5^{(1)})$. ($q-P(E_6^{(1)})$ and $q-P(E_7^{(1)})$ will also appear.)

q - $P(D_5^{(1)})$: q -**Painlevé VI** (Jimbo, Sakai (1996))

Under the condition $b_1 b_2 a_3 a_4 = q a_1 a_2 b_3 b_4$,

$$\frac{f(t)f(qt)}{b_3 b_4} = \frac{(g(t) - ta_1)(g(t) - ta_2)}{(g(t) - a_3)(g(t) - a_4)},$$

$$\frac{g(t)g(qt)}{a_3 a_4} = \frac{(f(qt) - tb_1)(f(qt) - tb_2)}{(f(qt) - b_3)(f(qt) - b_4)},$$

It arises from the connection matrix preserving deformation of the linear q -difference equations;

$$Y(qx) = (B_0 + B_1 x + B_2 x^2)Y(x), \quad Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

(B_0, B_1, B_2 : 2×2 matrices with non-degenerate conditions)

By the limit $q \rightarrow 1$, we have the usual Painlevé VI and Fuchsian differential equation with the singularities $\{0, 1, t, \infty\}$.

Description by Kajiwara, Noumi, Yamada

[6] K. Kajiwara, M. Noumi, Y. Yamada,

Geometric aspects of Painlevé equations. *J. Phys. A* **50** (2017), 073001.

q - $P(D_5^{(1)})$: Under the condition $\kappa_1^2 \kappa_2^2 = q\nu_1\nu_2 \dots \nu_8$,

$$f\bar{f} = \nu_3\nu_4 \frac{(g - \frac{\nu_5}{\kappa_2})(g - \frac{\nu_6}{\kappa_2})}{(g - \frac{1}{\nu_1})(g - \frac{1}{\nu_2})}, \quad g\underline{g} = \frac{1}{\nu_1\nu_2} \frac{(f - \frac{\kappa_1}{\nu_7})(f - \frac{\kappa_1}{\nu_8})}{(f - \nu_3)(f - \nu_4)}$$

$\bar{\cdot}$ time evolution, $\underline{\cdot}$ inverse time evolution.

$$\overline{\kappa_1} = \kappa_1/q, \quad \overline{\kappa_2} = q\kappa_2, \quad \overline{\nu_i} = \nu_i \quad (i = 1, \dots, 8).$$

Weyl group symmetry was described.

They described other q -Painlevé equations by using the same variables
($f, g, \kappa_1, \kappa_2, \nu_1, \dots, \nu_8$)

Unified description for q -Painlevé equations

A Lax pair of q - $P(D_5^{(1)})$ by Kajiwara, Noumi, Yamada

$$L_1 = \left\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \frac{\nu_5}{\kappa_2})(g - \frac{\nu_6}{\kappa_2})}{fg} \right\}$$

$$+ \frac{\nu_1\nu_2(\frac{z}{q} - \nu_3)(\frac{z}{q} - \nu_4)}{f - \frac{z}{q}}(g - T_z^{-1}) + \frac{(\frac{\kappa_1}{\nu_7} - z)(\frac{\kappa_1}{\nu_8} - z)}{q(f - z)} \left(\frac{1}{g} - T_z \right)$$

$$L_2 = \left(1 - \frac{f}{z} \right) T + T_z - \frac{1}{g}$$

T : time evolution $T(f) = \bar{f}$, $T_z : T_z h(z) = h(qz)$.

q -Painlevé equation q - $P(D_5^{(1)})$ is described as the compatibility condition for L_1 and L_2 .

The equation $L_1 y = 0$ is a second-order q -difference equation

$$\frac{\nu_1 \nu_2 \left(\frac{z}{q} - \nu_3\right) \left(\frac{z}{q} - \nu_4\right)}{\frac{z}{q} - f} \left(y\left(\frac{z}{q}\right) - g y(z) \right) + \frac{\left(\frac{\kappa_1}{\nu_7} - z\right) \left(\frac{\kappa_1}{\nu_8} - z\right)}{q(z - f)} \left(y(qz) - \frac{1}{g} y(z) \right) \\ + \left\{ z \frac{(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1 \nu_2 \nu_3 \nu_4 \left(g - \frac{\nu_5}{\kappa_2}\right) \left(g - \frac{\nu_6}{\kappa_2}\right)}{fg} \right\} y(z) = 0.$$

It is essentially equivalent to the linear q -difference equation of Jimbo and Sakai for $y_1(x)$.

$$Y(qx) = (B_0 + B_1 x + B_2 x^2) Y(x), \quad Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Plan of this talk

We observe relationships among a q -deformation of Heun's differential equation, q -Painlevé equation $q-P(D_5^{(1)})$ and the initial value space.
(Similar relationships for $q-P(E_6^{(1)})$, $q-P(E_7^{(1)})$ [9, 14])

We review the Weyl group symmetry $W(D_5^{(1)})$ and $\widetilde{W}(D_5^{(1)})$.

We investigate the symmetry of the linear q -difference equations which are associated with Lax pairs of some q -Painlevé equations [9, 14].

In particular, we look into the symmetry which is related to the q -middle convolution.

Some basic matters on the q -middle convolution will be also explained.

We observe a relationship between the time evolution on the q -Painlevé equations and the parallel translation from the Weyl group symmetry.

4 q -Heun equation and q -Painlevé equation

Heun's differential equation ($\gamma + \delta + \epsilon = \alpha + \beta + 1$)

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)} y = 0,$$

It is a standard form of Fuchsian differential equation with **four** regular singularities $\{0, 1, t, \infty\}$.

The parameter B is called an accessory parameter.

Hypergeometric differential equation is a standard form of Fuchsian differential equation with **three** regular singularities $\{0, 1, \infty\}$.

$$z(1-z) \frac{d^2y}{dz^2} + (\gamma - (\alpha + \beta + 1)z) \frac{dy}{dz} - \alpha\beta y = 0.$$

The hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z) (= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n)$ satisfies it.

4.1 q -hypergeometric equation

The basic hypergeometric series by Heine (1846)

$${}_2\phi_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} x^n, \quad (\lambda, q)_n = \prod_{i=0}^{n-1} (1 - \lambda q^i).$$

It satisfies the q -hypergeometric equation

$$(x - q)f(x/q) - ((a + b)x - q - c)f(x) + (abx - c)f(qx) = 0.$$

If $q \rightarrow 1$, then we obtain the hypergeometric differential equation.

Coefficients of $f(x/q)$, $f(x)$ and $f(qx)$ are polynomials of degree 1.

The q -Heun equation is defined by replacing the coefficients with polynomials of degree 2.

4.2 q -Heun equation

$$\begin{aligned} & \{a_2x^2 + a_1x + a_0\}g(x/q) - \{b_2x^2 + b_1x + b_0\}g(x) \\ & + \{c_2x^2 + c_1x + c_0\}g(xq) = 0, \quad (a_2a_0c_2c_0 \neq 0). \end{aligned}$$

It was introduced by Hahn (1971, [4]) as a q -difference analogue of Heun's differential equation.

It was rediscovered in 2016 ([12]) by considering degeneration of one particle Ruijsenaars-van Diejen system four times.

Fourth degenerate Ruijsenaars-van Diejen operator ($T_x^\pm g(x) = g(q^{\pm 1}x)$)
 $A^{\langle 4 \rangle}(x; h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta) = x^{-1}(x - q^{h_1+1/2}t_1)(x - q^{h_2+1/2}t_2)T_x^{-1}$
 $+ q^{\alpha_1+\alpha_2}x^{-1}(x - q^{l_1-1/2}t_1)(x - q^{l_2-1/2}t_2)T_x$
 $- \{(q^{\alpha_1} + q^{\alpha_2})x + q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}(q^{\beta/2} + q^{-\beta/2})t_1t_2x^{-1}\}.$

q -Heun equation is written as

$$A^{\langle 4 \rangle}(x; h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta)g(x) = Eg(x) \quad (E: \text{eigenvalue}),$$

As $q \rightarrow 1$, we obtain Heun's differential equation

$$\frac{d^2 y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)} y = 0,$$

[4] W. Hahn, *Funkcial. Ekvac.* **14** (1971), 73–78.

[12] K. Takemura, *J. Integrable Systems* **2** (2017), xyx008.

Heun's equation in terms of elliptic function
 [Inozemtsev system (one particle)]

$$\left\{ -\frac{d^2}{dx^2} + \sum_{i=0}^3 l_i(l_i + 1)\wp(x + \omega_i) \right\} f(x) = Ef(x)$$

⇔ equivalent

$$\frac{d^2 y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)} y = 0$$

Heun's differential equation

⇒ difference analogue

Ruijsenaars-van Diejen system (one particle)

$$Ag(x) = Eg(x)$$

⇓ trigonometric limit

$$A^{\langle 1 \rangle} g(x) = Eg(x)$$

⇓ degeneration

variant of q -Heun equation of degree 4

$$A^{\langle 2 \rangle} g(x) = Eg(x)$$

⇓ degeneration

variant of q -Heun equation of degree 3

$$A^{\langle 3 \rangle} g(x) = Eg(x)$$

⇓ degeneration

q -difference analogue

q -Heun equation (degree 2)

$$A^{\langle 4 \rangle} g(x) = Eg(x)$$

4.3 Linear q -difference eqn associated to q -Painlevé eqn

We investigate the linear q -difference equation

$$L_1 y(z) = 0,$$

where L_1 is one of the Lax Pair of q - $P(D_5^{(1)})$. It is written as

$$\begin{aligned} & \frac{\nu_1 \nu_2 \left(\frac{z}{q} - \nu_3\right) \left(\frac{z}{q} - \nu_4\right)}{\frac{z}{q} - f} \left(y\left(\frac{z}{q}\right) - g y(z) \right) + \frac{\left(\frac{\kappa_1}{\nu_7} - z\right) \left(\frac{\kappa_1}{\nu_8} - z\right)}{q(z - f)} \left(y(qz) - \frac{1}{g} y(z) \right) \\ & + \left\{ z \frac{(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1 \nu_2 \nu_3 \nu_4 \left(g - \frac{\nu_5}{\kappa_2}\right) \left(g - \frac{\nu_6}{\kappa_2}\right)}{fg} \right\} y(z) = 0. \end{aligned}$$

It is not the q -Heun equation, but we obtain the q -Heun equation by specializing the parameters f and g .

We substitute $f = \nu_3$ to the equation $L_1 y(z) = 0$. Then we have

$$\begin{aligned}
& (z - \nu_3)(z - q\nu_4)y(z/q) + \frac{1}{\nu_1\nu_2} \left(z - \frac{\kappa_1}{\nu_7}\right) \left(z - \frac{\kappa_1}{\nu_8}\right)y(qz) \\
& - \left[\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)z^2 - \left(\frac{(\nu_3\nu_7 - \kappa_1)(\nu_4\nu_7 - \kappa_1)}{\nu_1\nu_2\nu_3\nu_7\nu_7} \frac{1}{g} \right. \right. \\
& \left. \left. + \frac{\nu_3}{\nu_1} + \frac{\nu_3}{\nu_2} + \frac{q\nu_4\nu_5}{\kappa_2} + \frac{q\nu_4\nu_6}{\kappa_2}\right)z \right. \\
& \left. + \frac{q\nu_3\nu_4\nu_5^{1/2}\nu_6^{1/2}}{\kappa_2} \left\{ \left(\frac{\nu_6}{\nu_5}\right)^{1/2} + \left(\frac{\nu_6}{\nu_5}\right)^{-1/2} \right\} \right] y(z) = 0.
\end{aligned}$$

It is the q -Heun equation. The accessory parameter corresponds to the term $\left(\frac{(\nu_3\nu_7 - \kappa_1)(\nu_4\nu_7 - \kappa_1)}{\nu_1\nu_2\nu_3\nu_7\nu_7} \frac{1}{g} + \frac{\nu_3}{\nu_1} + \frac{\nu_3}{\nu_2} + \frac{q\nu_4\nu_5}{\kappa_2} + \frac{q\nu_4\nu_6}{\kappa_2}\right)$.

Namely, **the parameter g plays a role of the accessory parameter.**

We substitute $(f, g) = (f_1, f_1 g_1 + \nu_5/\kappa_2)$ into $L_1 y(z) = 0$, and set $f_1 = 0$. Then we obtain

$$\begin{aligned} & (z - q\nu_3)(z - q\nu_4)y(z/q) + \frac{1}{\nu_1\nu_2} \left(z - \frac{\kappa_1}{\nu_7} \right) \left(z - \frac{\kappa_1}{\nu_8} \right) y(qz) \\ & - \left[\left(\frac{1}{\nu_1} + \frac{1}{\nu_2} \right) z^2 + \left\{ g_1 q \nu_3 \nu_4 \left(1 - \frac{\nu_6}{\nu_5} \right) - \frac{q\nu_5(\nu_3 + \nu_4)}{\kappa_2} - \frac{\kappa_1 \kappa_2 (\nu_7 + \nu_8)}{\nu_1 \nu_2 \nu_5 \nu_7 \nu_8} \right\} z \right. \\ & \left. + \frac{q\kappa_1 \nu_3^{1/2} \nu_4^{1/2}}{\nu_1^{1/2} \nu_2^{1/2} \nu_7^{1/2} \nu_8^{1/2}} \left\{ \left(\frac{q\nu_6}{\nu_5} \right)^{1/2} + \left(\frac{q\nu_6}{\nu_5} \right)^{-1/2} \right\} \right] y(z) = 0. \end{aligned}$$

It is the q -Heun equation.

The blow-up $(f, g) = (f_1, f_1 g_1 + \nu_5/\kappa_2)$ at $(f, g) = (0, \nu_5/\kappa_2)$ is related to the initial-value space of q - $P(D_5^{(1)})$.

For details, see Sasaki-Takagi-T. [10].

In summary, the q -Heun equation is related to the q - $P(D_5^{(1)})$ through the Lax pair and the initial-value space.

Lax equation associated to $e\text{-}P(E_8^{(1)})$ (elliptic-difference)	\Rightarrow	Ruijsenaars-van Diejen system (one particle) Noumi-Ruijsenaars -Yamada (2020)	$Ay(x) = Ey(x)$
			\Downarrow trigonometric limit
Lax equation associated to $q\text{-}P(E_8^{(1)})$	\Rightarrow	Sasaki-Takagi-T. (2023)	$A^{\langle 1 \rangle}y(x) = Ey(x)$
			\Downarrow degeneration
Lax equation associated to $q\text{-}P(E_7^{(1)})$	\Rightarrow	variant of q -Heun equation of degree 4 T. (2017) STT (2023)	$A^{\langle 2 \rangle}y(x) = Ey(x)$
			\Downarrow degeneration
Lax equation associated to $q\text{-}P(E_6^{(1)})$	\Rightarrow	variant of q -Heun equation of degree 3 T. (2017) STT (2023)	$A^{\langle 3 \rangle}y(x) = Ey(x)$
			\Downarrow degeneration
Lax equation associated to $q\text{-}P(D_5^{(1)})$	\Rightarrow	q -Heun equation (degree 2) T. (2017) STT (2023)	$A^{\langle 4 \rangle}y(x) = Ey(x)$
\vdots		specialization of (f, g) to E	\vdots

Degenerate cases: Chihiro Sato and T. (2026) Phys. Scr.

In the case $q = 1$, the results corresponds to the relationship among the sixth Painlevé equation, 2×2 Fuchsian system with four singularities and Heun equation.

2×2 Fuchsian system with sing. $\{0, 1, t, \infty\}$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y, \quad Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}. \quad (1)$$

We determine elements of A_0, A_1, A_t under the following assumption:

Eigenvalues of A_i : $\theta_i, 0, (i = 0, 1, t)$

$$A_0 + A_1 + A_t = - \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad \theta_\infty = \kappa_1 - \kappa_2.$$

$$\text{Then (1,2)-element } a_{12}(z) = \frac{k(z-\lambda)}{z(z-1)(z-t)}.$$

On the case $\lambda \neq 0, 1, t, \infty$, by adding the relation $\mu = a_{11}(\lambda)$, the elements of A_0, A_1, A_t are determined by $\theta_0, \theta_1, \theta_t, \theta_\infty, \lambda, \mu, k$.

By eliminating $y_2(z)$ in Eq.(1), we have

$$\begin{aligned} & \frac{d^2 y_1}{dz^2} + \left(\frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_t}{z - t} - \frac{1}{z - \lambda} \right) \frac{dy_1}{dz} \\ & + \left(\frac{\kappa_1(\kappa_2 + 1)}{z(z - 1)} + \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} - \frac{t(t - 1)H}{z(z - 1)(z - t)} \right) y_1 = 0, \end{aligned} \quad (2)$$

$$\begin{aligned} H = \frac{1}{t(t - 1)} & [\lambda(\lambda - 1)(\lambda - t)\mu^2 - \{\theta_0(\lambda - 1)(\lambda - t) \\ & + \theta_1\lambda(\lambda - t) + (\theta_t - 1)\lambda(\lambda - 1)\}\mu + \kappa_1(\kappa_2 + 1)(\lambda - t)]. \end{aligned}$$

Five regular singularities $z = 0, 1, t, \infty, \lambda$;

$z = \lambda$: apparent singularity (non-log.)

The monodromy preserving condition for Eq. (2) is written as

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda},$$

and this is written as the sixth Painlevé equation on λ .

Namely the sixth Painlevé equation is obtained by the monodromy preserving deformation of Eq. (2) (or Eq. (1)).

If we set $\lambda = 0$ in Eq. (2), then we obtain Heun's differential equation.

We can obtain the Heun equation by other ways of restricting parameters, and it is related to the initial value space of Painlevé VI.

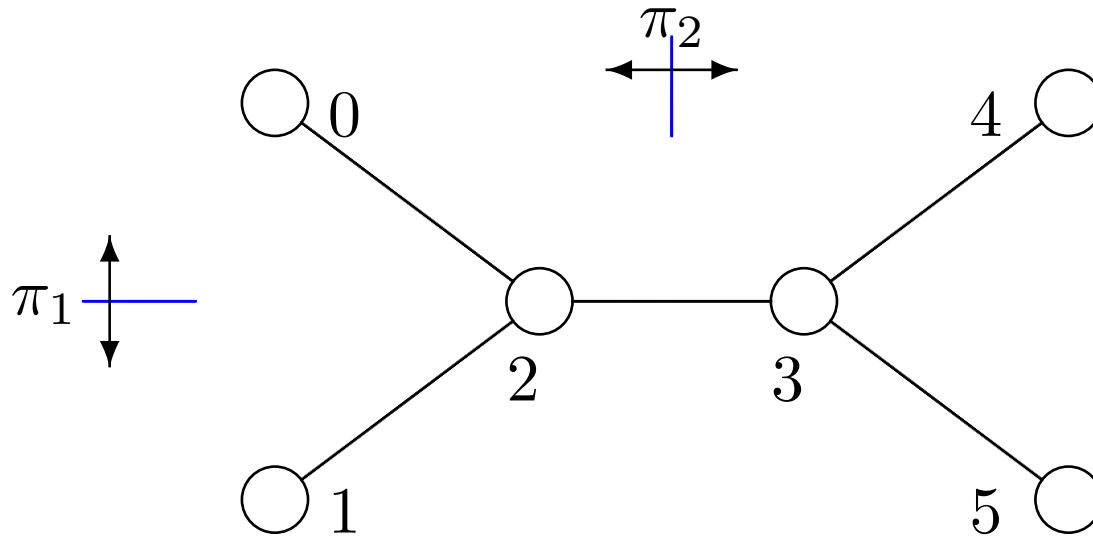
[11] K. Takemura, Middle convolution and Heun's equation, *SIGMA* 5 (2009), paper 040.

The equation $L_1 y(z) = 0$ in the q -deformed situation corresponds to Eq. (2) in the case $q = 1$.

The appearance of q -Heun equation by blow-up of the parameters (e.g. $(f, g) = (f_1, f_1 g_1 + \nu_5 / \kappa_2)$ and $f_1 = 0$) in the q -deformed situation corresponds to the restrictions to Heun's differential equation in the case $q = 1$.

5 Weyl group symmetry of $q-P(D_5^{(1)})$

Let $W(D_5^{(1)})$ be the affine Weyl group (or infinite Coxeter group) which corresponds to the following diagram.



Generators of $W(D_5^{(1)})$: s_0, s_1, \dots, s_5 .

Relations: $s_i^2 = \text{id}$ ($i = 0, 1, \dots, 5$), $s_0 s_1 = s_1 s_0$, $s_0 s_2 s_0 = s_2 s_0 s_2$ etc.

Extended affine Weyl group $\widetilde{W}(D_5^{(1)})$ is defined by adding the generators π_1, π_2 (+ relations $\pi_1^2 = \pi_2^2 = (\pi_1 \pi_2)^4 = \text{id}$, $\pi_1 s_0 = s_1 \pi_1$ etc.), which correspond to the diagram automorphism.

Kajiwara, Noumi, Yamada [6] introduced a representation of $\widetilde{W}(D_5^{(1)})$, which is associated to q - $P(D_5^{(1)})$.

Define the action of the generators $s_0, \dots, s_5, \pi_1, \pi_2$ to the variables $f, g, \nu_1, \dots, \nu_8, \kappa_1, \kappa_2, q$ by

$$s_0 : \nu_7 \leftrightarrow \nu_8, \quad s_1 : \nu_3 \leftrightarrow \nu_4$$

$$s_2 : \nu_3 \rightarrow \frac{\kappa_1}{\nu_7}, \quad \nu_7 \rightarrow \frac{\kappa_1}{\nu_3}, \quad \kappa_2 \rightarrow \frac{\kappa_1 \kappa_2}{\nu_3 \nu_7}, \quad g \rightarrow g \frac{f - \nu_3}{f - \frac{\kappa_1}{\nu_7}}$$

$$s_3 : \nu_1 \rightarrow \frac{\kappa_2}{\nu_5}, \quad \nu_5 \rightarrow \frac{\kappa_2}{\nu_1}, \quad \kappa_1 \rightarrow \frac{\kappa_1 \kappa_2}{\nu_1 \nu_5}, \quad f \rightarrow f \frac{g - \frac{1}{\nu_1}}{g - \frac{\nu_5}{\kappa_2}}$$

$$s_4 : \nu_1 \leftrightarrow \nu_2, \quad s_5 : \nu_5 \leftrightarrow \nu_6$$

$$\pi_1 : q \rightarrow 1/q, \nu_1 \rightarrow 1/\nu_1, \nu_2 \rightarrow 1/\nu_2, \nu_3 \rightarrow 1/\nu_7, \dots$$

$$\nu_8 \rightarrow 1/\nu_4, \kappa_1 \rightarrow 1/\kappa_1, \kappa_2 \rightarrow 1/\kappa_2, f \rightarrow f/\kappa_1, g \rightarrow 1/g$$

$$\pi_2 : q \rightarrow 1/q, \nu_1 \rightarrow 1/\nu_7, \dots$$

The representation of $\widetilde{W}(D_5^{(1)})$ is defined on the rational functions with the variables $f, g, \nu_1, \nu_2, \dots, \nu_8, \kappa_1, \kappa_2, q$ with the relation

$$\kappa_1^2 \kappa_2^2 = q \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8.$$

Recall that the principal variables of the q -Painlevé equations are f and g , and the variables $\nu_1, \nu_2, \dots, \nu_8, \kappa_1, \kappa_2$ are supplementary.

For the other q -Painlevé equations, the representations of the extended affine root systems are also defined on the rational functions with the variables $f, g, \nu_1, \nu_2, \dots, \nu_8, \kappa_1, \kappa_2, q$.

We recall the linear q -difference equation $L_1 y(z) = 0$ ass. to q - $P(D_5^{(1)})$.

$$\left\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \frac{\nu_5}{\kappa_2})(g - \frac{\nu_6}{\kappa_2})}{fg} \right\} y(z) \\ + \frac{\nu_1\nu_2(\frac{z}{q} - \nu_3)(\frac{z}{q} - \nu_4)}{f - \frac{z}{q}} (gy(z) - y(z/q)) \\ + \frac{(z - \frac{\kappa_1}{\nu_7})(z - \frac{\kappa_1}{\nu_8})}{q(f - z)} \left(\frac{1}{g} y(z) - y(qz) \right) = 0.$$

By the gauge-transformation, $y(z) = \tilde{y}(z) \frac{(q\nu_3/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty}$, we obtain the equation for $\tilde{y}(z)$, whose parameters are changed by

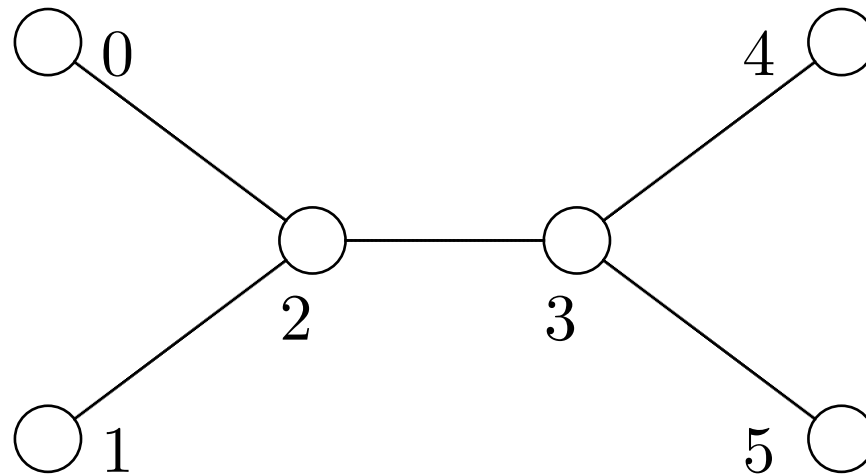
$$\tilde{\nu}_3 = \frac{\kappa_1}{\nu_7}, \quad \tilde{\nu}_7 = \frac{\kappa_1}{\nu_3}, \quad \tilde{\kappa}_2 = \frac{\kappa_1 \kappa_2}{\nu_3 \nu_7}, \quad \tilde{g} = g \frac{f - \nu_3}{f - \frac{\kappa_1}{\nu_7}}.$$

Namely we obtain the action of s_2 by this gauge-transformation.

The operator s_2 : gauge-transformation.

The operator s_3 is related to the q -middle convolution ([9]).

[9] S. Sasaki, S. Takagi, K. Takemura, q -middle convolution and q -Painlevé equation, *SIGMA*, **18** (2022), paper 056.



Middle convolution : Introduced by Katz (1996).

Dettweiler and Reiter reformulated for Fuchsian differential equations

$$\frac{dY(x)}{dx} = \left(\frac{A_1}{x - a_1} + \cdots + \frac{A_r}{x - a_r} \right) Y(x). \quad (A_1, \dots, A_r): m \times m \text{ matrices.}$$

Middle convolution: Convolution + quotient by some subspace

Convolution \Leftrightarrow Euler's integral transf. $y(x) \mapsto \int_{\Delta} y(s)(x - s)^{\lambda} ds$

q -middle convolution : Sakai and Yamaguchi (2011, 2017 IMRN)

$$Y(qx) = B(x)Y(x), \quad B(x) = B_{\infty} + \sum_{i=1}^N \frac{B_i}{1 - x/b_i}$$

We slightly modify the q -convolution. (Arai, T. arXiv:2503.11214)

Merits on reformulation:

Slightly simpler expression.

Convergence of the q -integral transformation ass. to the q -convolution.

One more parameter ξ in q -integration.

Composition: $mc_{\mu}^q \cdot mc_{\lambda}^q \simeq mc_{\lambda+\mu}^q$ (S-Y. $\bar{\Psi}_{\mu} \cdot \bar{\Psi}_{\lambda} \simeq \bar{\Psi}_{\log(q^{\lambda} + q^{\mu} - 1) / \log q}$)

Replacement of q -difference equations

$$Y(qx) = \left(B_\infty + \sum_{i=1}^N \frac{B_i}{1 - x/b_i} \right) Y(x).$$

Set $B_0 = I_m - B_\infty - B_1 - \dots - B_N$ and $b_0 = 0$.

Then the q -difference equation is equivalent to

$$\frac{Y(qx) - Y(x)}{-x} = \left[\frac{B_0}{x} + \sum_{i=1}^N \frac{B_i}{x - b_i} \right] Y(x) = \left[\sum_{i=0}^N \frac{B_i}{x - b_i} \right] Y(x). \quad (3)$$

Addition

Define the addition add_μ by

$$add_\mu : (B_0, B_1, \dots, B_N) \mapsto ((1 - q^\mu)I_m + q^\mu B_0, q^\mu B_1, \dots, q^\mu B_N).$$

If $Y(x) \in \mathbb{C}^m$ satisfies (3), then $V(x) = x^\mu Y(x)$ satisfies the equation induced by add_μ .

Addition is obtained from the gauge-transformation

6 Reformulation of q -middle convolution

Reformulated q -convolution is associated to

$$\frac{Y(qx) - Y(x)}{-x} = \sum_{i=0}^N \frac{B_i}{x - b_i} Y(x) \rightarrow \frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \sum_{i=0}^N \frac{G_i}{x - b_i} \tilde{Y}(x)$$

Definition 1 (Reformulation of q -convolution, Arai-T. (2025)).

Let $\mathbf{B} = (B_0, B_1, \dots, B_N)$ be the tuple of $m \times m$ matrices and $\lambda \in \mathbb{C}$. Define the q -convolution $c_\lambda^q : (B_0, B_1, \dots, B_N) \mapsto (G_0, G_1, \dots, G_N)$ by

$$G_0 = \begin{pmatrix} q^{-\lambda} B_0 + (1 - q^{-\lambda}) I_m & q^{-\lambda} B_1 & \cdots & q^{-\lambda} B_N \\ O & O & & O \\ \vdots & \vdots & & \vdots \end{pmatrix},$$

(G_0, G_1, \dots, G_N are $(N + 1)m \times (N + 1)m$ matrices)

$$\begin{aligned}
G_1 &= \begin{pmatrix} O & O & & O \\ q^{-\lambda} B_0 & q^{-\lambda} B_1 + (1 - q^{-\lambda}) I_m & \cdots & q^{-\lambda} B_N \\ O & O & & O \\ \vdots & \vdots & & \vdots \end{pmatrix}, \\
&\vdots \\
G_N &= \begin{pmatrix} \vdots & \vdots & & \vdots \\ O & O & & O \\ q^{-\lambda} B_0 & q^{-\lambda} B_1 & \cdots & q^{-\lambda} B_N + (1 - q^{-\lambda}) I_m \end{pmatrix}.
\end{aligned}$$

Sakai-Yamaguchi defined the q -convolution c_λ^{SY} for $(B_\infty; B_1, \dots, B_N)$,

$$Y(qx) = \left[B_\infty + \sum_{i=1}^N \frac{B_i}{1 - x/b_i} \right] Y(x).$$

On two q -convolutions, we have $c_\lambda^q = add_{-\lambda} \circ c_\lambda^{SY}$.

q -convolution and q -integral transformation

Jackson integral ($\xi \in \mathbb{C} \setminus \{0\}$)

$$\int_0^{\xi\infty} f(s) d_q s = (1 - q) \sum_{n=-\infty}^{\infty} q^n \xi f(q^n \xi).$$

Kernel function

Let $\tilde{P}_\lambda(x, s) = \tilde{P}_\lambda^{(1)}(x, s)$ or $\tilde{P}_\lambda^{(2)}(x, s)$, where

$$\tilde{P}_\lambda^{(1)}(x, s) = x^{-\lambda} \frac{(q^{\lambda+1} s/x; q)_\infty}{(qs/x; q)_\infty}, \quad \tilde{P}_\lambda^{(2)}(x, s) = s^{-\lambda} \frac{(x/s; q)_\infty}{(q^{-\lambda} x/s; q)_\infty}.$$

($(x; q)_\infty = (1 - x)(1 - qx)(1 - q^2 x) \cdots$)

$\tilde{P}_\lambda(x, s)$ satisfies the relation

$$q^\lambda \tilde{P}_\lambda(qx, s) = \tilde{P}_\lambda(x, s/q) = \frac{x - q^\lambda s}{x - s} \tilde{P}_\lambda(x, s).$$

q -convolution is related to the q -integral transformation whose kernel function is $\tilde{P}_\lambda(x, s)$.

q -convolution and q -integral transformation

Theorem 2. ([2], cf. [8, Theorem 2.1], [1, Corollary 2.3]) Set $b_0 = 0$.

Let $Y(x)$ be a solution to
$$\frac{Y(qx) - Y(x)}{-x} = \left[\sum_{i=0}^N \frac{B_i}{x - b_i} \right] Y(x).$$

Let $\xi \in \mathbb{C} \setminus \{0\}$. Set

$$\tilde{Y}_i(x) = \int_0^{\xi^\infty} \frac{\tilde{P}_\lambda(x, s)}{s - b_i} Y(s) d_q s, \quad (i = 0, \dots, N), \quad \tilde{Y}(x) = \begin{pmatrix} \tilde{Y}_0(x) \\ \vdots \\ \tilde{Y}_N(x) \end{pmatrix}.$$

If every element of $\tilde{Y}_i(x)$ converges for $i = 0, 1, \dots, N$ and

$$\lim_{K \rightarrow -\infty} \tilde{P}_\lambda(x, q^{K-1}\xi) Y(q^K \xi) = 0, \quad \lim_{L \rightarrow +\infty} \tilde{P}_\lambda(x, q^L \xi) Y(q^{L+1}\xi) = 0,$$

then the function $\tilde{Y}(x)$ satisfies

$$\frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \left[\sum_{i=0}^N \frac{G_i}{x - b_i} \right] \tilde{Y}(x).$$

A primitive version of Thm 2 was obtained by Sakai-Yamaguchi.

The parameter ξ was added in [Sasaki-Takagi-T(2022)], [Arai-T(2023)].

To prove the theorem, we introduce the finite summations

$$\tilde{Y}_i^{[K,L]}(x) = (1 - q)x^{-\lambda} \sum_{n=K}^L \frac{\tilde{P}_\lambda(x, s)}{s - b_i} sY(s)|_{s=q^n \xi}, \quad (i = 0, 1, \dots, N).$$

Proposition 3. (Inhomogeneous equation for finite summation)

$$\text{Set } \tilde{Y}^{[K,L]}(x) = \begin{pmatrix} \tilde{Y}_0^{[K,L]}(x) \\ \vdots \\ \tilde{Y}_N^{[K,L]}(x) \end{pmatrix}, \quad \tilde{f}^{[K,L]}(x) = (1 - q)q^{-\lambda}x^{-\lambda} \begin{pmatrix} \frac{x}{x-b_0} (\tilde{P}_\lambda(x, q^{K-1}\xi)Y(q^K\xi) - \tilde{P}_\lambda(x, q^L\xi)Y(q^{L+1}\xi)) \\ \vdots \\ \frac{x}{x-b_N} (\tilde{P}_\lambda(x, q^{K-1}\xi)Y(q^K\xi) - \tilde{P}_\lambda(x, q^L\xi)Y(q^{L+1}\xi)) \end{pmatrix}.$$

Then we have

$$\frac{\tilde{Y}^{[K,L]}(qx) - \tilde{Y}^{[K,L]}(x)}{-x} = \left[\sum_{i=0}^N \frac{G_i}{x - b_i} \right] \tilde{Y}^{[K,L]}(x) - \frac{1}{x} \tilde{f}^{[K,L]}(x).$$

In the situation of Prop 3, we assume that $\tilde{Y}^{[K,L]}(x)$ converges as $K \rightarrow -\infty$ and $L \rightarrow +\infty$ and $\tilde{P}_\lambda(x, q^{K-1}\xi)Y(q^K\xi) \rightarrow 0$ ($K \rightarrow \pm\infty$).

Then $\tilde{Y}^{[K,L]}(x)$ converges to $\tilde{Y}(x)$ as $K \rightarrow -\infty$ and $L \rightarrow +\infty$, and $\tilde{Y}(x)$ satisfies

$$\frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \left[\sum_{i=0}^N \frac{G_i}{x - b_i} \right] \tilde{Y}(x).$$

A sufficient condition for convergence is given by the inequalities on the eigenvalues of two matrices $I_m - B_0$ and $I_m - B_0 - B_1 - \cdots - B_N$.

Theorem 4. (Arai-T. 2025 [2]) Set $b_0 = 0$. Let $Y(x)$ be a solution to

$$\frac{Y(qx) - Y(x)}{-x} = \left[\sum_{i=0}^N \frac{B_i}{x - b_i} \right] Y(x).$$

If the absolute value of any eigenvalue of $I_m - B_0$ is strictly less than 1 and the absolute value of any eigenvalue of $I_m - B_0 - B_1 - \cdots - B_N$ is strictly more than $|q^\lambda|$, then the assumption of the convergence in Thm 2 holds and the integral representation $\tilde{Y}(x)$ satisfies

$$\frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \left[\sum_{i=0}^N \frac{G_i}{x - b_i} \right] \tilde{Y}(x).$$

q -convolution does not preserve irreducibility in general.

$(G_0, G_1, \dots, G_N: (N+1)m \times (N+1)m$ matrices)

Definition 5 (q -middle convolution, the same as Sakai-Yamaguchi).

We define the \mathbf{G} -invariant subspaces \mathcal{K} and \mathcal{L} of $(\mathbb{C}^m)^{N+1}$ as follows;

$$\mathcal{K} = \begin{pmatrix} \ker B_0 \\ \vdots \\ \ker B_N \end{pmatrix}, \quad \mathcal{L} = \ker(G_0 + G_1 + \dots + G_N).$$

We denote the action of G_k on the quotient space $(\mathbb{C}^m)^{N+1}/(\mathcal{K} + \mathcal{L})$ by \overline{G}_k ($k = 0, 1, \dots, N$). Then the q -middle convolution mc_λ^q is defined by the correspondence $(B_0, B_1, \dots, B_N) \mapsto (\overline{G}_0, \overline{G}_1, \dots, \overline{G}_N)$.

$$mc_\lambda^q : (B_0, B_1, \dots, B_N) \mapsto (\overline{G}_0, \overline{G}_1, \dots, \overline{G}_N).$$

7 Examples of q -middle convolution

7.1 q -hypergeometric equation

We obtain q -hypergeometric equation from a scalar q -difference eqn.

The function $y(x) = x^\mu \frac{(\alpha x; q)_\infty}{(\beta x; q)_\infty}$ satisfies $y(qx) = q^\mu \frac{1 - \beta x}{1 - \alpha x} y(x)$.

It is written as the following scalar q -difference equation.

$$\frac{y(qx) - y(x)}{-x} = \left[\frac{B_0}{x} + \frac{B_1}{x - 1/\alpha} \right] y(x), \quad B_0 = 1 - q^\mu, \quad B_1 = q^\mu \left(1 - \frac{\beta}{\alpha} \right).$$

We apply q -convolution for $\mathbf{B} = (B_0, B_1)$.

Write $c_\lambda^q(B_0, B_1) = (G_0, G_1) = \mathbf{G}$. Then

$$G_0 = \begin{pmatrix} q^{-\lambda}B_0 + 1 - q^{-\lambda} & q^{-\lambda}B_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - q^{-\lambda+\mu} & q^{-\lambda+\mu}(1 - \beta/\alpha) \\ 0 & 0 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0 & 0 \\ q^{-\lambda}B_0 & q^{-\lambda}B_1 + 1 - q^{-\lambda} \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 \\ q^{-\lambda} - q^{-\lambda+\mu} & q^{-\lambda+\mu}(1 - \beta/\alpha) + 1 - q^{-\lambda} \end{pmatrix}.$$

The corresponding q -difference equation is

$$\frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \left[\frac{G_0}{x} + \frac{G_1}{x - 1/\alpha} \right] \tilde{Y}(x), \quad \tilde{Y}(x) = \begin{pmatrix} \tilde{y}_0(x) \\ \tilde{y}_1(x) \end{pmatrix}.$$

The q -difference equation for $\tilde{y}_0(x)$ is written as

$$(q^{-\lambda}\beta x - q)\tilde{y}_0(x/q) + q^{\lambda-\mu}(\alpha x - q)\tilde{y}_0(qx) \\ - \{(q^{-\mu}\alpha + \beta)x - q - q^{\lambda-\mu+1}\}\tilde{y}_0(x) = 0. \quad (4)$$

By specializing the parameters, we obtain the standard form of the q -hypergeometric equation

$$(x - q)g(x/q) + (abx - c)g(qx) - \{(a + b)x - q - c\}g(x) = 0.$$

By Thm 2 (q -integral transformation related to q -convolution), we obtain q -integral representations of solutions to the equation

$$\frac{\tilde{Y}(qx) - \tilde{Y}(x)}{-x} = \left[\frac{G_0}{x} + \frac{G_1}{x - 1/\alpha} \right] \tilde{Y}(x), \quad \tilde{Y}(x) = \begin{pmatrix} \tilde{y}_0(x) \\ \tilde{y}_1(x) \end{pmatrix}.$$

The function

$$\tilde{y}_0(x) = (1 - q)x^{-\lambda} \sum_{n=-\infty}^{\infty} (q^n \xi)^\mu \frac{(q^{\lambda+n+1} \xi/x, q^n \xi \alpha; q)_\infty}{(q^{n+1} \xi/x, q^n \xi \beta; q)_\infty}$$

satisfies q -hypergeometric equation (4) for any ξ , if $\mu > 0$ and $|q|^{\lambda-\mu} |\alpha/\beta| < 1$.

We specialize the parameter ξ to obtain one-sided series.

- The case $\xi = 1/\alpha$. ($n \in \mathbb{Z}_{\leq 0} \Rightarrow (q^n; q)_\infty = 0$)

$$\begin{aligned}
\tilde{y}_0(x) &= (1 - q)x^{-\lambda} \sum_{n=1}^{\infty} (q^n/\alpha)^\mu \frac{(q^{\lambda+n+1}/(\alpha x), q^n; q)_\infty}{(q^{n+1}/(\alpha x), q^n \beta/\alpha; q)_\infty} \\
&= (1 - q) \frac{q^\mu}{\alpha^\mu} \frac{(q^{\lambda+2}/(\alpha x), q; q)_\infty}{(q^2/(\alpha x), q\beta/\alpha; q)_\infty} x^{-\lambda} {}_2\phi_1 \left(\begin{matrix} q^2/(\alpha x), q\beta/\alpha \\ q^{\lambda+2}/(\alpha x) \end{matrix} ; q, q^\mu \right) \\
&= (1 - q) \frac{q^\mu}{\alpha^\mu} \frac{(q^{\mu+1}\beta/\alpha, q; q)_\infty}{(q^\mu, q\beta/\alpha; q)_\infty} x^{-\lambda} {}_2\phi_1 \left(\begin{matrix} q^\lambda, q^\mu \\ q^{\mu+1}\beta/\alpha \end{matrix} ; q, q^2/(\alpha x) \right).
\end{aligned}$$

- The case $\xi = q^{-\lambda}x$.

$$\begin{aligned}
\hat{y}_0(x) &= (1 - q)q^{-\lambda\mu} x^\mu \frac{(q^{-\lambda}\alpha x, q; q)_\infty}{(q^{-\lambda}\beta x, q^{1-\lambda}; q)_\infty} {}_2\phi_1 \left(\begin{matrix} q^{-\lambda}\beta x, q^{1-\lambda} \\ q^{-\lambda}\alpha x \end{matrix} ; q, q^\mu \right) \\
&= (1 - q)q^{-\lambda\mu} \frac{(q^{1-\lambda+\mu}, q; q)_\infty}{(q^\mu, q^{1-\lambda}; q)_\infty} x^\mu {}_2\phi_1 \left(\begin{matrix} \alpha/\beta, q^\mu \\ q^{1-\lambda+\mu} \end{matrix} ; q, q^{-\lambda}\beta x \right).
\end{aligned}$$

7.2 Other q -hypergeometric type equation

We can obtain variants of q -hypergeometric equation and the generalized q -hypergeometric equations of order 3 by applying q -middle convolutions and additions appropriately.

Solutions in terms of Jackson integrals are also obtained.

By specializing the parameter ξ , we obtain solutions in terms of ${}_3\phi_2$ etc.

[1] Y. Arai, K. Takemura, On q -middle convolution and q -hypergeometric equations, *SIGMA* **19** (2023), paper 037, arXiv:2209.02227.

[2] Y. Arai, K. Takemura, Reformulation of q -middle convolution and applications, arXiv:2503.11214.

7.3 Linear q -eqn of Jimbo-Sakai and q -middle convolution

[9] S. Sasaki, S. Takagi, K. Takemura, q -middle convolution and q -Painlevé equation, *SIGMA*, **18** (2022), paper 056.

Linear q -difference equation of Jimbo-Sakai:

$$Y(qx, t) = A(x, t)Y(x, t), \quad Y(x, t) = \begin{bmatrix} y_1(x, t) \\ y_2(x, t) \end{bmatrix},$$

$$A(x, t) = A_0(t) + A_1(t)x + A_2x^2 = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad A_2 = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix},$$

$$\det A(x, t) = \chi_1\chi_2(x - ta_1)(x - ta_2)(x - a_3)(x - a_4),$$

The eigenvalues of $A_0(t)$ are $t\theta_1, t\theta_2$. (Assume $\chi_1\chi_2a_1a_2a_3a_4 = \theta_1\theta_2$)

We introduce the parameters w, y, z and impose the condition.

$$a_{12}(x) = \chi_2w(x - y), \quad a_{11}(x)|_{x=y} = (y - ta_1)(y - ta_2)/(qz).$$

Then the elements of $A(x, t)$ are determined as

$$A(x, t) = \begin{pmatrix} \chi_1((x - y)(x - \alpha) + z_1) & \chi_2 w(x - y) \\ \chi_1 w^{-1}(\gamma x + \delta) & \chi_2((x - y)(x - \beta) + z_2) \end{pmatrix}$$

where

$$\alpha = \frac{1}{\chi_1 - \chi_2} [y^{-1}((\theta_1 + \theta_2)t - \chi_1 z_1 - \chi_2 z_2) - \chi_2((a_1 + a_2)t + a_3 + a_4 - 2y)]$$

$$\beta = \frac{1}{\chi_1 - \chi_2} [-y^{-1}((\theta_1 + \theta_2)t - \chi_1 z_1 - \chi_2 z_2) + \chi_1((a_1 + a_2)t + a_3 + a_4 - 2y)]$$

$$\gamma = z_1 + z_2 + (y + \alpha)(y + \beta) + (\alpha + \beta)y - a_1 a_2 t^2 - (a_1 + a_2)(a_3 + a_4)t - a_3 a_4$$

$$\delta = y^{-1}(a_1 a_2 a_3 a_4 t^2 - (\alpha y + z_1)(\beta y + z_2))$$

and

$$z_1 = \frac{(y - ta_1)(y - ta_2)}{q\chi_1 z}, \quad z_2 = q\chi_1(y - a_3)(y - a_4)z.$$

By a gauge transformation, we change $Y(qx, t) = A(x, t)Y(x, t)$ to

$$Y(qx) = B(x)Y(x), \quad B(x) = B(x, t) = \frac{A(x, t)}{c_0(x - ta_1)(x - ta_2)},$$

where $c_0 \in \mathbb{C} \setminus \{0\}$. Write

$$\frac{Y(qx) - Y(x)}{-x} = \left[\frac{B_0}{x} + \frac{B_1}{x - ta_1} + \frac{B_2}{x - ta_2} \right] Y(x).$$

Then, $\det B_1 = 0$, $\det B_2 = 0$.

$$\det B_0 = 0 \Leftrightarrow c_0 = \frac{\theta_1}{ta_1 a_2} \text{ or } \frac{\theta_2}{ta_1 a_2}.$$

$$\text{Set } c_0 = \frac{\theta_1}{ta_1 a_2}.$$

There exist non-zero vectors $\begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix}$, $\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$, $\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ s.t.

$$B_0 \begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B_2 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We apply the q -middle convolution mc_λ^q to (B_0, B_1, B_2) .

$$mc_\lambda^q(\mathbb{C}^2) = (\mathbb{C}^2)^{\otimes 3} / (\mathcal{K} + \mathcal{L}), \dim \mathcal{K} = 3.$$

If $q^\lambda = \frac{\kappa_1 a_1 a_2 t}{\theta_1}$ or $\frac{\kappa_2 a_1 a_2 t}{\theta_1}$, then $\dim \mathcal{L} = 1$ and $\dim(mc_\lambda^q(\mathbb{C}^2)) = 2$.

Set $q^\lambda = \frac{\kappa_2 a_1 a_2 t}{\theta_1}$. Basis of \mathcal{K} , \mathcal{L} are given as

$$\mathcal{K} = \mathbb{C} \left\langle \begin{pmatrix} v_{01} \\ v_{02} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ v_{11} \\ v_{12} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_{21} \\ v_{22} \end{pmatrix} \right\rangle, \quad \mathcal{L} = \mathbb{C} \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

q -convolution is written as $c_\lambda^q(B_0, B_1, B_2) = (G_0, G_1, G_2)$,

$$G_0 = \begin{pmatrix} B_0 - (1 - q^\lambda)I_2 & B_1 & B_2 \\ & O & \\ & & O \end{pmatrix},$$

$$G_1 = \begin{pmatrix} & O & & \\ B_0 & B_1 - (1 - q^\lambda)I_2 & B_2 & \\ & & & O \end{pmatrix}, \quad G_2 = \begin{pmatrix} & & & O \\ & & & O \\ B_0 & B_1 & B_2 - (1 - q^\lambda)I_2 & \end{pmatrix}.$$

To obtain the matrix elements of $mc_\lambda^q(B_0, B_1, B_2)$ on the quotient space $(\mathbb{C}^2)^{\otimes 3}/(\mathcal{K} + \mathcal{L}) \simeq \mathbb{C}^2$, we consider a simultaneous similarity transformation $P^{-1}G_iP$, $i = 0, 1, 2$. Set

$$P = \begin{pmatrix} 0 & 0 & 0 & v_{01} & 0 & 0 \\ g_1 & g_2 & 1 & v_{02} & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{11} & 0 \\ 1 & 1 & 1 & 0 & v_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & v_{21} \\ 0 & 0 & 1 & 0 & 0 & v_{22} \end{pmatrix}, \quad \begin{aligned} \bar{G}_0 &= P^{-1}G_0P, \\ \bar{G}_1 &= P^{-1}G_1P, \\ \bar{G}_2 &= P^{-1}G_2P. \end{aligned}$$

We have the expressions

$$\bar{G}_0 = \begin{pmatrix} \tilde{G}_0 & O \\ * & * \end{pmatrix}, \quad \bar{G}_1 = \begin{pmatrix} \tilde{G}_1 & O \\ * & * \end{pmatrix}, \quad \bar{G}_2 = \begin{pmatrix} \tilde{G}_2 & O \\ * & * \end{pmatrix},$$

where $\tilde{G}_0, \tilde{G}_1, \tilde{G}_2$ are 2×2 matrices. We can choose the values g_1, g_2 such that $\tilde{G}_0 + \tilde{G}_1 + \tilde{G}_2$ is a diagonal matrix.

Proposition 6.

If $q^\lambda = \frac{\kappa_2 a_1 a_2 t}{\theta_1}$, then $mc_\lambda^q(B_0, B_1, B_2) \simeq (\tilde{G}_0, \tilde{G}_1, \tilde{G}_2)$.

Write

$$\frac{\tilde{G}(qx) - \tilde{G}(x)}{-x} = \left[\frac{\tilde{G}_0}{x} + \frac{\tilde{G}_1}{x - ta_1} + \frac{\tilde{G}_2}{x - ta_2} \right] \tilde{G}(x),$$

$$\tilde{G}(qx) = \bar{F}(x) \tilde{G}(x), \quad \tilde{A}(x) = \tilde{c}_0(x - ta_1)(x - ta_2) \bar{F}(x).$$

We choose $\tilde{c}_0 \in \mathbb{C} \setminus \{0\}$ appropriately.

By comparing $\tilde{A}(x)$ with $A(x)$, we obtain the mapping of the parameters y, z, a_1, \dots on the q -middle convolution mc_λ^q as follows.

$$\chi_1 \rightarrow \frac{\chi_1 \chi_2 a_1 a_2 t}{\theta_1}, \quad \chi_2 \rightarrow \chi_2, \quad t\theta_1 \rightarrow \frac{\chi_2 a_1 a_2 t^2 \theta_2}{\theta_1}, \quad t\theta_2 \rightarrow \frac{\chi_2^2 a_1^2 a_2^2 t^3}{\theta_1}$$

$$a_1 \rightarrow a_1, \quad a_2 \rightarrow a_2, \quad a_3 \rightarrow \frac{\chi_2 a_1 a_2 a_3 t}{\theta_1}, \quad a_4 \rightarrow \frac{\chi_2 a_1 a_2 a_4 t}{\theta_1}$$

$$y \rightarrow \tilde{y} = \frac{qz \frac{(y-a_3)(y-a_4)}{(y-ta_1)(y-ta_2)} - \frac{1}{\chi_1}}{qz \frac{(y-a_3)(y-a_4)}{(y-ta_1)(y-ta_2)} - \frac{\theta_1}{\chi_1 \chi_2 a_1 a_2 t}} y,$$

$$z \rightarrow \tilde{z} = \frac{(y-a_3)(y-a_4)}{(y-ta_1)(y-ta_2)} \frac{(\tilde{y}-ta_1)(\tilde{y}-ta_2)}{\left(\tilde{y} - \frac{\chi_2 a_1 a_2 a_3 t}{\theta_1}\right) \left(\tilde{y} - \frac{\chi_2 a_1 a_2 a_4 t}{\theta_1}\right)} z.$$

We can make a correspondence between the variables y, z, a_1, \dots of Jimbo-Sakai and f, g, ν_1, \dots of Kajiwara-Noumi-Yamada.

Then, the transformation of the parameters induced by the q -middle convolution is written as

$$\nu_1 \rightarrow q \frac{\nu_1 \nu_2 \nu_5}{\kappa_2}, \quad \nu_2 \rightarrow \nu_2, \quad \frac{\kappa_2}{\nu_5} \rightarrow q \frac{\nu_2 \nu_5}{\nu_6}, \quad \frac{\kappa_2}{\nu_6} \rightarrow q^2 \frac{\nu_2^2 \nu_5}{\kappa_2},$$

$$\frac{\kappa_1}{\nu_7} \rightarrow \frac{\kappa_1}{\nu_7}, \quad \frac{\kappa_1}{\nu_8} \rightarrow \frac{\kappa_1}{\nu_8}, \quad \nu_3 \rightarrow q \frac{\nu_2 \nu_3 \nu_5}{\kappa_2}, \quad \nu_4 \rightarrow q \frac{\nu_2 \nu_4 \nu_5}{\kappa_2},$$

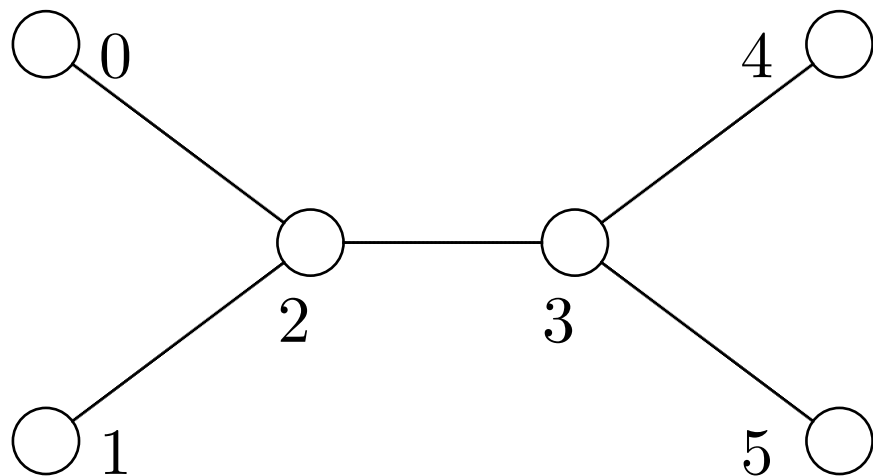
$$f \rightarrow \tilde{f} = \frac{\frac{(f-\nu_3)(f-\nu_4)}{\left(f-\frac{\kappa_1}{\nu_7}\right)\left(f-\frac{\kappa_1}{\nu_8}\right)} g - \frac{1}{\nu_1}}{\frac{(f-\nu_3)(f-\nu_4)}{\left(f-\frac{\kappa_1}{\nu_7}\right)\left(f-\frac{\kappa_1}{\nu_8}\right)} g - \frac{\kappa_2}{q\nu_1\nu_2\nu_5}} f,$$

$$g \rightarrow \tilde{g} = \frac{\left(\tilde{f} - \frac{\kappa_1}{\nu_7}\right)\left(\tilde{f} - \frac{\kappa_1}{\nu_8}\right)}{\left(\tilde{f} - q\frac{\nu_2\nu_3\nu_5}{\kappa_2}\right)\left(\tilde{f} - q\frac{\nu_2\nu_4\nu_5}{\kappa_2}\right)} \frac{(f-\nu_3)(f-\nu_4)}{\left(f-\frac{\kappa_1}{\nu_7}\right)\left(f-\frac{\kappa_1}{\nu_8}\right)} g$$

Theorem 7. The transformation of the parameters induced by the q -middle convolution mc_λ^q ($q^\lambda = \kappa_2 a_1 a_2 t / \theta_1$) is written as

$$s_2 s_1 s_0 s_2 s_5 s_3 s_2 s_0 s_1 s_2.$$

Recall that



$$s_2 : \nu_3 \rightarrow \frac{\kappa_1}{\nu_7}, \quad \nu_7 \rightarrow \frac{\kappa_1}{\nu_3}, \quad \kappa_2 \rightarrow \frac{\kappa_1 \kappa_2}{\nu_3 \nu_7}, \quad g \rightarrow g \frac{f - \nu_3}{f - \frac{\kappa_1}{\nu_7}}$$

$$s_3 : \nu_1 \rightarrow \frac{\kappa_2}{\nu_5}, \quad \nu_5 \rightarrow \frac{\kappa_2}{\nu_1}, \quad \kappa_1 \rightarrow \frac{\kappa_1 \kappa_2}{\nu_1 \nu_5}, \quad f \rightarrow f \frac{g - \frac{1}{\nu_1}}{g - \frac{\nu_5}{\kappa_2}}$$

$$s_0 : \nu_7 \leftrightarrow \nu_8, \quad s_1 : \nu_3 \leftrightarrow \nu_4, \quad s_4 : \nu_1 \leftrightarrow \nu_2, \quad s_5 : \nu_5 \leftrightarrow \nu_6.$$

A proof of the theorem is based on

$$s_2 s_0 s_1 s_2(g) = \frac{f - \nu_3}{f - \frac{\kappa_1}{\nu_7}} \frac{f - \nu_4}{f - \frac{\kappa_1}{\nu_8}} g, \quad s_2 s_0 s_1 s_2 s_5 s_3(f) = \tilde{f}.$$

We calculated the q -integral transformation associated to the q -middle convolution mc_λ^q ($q^\lambda = \kappa_2 a_1 a_2 t / \theta_1$).

Let $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ be a solution to the original q -difference equation associated to (B_0, B_1, B_2) . Then, the function

$$\widehat{Y}(x) = \begin{pmatrix} \widehat{Y}_0(x) \\ \widehat{Y}_1(x) \\ \widehat{Y}_2(x) \end{pmatrix} = \begin{pmatrix} \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s} y_1(s) d_q s \\ \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s} y_2(s) d_q s \\ \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s-ta_1} y_1(s) d_q s \\ \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s-ta_1} y_2(s) d_q s \\ \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s-ta_2} y_1(s) d_q s \\ \int_0^{\xi\infty} \frac{P_\lambda(x,s)}{s-ta_2} y_2(s) d_q s \end{pmatrix}$$

satisfies the q -difference equation which is associated to the q -convolution c_λ^q , if the condition for convergence holds.

Write $\check{Y}(x) = P^{-1}\widehat{Y}(x) = \begin{pmatrix} \check{y}_1(x) \\ \vdots \\ \check{y}_6(x) \end{pmatrix}$. Then, $\check{Y}(x) = \begin{pmatrix} \check{y}_1(x) \\ \check{y}_2(x) \end{pmatrix}$

satisfies the q -difference equation which is associated to the q -middle convolution mc_λ^q , if the condition for convergence holds.

By calculating the integral representations of $\check{y}_1(x)$ and $\check{y}_2(x)$, we have

Theorem 8. The function $\check{Y}(x)$ determined by

$$\check{y}_1(x) = \frac{(a_1 - a_2)(\kappa_1 qz - 1)(\kappa_2 a_1 a_2 t - \theta_1)t}{\kappa_2 w(\kappa_1 a_1 a_2 t - \theta_1)qz} \int_0^{\xi^\infty} \frac{y_1(s)}{s - x} \tilde{P}_\lambda(x, s) d_q s,$$

$$\begin{aligned} \check{y}_2(x) = & \frac{1}{qwyz\chi_2 a_1 a_2} \int_0^{\xi^\infty} \left\{ \frac{qz\theta_1}{s - qy} - \frac{ta_1 a_2}{s - y} \right. \\ & \left. + \left(\frac{qz\theta_1 - ta_1 a_2}{\chi_1 ta_1 a_2 - \theta_1} - \frac{q^2 yz}{s - qy} \right) \frac{\chi_2 ta_1 a_2 - \theta_1}{x - s} \right\} y_1(s) \tilde{P}_\lambda(x, s) d_q s. \end{aligned}$$

formally satisfies the q -difference equation which is associated to the q -middle convolution mc_λ^q ($q^\lambda = \kappa_2 a_1 a_2 t / \theta_1$).

q -integral transformation for q -Heun equation

Set $y = a_3$ in Theorem 8.

The original q -difference equation is written as

$$(x - a_1t)(x - a_2t)y_1(qx) + \frac{\kappa_1\kappa_2a_1^2a_2^2t^2(x - a_3)(x - a_4q)}{\theta_1^2q}y_1(x/q) - \left\{ \frac{a_1a_2t(\kappa_2 + \kappa_1q)}{\theta_1q}x^2 + E'x + a_1a_2t^2\left(1 + \frac{\theta_2}{\theta_1}\right) \right\}y_1(x) = 0.$$

If $q^\lambda = \frac{\kappa_2a_1a_2t}{\theta_1}$, then $\check{y}_1(x) = \int_0^{\xi\infty} \frac{y_1(s)}{s - x} \frac{(q^{\lambda+1}s/x; q)_\infty}{(qs/x; q)_\infty} d_qs$ satisfies

$$(x - a_1t)(x - a_2t)\check{y}_1(qx) + \frac{\kappa_1a_1a_2t}{q\theta_1} \left(x - \frac{\kappa_2a_1a_2a_3t}{\theta_1}\right) \left(x - \frac{q\kappa_2a_1a_2a_4t}{\theta_1}\right)\check{y}_1(x/q) - \left\{ \frac{q\kappa_1a_1a_2t + \theta_1}{q\theta_1}x^2 + E'x + \frac{\kappa_2a_1^2a_2^2t^2(\theta_1t + a_3a_4\kappa_1)}{\theta_1^2} \right\}\check{y}_1(x) = 0.$$

These equations are q -Heun equations.

Theorem 9. [Sasaki-Takagi-T. 2022]

Assume that $l_1 l_2 l_3 l_4 = h_1 h_2 h_3 q^2$ and $y(x)$ satisfies the q -Heun equation

$$(x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q) + l_3 l_4 (x - l_1 q^{-1/2})(x - l_2 q^{-1/2})g(qx) \\ - \{(l_3 + l_4)x^2 + l_3 l_4 E x + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2} (h_3^{1/2} + h_3^{-1/2})\}g(x) = 0.$$

If λ satisfies $q^\lambda = q/l_4$, then

$$\check{y}(x) = \int_0^{\xi^\infty} \frac{y(s)}{s-x} \frac{(q^{\lambda+1} s/x; q)_\infty}{(qs/x; q)_\infty} d_q s \text{ formally satisfies}$$

$$(x - h'_1 q^{1/2})(x - h'_2 q^{1/2})g(x/q) + l'_3 l'_4 (x - l'_1 q^{-1/2})(x - l'_2 q^{-1/2})g(qx) \\ - \{(l'_3 + l'_4)x^2 + l'_3 l'_4 E x + (l'_1 l'_2 l'_3 l'_4 h'_1 h'_2)^{1/2} ((h'_3)^{1/2} + (h'_3)^{-1/2})\}g(x) = 0, \\ l'_1 = l_1, l'_2 = l_2, l'_3 = l_3, l'_4 = q, h'_1 = qh_1/l_4, h'_2 = qh_2/l_4, h'_3 = l_4/(qh_3).$$

We can calculate the q -integral transformation from the q -middle convolution mc_λ^q for the other parameter $q^\lambda = \kappa_1 a_1 a_2 t / \theta_1$.

Then, the Weyl group symmetry is written as s_3 in $W(D_5^{(1)})$, although we cannot obtain any integral transformation of q -Heun equation by restricting the parameters.

In summary, formal integral transformation of q -Heun equation was obtained from the q -middle convolution mc_λ^q where $q^\lambda = \kappa_2 a_1 a_2 t / \theta_1$, whose symmetry is $s_2 s_1 s_0 s_2 s_5 s_3 s_2 s_0 s_1 s_2$ in $W(D_5^{(1)})$.

However, it is not simple to obtain argument for convergence.

By considering the q -integral transformation by the kernel identity, we can obtain q -integral transformations of q -Heun equation with attention to convergence.

[15] T., Kernel function, q -integral transformation and q -Heun equations, *SIGMA* **20** (2024), paper 083.

Theorem 10 ([15], q -integral transformations of q -Heun equations).

Assume that the function $h(s)$ satisfies the q -Heun equation

$$A^{(4)}(s; h'_1, h'_2, l'_1, l'_2, \alpha'_1, \alpha'_2, \beta')h(s) = E'h(s) \text{ and}$$

$$\lim_{L \rightarrow +\infty} \frac{h(s)}{s^{(h'_1+h'_2-l'_1-l'_2-\alpha'_1-\alpha'_2+\beta'+2)/2}} \Big|_{s=q^L \xi} = C_1, \quad \lim_{K \rightarrow -\infty} \frac{h(s)}{s^{-\alpha'_1}} \Big|_{s=q^K \xi} = C_2$$

for some constants E', C_1, C_2 .

Let $h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta, E, \chi, \mu, \mu_0$ be the parameters which satisfy

$$\chi = (l'_1 + l'_2 - h'_1 - h'_2 - \alpha'_1 + \alpha'_2 - \beta')/2, \quad \mu = \mu_0 + 1 + \chi,$$

$$E = q^{\mu_0 + \alpha_1 - \alpha'_1} E', \quad \beta = -\beta' - \chi, \quad \alpha_2 = \alpha_1 - \alpha'_1 + \alpha'_2 - \chi,$$

$$l_i = l'_i + \mu_0, \quad h_i = h'_i + \mu_0 + \chi, \quad i = 1, 2.$$

Define the kernel functions by

$$P_{\mu, \mu_0}^{(1)}(x, s) = \frac{(q^\mu s/x; q)_\infty}{(q^{\mu_0} s/x; q)_\infty}, \quad P_{\mu, \mu_0}^{(2)}(x, s) = \left(\frac{x}{s}\right)^{\mu - \mu_0} \frac{(q^{-\mu_0 + 1} x/s; q)_\infty}{(q^{-\mu + 1} x/s; q)_\infty}.$$

(i) The Jackson integral

$$g(x) = x^{-\alpha_1} \int_0^{\xi^\infty} s^{-(h'_1+h'_2-l'_1-l'_2-\alpha'_1-\alpha'_2+\beta'+2)/2} h(s) P_{\mu, \mu_0}^{(1)}(x, s) d_q s$$

converges and it satisfies

$$A^{\langle 4 \rangle}(x; h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta)g(x) = Eg(x) + (1 - q)(k_2(x) - k_1(x)),$$

$$k_1(x) = C_1 x^{-\alpha_1} q^{\mu_0 + \alpha_1 + h'_1 + h'_2 + \chi} (q^{\beta'} - 1)t_1 t_2,$$

$$k_2(x) = C_2 x^{-\alpha_1} \frac{\vartheta_q(q^{-\mu_0 - \chi} x / \xi)}{\vartheta_q(q^{-\mu_0 + 1} x / \xi)} \xi^{\chi + 1} q^{\mu_0 + \alpha_1} (q^{\alpha'_2 - \alpha'_1} - 1).$$

(ii) The Jackson integral

$$g(x) = x^{-\alpha_1} \int_0^{\xi^\infty} s^{-(h'_1+h'_2-l'_1-l'_2-\alpha'_1-\alpha'_2+\beta'+2)/2} h(s) P_{\mu, \mu_0}^{(2)}(x, s) d_q s$$

converges and it satisfies

$$A^{\langle 4 \rangle}(x; h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta)g(x) = Eg(x) + (1 - q)(k_2(x) - k_1(x)),$$

$$k_1(x) = C_1 x^{-\alpha_1 + \chi + 1} \xi^{-\chi - 1} \frac{\vartheta_q(q^{-\mu_0 + 1} x / \xi)}{\vartheta_q(q^{-\mu_0 - \chi} x / \xi)} q^{\mu_0 + \alpha_1 + h'_1 + h'_2 + \chi} (q^{\beta'} - 1) t_1 t_2,$$

$$k_2(x) = C_2 x^{-\alpha_1 + \chi + 1} q^{\mu_0 + \alpha_1} (q^{\alpha'_2 - \alpha'_1} - 1).$$

In the theorem, if $C_1 = C_2 = 0$, then

$$A^{\langle 4 \rangle}(x; h_1, h_2, l_1, l_2, \alpha_1, \alpha_2, \beta)g(x) = Eg(x),$$

and we obtain the q -integral transformations of solutions to the q -Heun equations with different parameters.

Murakami and T. (2026) obtained special solutions of q -Heun equation which are expressed as finite sum of q -hypergeometric functions by using the q -integral transformations.

8 Symmetry of Linear q -eqn. assoc. to q - $P(D_5^{(1)})$

We recall the linear q -diff. equation $L_1 y(z) = 0$ assoc. to q - $P(D_5^{(1)})$.

$$\left\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \frac{\nu_5}{\kappa_2})(g - \frac{\nu_6}{\kappa_2})}{fg} \right\} y(z) \\ + \frac{\nu_1\nu_2(\frac{z}{q} - \nu_3)(\frac{z}{q} - \nu_4)}{f - \frac{z}{q}} (gy(z) - y(z/q)) \\ + \frac{(z - \frac{\kappa_1}{\nu_7})(z - \frac{\kappa_1}{\nu_8})}{q(f - z)} \left(\frac{1}{g} y(z) - y(qz) \right) = 0.$$

s_2 is related to the gauge-transf. $y(z) = \tilde{y}(z) \frac{(q\nu_3/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty}$.

s_3 is related to the q -middle convolution.

Another gauge-transformation

Let $y(z)$ be a solution to the linear q -difference equation $L_1 y(z) = 0$, and set $y(z) = z^d \tilde{y}(z)$.

Then, the function $\tilde{y}(z)$ satisfies the equation $\tilde{L}_1 \tilde{y}(z) = 0$, whose parameters are changed by

$$(\nu_1, \nu_2, \nu_5/\kappa_2, \nu_6/\kappa_2, g) \mapsto (q^{-d}\nu_1, q^{-d}\nu_2, q^d\nu_5/\kappa_2, q^d\nu_6/\kappa_2, q^d g).$$

We denote by $G[a]$ the transformation of the parameters

$$(\nu_1, \nu_2, \nu_5/\kappa_2, \nu_6/\kappa_2, g) \mapsto (\nu_1/a, \nu_2/a, a\nu_5/\kappa_2, a\nu_6/\kappa_2, ag).$$

Dilation $z \mapsto cz$

Set $z = u/c$ and $y(z) = \tilde{y}(u) = \tilde{y}(cz)$.

If $y(z)$ satisfies $L_1 y(z) = 0$, then $\tilde{y}(u)$ satisfies $\tilde{L}_1 \tilde{y}(u) = 0$, whose parameters are changed by

$$(\nu_3, \nu_4, \nu_7/\kappa_1, \nu_8/\kappa_1, f) \mapsto (c\nu_3, c\nu_4, \nu_7/\kappa_1/c, \nu_8/\kappa_1/c, cf)$$

We denote this transformation of the parameters by $D[c]$.

By the way, the number of the parameters $\nu_1, \nu_2, \dots, \nu_8; \kappa_1, \kappa_2$ are too much for q -Painlevé equation q - $P(D_5^{(1)})$ and the Weyl group symmetry $\widetilde{W}(D_5^{(1)})$.

The gauge-transformation $G[a]$ and the dilation $D[c]$ are extra symmetry.

We introduce an equivalent of the parameters by taking the gauge-transformation $G[a]$ and the dilation $D[c]$ into account.

$$\begin{aligned}
& (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g) \\
& \sim (\nu'_1, \nu'_2, \nu'_3, \nu'_4, \nu'_5, \nu'_6, \nu'_7, \nu'_8; \kappa'_1, \kappa'_2; f', g') \\
& \Leftrightarrow \exists a, b, c, d \in \mathbb{C}(\nu_1, \nu_2, \dots, \nu_8, \kappa_1, \kappa_2, f, g) \setminus \{0\} \text{ such that} \\
& (\nu'_1, \nu'_2, \nu'_3, \nu'_4, \nu'_5, \nu'_6, \nu'_7, \nu'_8; \kappa'_1, \kappa'_2; f', g') \\
& = (\nu_1/a, \nu_2/a, c\nu_3, c\nu_4, ab\nu_5, ab\nu_6, \nu_7/(cd), \nu_8/(cd); \kappa_1/d, b\kappa_2; cf, ag)
\end{aligned}$$

This part is not presented in our previous papers [9, 10, 14]

It is shown that the actions $s_0, s_1, s_2, s_3, s_4, s_5, \pi_1, \pi_2$ are compatible with the equivalence.

More precisely, we have the following proposition.

Proposition 11. (Sato, T., Yamashita, *in preparation*)

Let $(\nu'_1, \nu'_2, \nu'_3, \nu'_4, \nu'_5, \nu'_6, \nu'_7, \nu'_8; \kappa'_1, \kappa'_2; f', g')$ be a tuple of the parameters which is equivalent to

$(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g)$.

The actions of the generators of $\widetilde{W}(D_5^{(1)})$ to

$(\nu'_1, \nu'_2, \nu'_3, \nu'_4, \nu'_5, \nu'_6, \nu'_7, \nu'_8; \kappa'_1, \kappa'_2; f', g')$ are written as the same form of the actions to $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g)$ up to the equivalence.

Recall that

$$\begin{aligned} & (s_2(\nu_1), s_2(\nu_2), s_2(\nu_3), \dots, s_2(\nu_8); s_2(\kappa_1), s_2(\kappa_2); s_2(f), s_2(g)) \\ &= (\nu_1, \nu_2, \kappa_1/\nu_7, \nu_4, \nu_5, \nu_6, \kappa_1/\nu_3, \nu_8; \kappa_1, \kappa_1\kappa_2/(\nu_3\nu_7); f, g \frac{f - \nu_3}{f - \frac{\kappa_1}{\nu_7}}). \end{aligned}$$

We have $s_2(\nu'_3) = s_2(c)s_2(\nu_3) = s_2(c)\kappa_1/\nu_7 = (\kappa'_1/\nu'_7)(s_2(c)/c)$.

$$\begin{aligned}
& (s_2(\nu'_1), s_2(\nu'_2), s_2(\nu'_3), s_2(\nu'_4), s_2(\nu'_5), s_2(\nu'_6), s_2(\nu'_7), s_2(\nu'_8); \\
& \quad s_2(\kappa'_1), s_2(\kappa'_2); s_2(f'), s_2(g')) \\
& = (\nu'_1 a / s_2(a), \nu'_2 a / s_2(a), (\kappa'_1 / \nu'_7)(s_2(c) / c), \nu'_4 s_2(c) / c, \nu'_5 s_2(ab) / (ab), \\
& \quad \nu'_6 s_2(ab) / (ab), (\kappa'_1 / \nu'_3) cd / s_2(cd), \nu'_8 cd / s_2(cd); \kappa'_1 d / s_2(d), \\
& \quad \kappa'_1 \kappa'_2 / (\nu'_3 \nu'_7) s_2(b) cd / (bcd); f' s_2(c) / c, g' \frac{f' - \nu'_3}{f' - \frac{\kappa'_1}{\nu'_7}} s_2(a) / a) \\
& \sim (\nu'_1, \nu'_2, \kappa'_1 / \nu'_7, \nu'_4, \nu'_5, \nu'_6, \kappa'_1 / \nu'_3, \nu'_8; \kappa'_1, \kappa'_1 \kappa'_2 / (\nu'_3 \nu'_7); f', g' \frac{f' - \nu'_3}{f' - \frac{\kappa'_1}{\nu'_7}})
\end{aligned}$$

Therefore, the realization of the action s_2 to $(\nu'_1, \nu'_2, \dots; f', g')$ is similar to the one to the action s_2 to $(\nu_1, \nu_2, \dots; f, g)$ up to the equivalence.

We can also show similar relations for the actions $s_0, s_1, s_3, s_4, s_5, \pi_1, \pi_2$.

9 q - $P(D_5^{(1)})$ in terms of Weyl group $\widetilde{W}(D_5^{(1)})$

In Sakai's geometric realization, the time evolution of discrete Painlevé equations is described by the parallel translation in the extended affine Weyl group.

$W(D_5)$: finite Weyl group of type D_5 .

Q : Root lattice of type D_5 , P : Weight lattice of type D_5 .

On affine Weyl group $W(D_5^{(1)})$ and extended affine Weyl group $\widetilde{W}(D_5^{(1)})$, we have

$$W(D_5^{(1)}) \simeq W(D_5) \ltimes Q, \quad \widetilde{W}(D_5^{(1)}) \simeq W(D_5) \ltimes P$$

By choosing a special element of P as a parallel translation (i.e. $(\pi_2\pi_1s_2s_1s_0s_2)^2$) on Sakai's representation of $\widetilde{W}(D_5^{(1)})$ coming from the algebraic surface related with the initial value space, we would obtain the q -Painlevé equation with the symmetry $D_5^{(1)}$.

On the other hand, in Kajiwara Noumi Yamada's realization [6], the expression of the parallel translation in the affine Weyl group is not compatible with the discrete Painlevé equation at a glance.

In the case q - $P(D_5^{(1)})$, a candidate of the parallel translation is $(\pi_2\pi_1s_2s_1s_0s_2)^2$.

$$(\pi_2\pi_1s_2s_1s_0s_2)^2(f) = \frac{q\nu_7\nu_8\nu_3\nu_4}{\kappa_1 f} \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{(g - 1/\nu_1)(g - 1/\nu_2)}.$$

It is different from $\bar{f} = \frac{\nu_3\nu_4}{f} \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{(g - 1/\nu_1)(g - 1/\nu_2)}$.

Moreover,

$$(\pi_2\pi_1s_2s_1s_0s_2)^2(\nu_3) = \frac{q\nu_7\nu_8}{\kappa_1} \nu_3, \quad (\pi_2\pi_1s_2s_1s_0s_2)^2(\nu_1) = \frac{\kappa_2}{\nu_1\nu_2} \nu_1, \dots$$

To fix the difference, we introduce the transformation Ξ by

$$\begin{aligned} & (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g) \mapsto \\ & \left(\frac{\nu_5\nu_6}{\kappa_2}\nu_1, \frac{\nu_5\nu_6}{\kappa_2}\nu_2, \frac{\kappa_1}{q\nu_3\nu_4}\nu_3, \frac{\kappa_1}{q\nu_3\nu_4}\nu_4, \frac{\nu_1\nu_2}{\kappa_2}\nu_5, \frac{\nu_1\nu_2}{\kappa_2}\nu_6, \right. \\ & \quad \left. \frac{\kappa_1}{q\nu_7\nu_8}\nu_7, \frac{\kappa_1}{q\nu_7\nu_8}\nu_8; \frac{\kappa_1^2}{q^2\nu_3\nu_4\nu_7\nu_8}\kappa_1, \frac{\nu_1\nu_2\nu_5\nu_6}{\kappa_2^2}\kappa_2; \frac{\kappa_1}{q\nu_3\nu_4}f, \frac{\kappa_2}{\nu_5\nu_6}g \right). \end{aligned}$$

It originates from the gauge-transformation and the dilation. Namely,

$$\Xi(\zeta) = G\left[\frac{\kappa_2}{\nu_5\nu_6}\right]D\left[\frac{\kappa_1}{q\nu_7\nu_8}\right](\zeta),$$

for $\zeta \in \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5/\kappa_2, \nu_6/\kappa_2, \nu_7/\kappa_1, \nu_8/\kappa_1, f, g\}$.

The transformation Ξ is within the equivalence of the parameters in Proposition 11.

Theorem 12. (*T.* arXiv:2201.07529, RIMS **B91** (2023) [14]) Set

$$T = \Xi \cdot (\pi_2 \pi_1 s_2 s_1 s_0 s_2)^2.$$

Then,

$$T(f) = \frac{\nu_3 \nu_4 (g - \nu_5 / \kappa_2)(g - \nu_6 / \kappa_2)}{f (g - 1/\nu_1)(g - 1/\nu_2)},$$

$$T(g) = \frac{1}{gT(\nu_1)T(\nu_2)} \frac{(T(f) - T(\kappa_1/\nu_7))(T(f) - T(\kappa_1/\nu_8))}{(T(f) - T(\nu_3))(T(f) - T(\nu_4))},$$

and we recover the time evolution of the q -Painlevé equation $q\text{-}P(D_5^{(1)})$.
On the parameters, we have

$$T(\nu_i) = \nu_i \quad (i = 1, 2, 3, 4, 5, 6, 7, 8), \quad T(\kappa_1) = q^{-1} \kappa_1, \quad T(\kappa_2) = q \kappa_2.$$

We can obtain similar results for $q\text{-}P(E_6^{(1)})$ and $q\text{-}P(E_7^{(1)})$ (see [14]).
There is a mistake for the expression Ξ in the case $q\text{-}P(E_6^{(1)})$ in [14].

Kajiwara-Noumi-Yamada: $\widetilde{W}(D_5^{(1)})$ acts on the variables $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g)$.

Sakai: $\widetilde{W}(D_5^{(1)})$ acts on the variables $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4; f, g)$.

The number of the parameters of Kajiwara-Noumi-Yamada realization is too much, and it may cause the difference between the time evolution and the Weyl group symmetry.

By considering equivalence of the parameters, we can deduce the number of the parameters.

Then, we can fix the difference between the time evolution and the Weyl group symmetry by finding the transformation Ξ .

Summary

- Painlevé equations and discrete Painlevé equations.
 - Hypergeometric differential equation, Heun's differential equation, q -hypergeometric equation and q -Heun equation
 - q -Heun equation, degenerate Ruijsenaars-van Diejen systems and q -Painlevé equations (q - $P(D_5^{(1)})$) etc.)
 - A Lax pair (L_1, L_2) and Weyl group symmetries $W(D_5^{(1)})$ and $\widetilde{W}(D_5^{(1)})$.
 - Symmetry of the equation $L_1 y(z) = 0$.
 - q -middle convolution
 - A realization of q - $P(D_5^{(1)})$ in terms of the symmetry $\widetilde{W}(D_5^{(1)})$.
- We needed to adjust the parameters by the equivalence.
- Similar results for q - $P(E_6^{(1)})$ and q - $P(E_7^{(1)})$.

Problems and related works

- The problem to study the q -middle convolution for the linear q -difference equation $L_1 y(z) = 0$ of Kajiwara-Noumi-Yamada on $q-P(E_6^{(1)})$ and $q-P(E_7^{(1)})$ should be clarified in a near future. (The work by Fujii-Nobukawa ([3], arXiv:2207.12777) might be related to this problem)
- q -integral transformation for variants of the q -Heun equation (T. [15]).
- Linear 3×3 q -difference equation on $q-P(E_6^{(1)})$ and q -middle convolution by Park (2023 SIGMA).
- Linear 3×3 q -difference equation, $E_8^{(1)}$ symmetry and q -middle convolution by Nobukawa (arXiv:2601.06070).

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