A generalization of the q-Garnier system and its Lax form

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Discrete Painlevé equation

In 1990's, Grammaticos and his collaborators proposed a discrete analogue of the Painlevé property called the singularity confinement.

Example

Consider a difference equation

$$x_{n+1} + x_{n-1} = \frac{a x_n}{1 - x_n^2}, \quad x_0 = p, \quad x_1 = 1 + \varepsilon.$$

Then we obtain

$$x_2 = -\frac{a}{2\varepsilon} - \frac{a + 4p}{4} + O(\varepsilon), \quad x_3 = -1 + \varepsilon + O(\varepsilon^2), \quad x_4 = -p + O(\varepsilon).$$

Taking a limit $\varepsilon \to 0$*, we can find that a singularity appears at* x_2 *and disappears at* x_4*.*

That became a trigger for the discovery of various discrete Painlevé equations.

Problem

How many 2nd order discrete Painlevé equations exist?

An answer to this problem was given as follows.

Fact ([Sakai 01])

The 2nd order continuous/discrete Painlevé equations are classified by the geometry of rational surfaces called the initial value spaces as follows:

	Symmetry/Surface type					
elliptic	E_8/A_0					
multiplicative	E_8/A_0	E_7/A_1	E_6/A_2	D_5/A_3	A_4/A_4	E_3/A_5
	E_2/A_6	$\underset{\lvert a\rvert^2=8}{A_1}/A_7$	A_1/A_7	A_0/A_8		
additive	E_8/A_0	E_7/A_1	E_6/A_2	D_4/D_4	A_3/D_5	$2A_1/D_6$
	A_2/E_6	$\underset{\lvert a\rvert^2=4}{A_1}/D_7$	A_1/E_7	A_0/D_8	A_0/E_8	

Here the symbols E_3 and E_2 stand for $A_2 + A_1$ and $A_1 + \underset{\lvert a\rvert^2=14}{A_1}$ respectively.
Blue-colored types correspond to the continuous Painlevé equations.

Problem

Can we classify continuous/discrete Painlevé type systems of order ≥ 3*?*

Several higher order generalizations have been proposed from both continuous side;

- Isomonodromy deformation of the Fuchsian equations (Garnier, Sakai, S, etc.)
- Similarity reduction of the infinite dimensional integrable hierarchies (Adler, Noumi-Yamada, Gordoa-Joshi-Pickering, Fuji-S, Tsuda, etc.)
- Okamoto initial value space and affine Weyl group symmetry (Sasano, etc.)

and discrete side:

- Discrete analogue of the isomonodromy deformations (Sakai, Nagao-Yamada, etc.)
- Similarity reduction of the discrete integrable hierarchies (Tsuda, S, etc.)
- Birational representation of the extended affine Weyl groups (Kajiwara-Noumi-Yamada, Masuda, Okubo-S, etc.)

However there doesn't exist any theory which governs all of them.

(q-)Painlevé VI equation and its higher order generalizations

The Painlevé VI equation is described as the Hamiltonian system

$$t(t-1)\frac{dq}{dt}=\frac{\partial H_{\mathrm{VI}}}{\partial p},\quad t(t-1)\frac{dp}{dt}=-\frac{\partial H_{\mathrm{VI}}}{\partial q},$$

$$H_{\mathrm{VI}}[\kappa_0,\kappa_1,\kappa_t,\kappa;q,p]=q(q-1)(q-t)p\left(p-\frac{\kappa_0}{q}-\frac{\kappa_1}{q-1}-\frac{\kappa_t-1}{q-t}\right)+\kappa q_i.$$

In 1996, Jimbo and Sakai proposed a q-analogue of the Painlevé VI equation, which is described as

$$\frac{f\overline{f}}{a_3a_4}=\frac{(\overline{g}-tb_1)(\overline{g}-tb_2)}{(\overline{g}-b_3)(\overline{g}-b_4)},\quad \frac{g\overline{g}}{b_3b_4}=\frac{(f-ta_1)(f-ta_2)}{(f-a_3)(f-a_4)},$$

where $a_1a_2b_3b_4=qb_1b_2a_3a_4$.

	Symmetry/Surface type					
elliptic	E_8/A_0					
multiplicative	E_8/A_0	E_7/A_1	E_6/A_2	D_5/A_3	A_4/A_4	E_3/A_5
	E_2/A_6	$A_{1\,\|a\|^2=8}/A_7$	A_1/A_7	A_0/A_8		
additive	E_8/A_0	E_7/A_1	E_6/A_2	D_4/D_4	A_3/D_5	$2A_1/D_6$
	A_2/E_6	$A_{1\,\|a\|^2=4}/D_7$	A_1/E_7	A_0/D_8	A_0/E_8	

The Painlevé VI equation is obtained as the isomonodromy deformation of the Fuchsian equation. We propose higher order generalizations from this point of view.

Fact ([Oshima 08])

Irreducible Fuchsian equations with a fixed number of accessory parameters can be reduced to finite types of systems by the Katz's two operations (addition and middle convolution).

Fact ([Haraoka-Filipuk 07])

The isomonodromy deformation equation of the Fuchsian equation is invariant under the Katz's two operations.

Thanks to them, we have a good classification theory of isomonodromy deformation equations of Fuchsian equations.

We list 4 types of representative isomonodromy deformation equations below:

- Garnier system
  - Isomonodromy deformation [Garnier 1912]
- Sasano system
  - Okamoto initial value space and affine Weyl group symmetry [Sasano 07]
  - Similarity reduction of the integrable hierarchy [Fuji-S 08]
  - Isomonodromy deformation [Sakai 10][Fuji-Inoue-Shinomiya-S 13]
- FST system
  - Similarity reduction of the integrable hierarchy [Fuji-S 09][S 13][Tsuda 14]
  - Isomonodromy deformation [Sakai 10]
- Matrix Painlevé system
  - Isomonodromy deformation [Sakai 10][Kawakami 15]

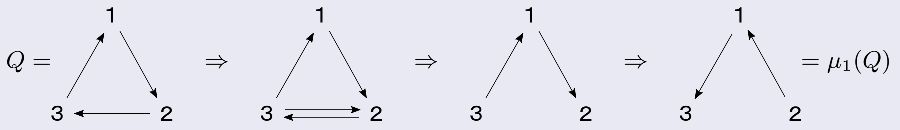
And their q-analogues are proposed recently (but there is no classification theory):

- q-Garnier system or q-FST system
  - q-Analogue of the isomonodromy deformation [Sakai 05][Park 18]
  - Similarity reduction of the discrete integrable hierarchy [Tsuda 10][S 15][S 17]
  - Pade method [Nagao-Yamada 18]
  - Birational representation of the extended affine Weyl group [Okubo-S 20]
- q-Sasano system
  - Birational representation of the extended affine Weyl group [Masuda 15]
- q-Matrix Painlevé system
  - q-Analogue of the isomonodromy deformation [Kawakami 20]

Let Q be a quiver without loops and 2-cycles and I a vertex set. We define a mutation μ_i at $i \in I$ as follows:

1. If there are k_1 arrows from i_1 to i and k_2 arrows from i to i_2, then we add $k_1 k_2$ arrows from i_1 to i_2.
2. If 2-cycles appear via the first operation, then we remove all of them.
3. We reverse the directions of all arrows touching i.

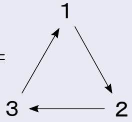
Example



We define a skew-symmetric matrix $\Lambda = (\lambda_{i,j})_{i,j \in I}$ corresponding to Q as follows:

1. If there are k arrows from i to j, then we set $\lambda_{i,j} = k$ and $\lambda_{j,i} = -k$.
2. If there is no arrow between i and j, then we set $\lambda_{i,j} = \lambda_{j,i} = 0$.

Example

$Q =$ 

$$\Leftrightarrow \quad \Lambda = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Let $y = (y_i)_{i \in I}$ be a tuples of coefficients. We define an action of μ_i on y by

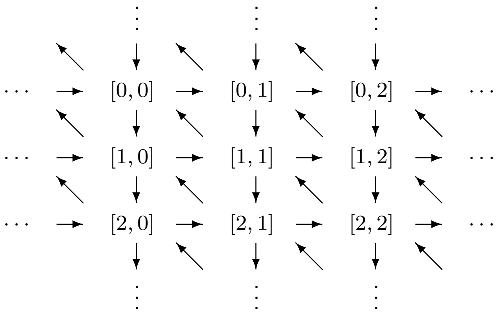
$$\mu_i(y_j) = \begin{cases} y_i^{-1} & (j = i) \\ y_j \left(1 + y_i^{-1}\right)^{\lambda_{ij}} & (\lambda_{ij} > 0) \\ y_j (1 + y_i)^{-\lambda_{ij}} & (\lambda_{ij} < 0) \\ y_j & (j \neq i,\ \lambda_{ij} = 0) \end{cases} .$$

Example

Q and Λ are the same as those in the previous examples.

$$(y_1, y_2, y_3) \xrightarrow{\mu_1} \left(\frac{1}{y_1}, \frac{y_1 y_2}{1 + y_1}, y_3(1 + y_1) \right)$$

Extended affine Weyl group of type $(A_{mn-1} \times A_{m-1} \times A_{m-1})^{(1)}$



Consider the above quiver. We always assume that

$$[j, i] = [j + mn, i] = [j, i + m] \quad (m, n \in \mathbb{N}, m > 1, mn > 2).$$

Let $y_{[j,i]}$ $(j \in \mathbb{Z}_{mn}, i \in \mathbb{Z}_m)$ be coefficients. We define multiplicative simple roots by

$$a_j = \prod_{i=0}^{m-1} y_{[j,i]} \quad (j \in \mathbb{Z}_{mn}), \quad b_i = \prod_{j=0}^{mn-1} y_{[j,i]}, \quad b'_i = \prod_{j=0}^{mn-1} y_{[j,i+j]} \quad (i \in \mathbb{Z}_m),$$

$$q = \prod_{j=0}^{mn-1} a_j = \prod_{i=0}^{m-1} b_i = \prod_{i=0}^{m-1} b'_i = \prod_{j=0}^{mn-1} \prod_{i=0}^{m-1} y_{[j,i]}.$$

We first define simple reflections r_j $(j \in \mathbb{Z}_{mn})$ by

$$r_j = \mu_{[j,0]}\,\mu_{[j,1]} \ldots \mu_{[j,m-2]}\,([j,m-2]\ [j,m-1])\,\mu_{[j,m-2]} \ldots \mu_{[j,1]}\,\mu_{[j,0]}.$$

Their actions on the coefficients and the simple roots are described as

$$r_j(y_{[j-1,i]}) = y_{[j-1,i]}\,y_{[j,i+1]}\,\frac{P_{[j,i+2]}}{P_{[j,i+1]}}, \quad r_j(y_{[j,i]}) = \frac{1}{y_{[j,i+1]}}\,\frac{P_{[j,i]}}{P_{[j,i+2]}},$$

$$r_j(y_{[j+1,i]}) = y_{[j,i]}\,y_{[j+1,i]}\,\frac{P_{[j,i+1]}}{P_{[j,i]}},$$

$$r_j(a_{j-1}) = a_{j-1}\,a_j, \quad r_j(a_j) = \frac{1}{a_j}, \quad r_j(a_{j+1}) = a_j\,a_{j+1},$$

where

$$P_{j,i} = \sum_{k=0}^{m-1}\prod_{l=0}^{k-1} y_{[j,i+l]} = 1 + y_{[j,i]} + y_{[j,i]}\,y_{[j,i+1]} + \ldots + y_{[j,i]}\,y_{[j,i+1]}\ \cdots\ y_{[j,i+m-2]}.$$

We next define simple reflections s_i, s_i' $(i \in \mathbb{Z}_m)$ by

$$s_i = \mu_{[0,i]}\,\mu_{[1,i]} \dots \mu_{[mn-2,i]}\,([mn-2,i]\ [mn-1,i])\,\mu_{[mn-2,i]} \dots \mu_{[1,i]}\,\mu_{[0,i]},$$
$$s_i' = \mu_{[0,i]}\,\mu_{[1,i+1]} \dots \mu_{[-2,i-2]}\,([-2,i-2]\ [-1,i-1])\,\mu_{[-2,i-2]} \dots \mu_{[1,i+1]}\,\mu_{[0,i]}.$$

Their actions on the coefficients and the simple roots are described as

$$s_i(y_{[j,i]}) = \frac{1}{y_{[j+1,i]}}\frac{Q_{[j,i]}}{Q_{[j+2,i]}}, \quad s_i(y_{[j,i+1]}) = y_{[j,i]}\,y_{[j,i+1]}\,y_{[j+1,i]}\,\frac{Q_{[j+2,i]}}{Q_{[j,i]}},$$
$$s_i(b_i) = \frac{1}{b_i}, \quad s_i(b_{i+1}) = b_i^2\,b_{i+1},$$

for $m = 2$ and

$$s_i(y_{[j,i-1]}) = y_{[j,i-1]}\,y_{[j+1,i]}\,\frac{Q_{[j+2,i]}}{Q_{[j+1,i]}}, \quad s_i(y_{[j,i]}) = \frac{1}{y_{[j+1,i]}}\frac{Q_{[j,i]}}{Q_{[j+2,i]}},$$
$$s_i(y_{[j,i+1]}) = y_{[j,i]}\,y_{[j,i+1]}\,\frac{Q_{[j+1,i]}}{Q_{[j,i]}},$$
$$s_i(b_{i-1}) = b_{i-1}\,b_i, \quad s_i(b_i) = \frac{1}{b_i}, \quad s_i(b_{i+1}) = b_i\,b_{i+1},$$

for $m \geq 3$, where

$$Q_{[j,i]} = \sum_{k=0}^{mn-1}\prod_{l=0}^{k-1} y_{[j+l,i]}, \quad y'_{[j,i]} = y_{[-j,i-j]}.$$

In the last we define Dynkin diagram automorphisms π_1, π_2 by

$$\begin{aligned}\pi_1 = &([0,0]\ [1,1]\ \dots\ [m-1,m-1]\ [m,0]\ \dots\ [mn-1,m-1])\\ &\times([0,1]\ [1,2]\ \dots,[m-1,0]\ [m,1]\ \dots\ [mn-1,0])\\ &\times\dots\\ &\times([0,m-1]\ [1,0]\ \dots\ [m-1,m-2]\ [m,m-1]\ \dots\ [mn-1,m-2]),\\ \pi_2 = &([0,0]\ [0,1]\ \dots\ [0,m-1])\\ &\times([1,0]\ [1,1]\ \dots\ [1,m-1])\\ &\times\dots\\ &\times([mn-1,0]\ [mn-1,1]\ \dots\ [mn-1,m-1]).\end{aligned}$$

They act on the coefficients and the simple roots as

$$\begin{aligned}&\pi_1(y_{[j,i]}) = y_{[j+1,i+1]}, \quad \pi_1(a_j) = a_{j+1}, \quad \pi_1(b_i) = b_{i+1},\\ &\pi_2(y_{[j,i]}) = y_{[j+1,i]}, \quad \pi_2(b_i) = b_{i+1}, \quad \pi_2(b'_i) = b'_{i+1}.\end{aligned}$$

Fact ([Masuda-Okubo-Tsuda 18])

Let

$$G = \langle r_0, \ldots, r_{mn-1} \rangle, \quad H = \langle s_0, \ldots, s_{m-1} \rangle, \quad H' = \langle s'_0, \ldots, s'_{m-1} \rangle,$$

Then G, H and H' are isomorphic to the affine Weyl groups of type $A^{(1)}_{mn-1}$, $A^{(1)}_{m-1}$ and $A^{(1)}_{m-1}$ respectively. Furthemore, any two groups are mutually commutative, namely

$$GH = HG, \quad GH' = H'G, \quad HH' = H'H.$$

Proposition ([Okubo-S 20])

The Dynkin diagram automorphisms π_1, π_2 satisfy fundamental relations

$$\pi_1^{mn} = 1, \quad \pi_2^{m} = 1, \quad \pi_1 \pi_2 = \pi_2 \pi_1,$$
$$r_j \pi_1 = \pi_1 r_{j+1}, \quad s_i \pi_1 = \pi_1 s_{i+1}, \quad s'_i \pi_1 = \pi_1 s'_i,$$
$$r_j \pi_2 = \pi_2 r_j, \quad s_i \pi_2 = \pi_2 s_{i+1}, \quad s'_i \pi_2 = \pi_2 s'_{i+1}.$$

Hence we can regard a group $\langle G, H, H' \rangle \rtimes \langle \pi_1, \pi_2 \rangle$ as an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$.

Let us introduce an independent variable z satisfying

$$r_j(z) = z, \quad s_i(z) = z, \quad s'_i(z) = z, \quad \pi_1(z) = z, \quad \pi_2(z) = q^{1/m} z.$$

We also set

$$\zeta = z \prod_{j=1}^{mn-1} \prod_{i=0}^{m-2} a_j^{(mn-j)/m} b_i^{(i+1)/m}.$$

Let E_{j_1,j_2} be a $mn \times mn$ matrix with 1 in (j_1, j_2)-th entry and 0 elsewhere. Consider matrices

$$\Pi_1 = \zeta^{\log_q \frac{1}{a_1}} \left(\sum_{j=1}^{mn-1} \prod_{k=0}^{j-1} \frac{1}{y_{[1,k]}} E_{j,j+1} + \frac{b_{m-1}}{a_1^n q} \zeta E_{mn,1} \right),$$

and

$$\Pi_2 = \sum_{j=1}^{mn} \prod_{k=1}^{j-1} y_{[k,j-1]} E_{j,j} + \sum_{j=1}^{mn-1} E_{j,j+1} + \frac{b_{m-1}}{q} \zeta E_{mn,1}.$$

We also set

$$M = \pi_2^{m-1}(\Pi_2)\, \pi_2^{m-2}(\Pi_2) \, \dots \, \pi_2(\Pi_2)\, \Pi_2.$$

Example $(mn = 6)$

$$\frac{\Pi_1}{\zeta^{\log_q \frac{1}{a_1}}} = \begin{pmatrix} 0 & \frac{1}{y_{[1,0]}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{y_{[1,0]}y_{[1,1]}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{y_{[1,0]}\cdots y_{[1,2]}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{y_{[1,0]}\cdots y_{[1,3]}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{y_{[1,0]}\cdots y_{[1,4]}} \\ \frac{b_{m-1}\zeta}{a_1^n q} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & y_{[1,1]} & 1 & 0 & 0 & 0 \\ 0 & 0 & y_{[1,2]}y_{[2,2]} & 1 & 0 & 0 \\ 0 & 0 & 0 & y_{[1,3]}\cdots y_{[3,3]} & 1 & 0 \\ 0 & 0 & 0 & 0 & y_{[1,4]}\cdots y_{[4,4]} & 1 \\ \frac{b_{m-1}\zeta}{q} & 0 & 0 & 0 & 0 & y_{[1,5]}\cdots y_{[5,5]} \end{pmatrix}.$$

Let I be the identity matrix. Consider matrices

$$R_0 = \zeta^{\log_q a_0} \left(\sum_{j=1}^{mn} \frac{P_{[0,j-1]} \prod_{k=0}^{j-2} y_{[0,k]}}{P_{[0,0]}} E_{j,j} + \frac{q\,(1-a_0)}{y_{[0,m-1]}\, P_{[0,0]}} \frac{1}{\zeta} E_{1,mn} \right),$$

$$R_j = r_j(\Pi_1^{-1})\, \pi_1(R_{j-1})\, \Pi_1 \quad (j = 1, \dots, mn-1).$$

Example $(mn = 6)$

$$\frac{R_1}{\zeta^{\log_q \frac{1}{a_1}}} = \begin{pmatrix} \frac{y_{[1,1]} P_{[1,2]}}{P_{[1,1]}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1-a_1}{P_{[1,1]}} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{y_{[1,1]} P_{[1,2]}}{y_{[1,2]} P_{[1,3]}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{y_{[1,1]} P_{[1,2]}}{y_{[1,2]} y_{[1,3]} P_{[1,4]}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{y_{[1,1]} P_{[1,2]}}{y_{[1,2]} \cdots y_{[1,4]} P_{[1,5]}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{y_{[1,1]} P_{[1,2]}}{y_{[1,2]} \cdots y_{[1,5]} P_{[1,0]}} \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-a_2) y_{[1,1]}}{P_{[2,2]}} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also consider matrices

$$
\begin{aligned}
S_0 = I &+ \sum_{k=0}^{n-1} \left(\frac{Q_{[mk+2,0]} \prod_{l=2}^{mk+1} y_{[l,0]}}{Q_{[2,0]}} - 1 \right) E_{mk+1,mk+1} \\
&+ \sum_{k=0}^{n-1} \left(\frac{Q_{[1,0]}}{Q_{[mk+2,0]} \prod_{l=1}^{mk+1} y_{[l,0]}} - 1 \right) E_{mk+2,mk+2} \\
&+ \sum_{k=0}^{n-1} \frac{b_0 - 1}{y_{[1,0]}\, Q_{[2,0]}}\, E_{mk+1,mk+2}, \\
S_i = s_i&(\Pi_2^{-1})\, \pi_2(S_{i-1})\, \Pi_2 \quad (i = 1, \ldots, m-1),
\end{aligned}
$$

and

$$
\begin{aligned}
S'_0 &= I + \sum_{k=0}^{n-1} \left(\frac{y'_{[0,0]}\, Q'_{[1,0]}}{Q'_{[0,0]}} - 1 \right) E_{mk+2,mk+2}, \\
S'_i &= s'_i(\Pi_2^{-1})\, \pi_2(S'_{i-1})\, \Pi_2 \quad (i = 1, \ldots, m-1).
\end{aligned}
$$

Example $(m = 3, n = 2)$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{y_{[2,1]} Q_{[3,1]}}{Q_{[2,1]}} & \frac{b_1 - 1}{y_{[1,1]} Q_{[2,1]}} & 0 & 0 & 0 \\ 0 & 0 & \frac{Q_{[1,1]}}{y_{[1,1]} y_{[2,1]} Q_{[3,1]}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{y_{[2,1]} \cdots y_{[5,1]} Q_{[0,1]}}{Q_{[2,1]}} & \frac{b_1 - 1}{y_{[1,1]} Q_{[2,1]}} \\ 0 & 0 & 0 & 0 & 0 & \frac{Q_{[1,1]}}{y_{[1,1]} \cdots y_{[5,1]} Q_{[0,1]}} \end{pmatrix},$$

$$S_1' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{y'_{[0,1]} Q'_{[1,1]}}{Q'_{[0,1]}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{y'_{[0,1]} Q'_{[1,1]}}{Q'_{[0,1]}} \end{pmatrix}.$$

Remark

The matrix S'_{m-1} is rational in ζ, is not diagonal and hence is much more complicated than the others. The cause has not been clarified yet.

Denoting by $T_{q,z} = \pi_2^m$, we arrive at the following results.

Theorem ([S 21])

The compatibility condition of a system of linear q-difference equations

$$
\begin{aligned}
&T_{q,z}(\psi) = M\,\psi,\\
&\pi_1(\psi) = \Pi_1\,\psi, \quad \pi_2(\psi) = \Pi_2\,\psi,\\
&r_j(\psi) = R_j\,\psi \quad (j \in \mathbb{Z}_{mn}), \quad s_i(\psi) = S_i\,\psi, \quad s_i'(\psi) = S_i'\,\psi \quad (i \in \mathbb{Z}_m),
\end{aligned}
$$

is equivalent to the action of the Dynkin diagram automorphisms and the simple reflections given in the previous section.

Remark

Our Lax form gives a similarity reduction of a q-Drinfeld-Sokolov hierarchy of type $A_{mn-1}^{(1)}$ corresponding to the partition $(n,\ldots,n)$ of $mn \in \mathbb{N}$.

Remark

Our Lax form (with $mn \times mn$ matrices) is transformed to that with $m \times m$ matrices via a q-Laplace transformation. The reduced system has been already proposed by Nagao, Park and Yamada.

Case $m = 2$

This case has been already investigated well.

Theorem ([Okubo-S 20][S 21])

We set

$$\tau_1 = s_0\, s_0'\, \pi_2,$$
$$\tau_2 = r_0\, \ldots\, r_{2n-2}\, \pi_1\, r_n\, \ldots\, r_{2n-1}\, r_0\, \ldots\, r_{n-2}\, \pi_1,$$
$$\tau_3 = (r_0\, r_2\, \ldots\, r_{2n-2}\, \pi_1)^2,$$
$$\tau_4 = s_0\, r_0\, \ldots\, r_{2n-2}\, \pi_1.$$

Then they provide higher order q-Painlevé systems as follows.

- τ_1*: q-FST system*
- τ_2*: (a direction of) Sakai's q-Garnier system*
- τ_3*: Tsuda's q-Painlevé system arising from the q-LUC hierarchy*
- τ_4*: Nagao-Yamada's "variation" of the q-Garnier system*

Moreover we have clarified a relationship between those q-Painlevé systems and the q-hypergeometric functions ${}_n\phi_{n-1}$ or ϕ_D.

Case $(m,n)=(3,2)$

The matrix M is described as

$$M=\begin{pmatrix} 1 & P_{[1,1]} & P^*_{[1,2]} & 1 & 0 & 0 \\ 0 & a_1 & \frac{a_1 P_{[2,2]}}{y_{[1,1]}} & y_{[1,0]} P^*_{[2,0]} & 1 & 0 \\ 0 & 0 & a_1 a_2 & \frac{a_1 a_2 P_{[3,0]}}{y_{[1,2]} y_{[2,2]}} & y_{[1,1]} y_{[2,1]} P^*_{[3,1]} & 1 \\ \frac{\zeta}{b_0 b_1} & 0 & 0 & a_1 \dots a_3 & \frac{a_1 \dots a_3 P_{[4,1]}}{y_{[1,0]} \cdots y_{[3,0]}} & y_{[1,2]} \dots y_{[3,2]} P^*_{[4,2]} \\ \frac{P^*_{[5,0]} \zeta}{b_1 y_{[5,0]} y_{[0,0]}} & \frac{\zeta}{b_1} & 0 & 0 & a_1 \dots a_4 & \frac{a_1 \dots a_4 P_{[5,2]}}{y_{[1,1]} \cdots y_{[4,1]}} \\ \frac{P_{[0,0]} \zeta}{y_{[0,0]} y_{[0,1]}} & \frac{P^*_{[0,1]} \zeta}{y_{[0,1]}} & \zeta & 0 & 0 & a_1 \dots a_5 \end{pmatrix},$$

where

$$P_{[i,j]} = 1 + y_{[i,j]} + y_{[i,j]}\, y_{[i,j+1]}, \quad P^*_{[i,j]} = 1 + y_{[i,j]} + y_{[i,j]}\, y_{[i+1,j]}.$$

We consider a translation

$$\tau_1 = s_0\, s_1\, s_0'\, s_1'\, \pi_2.$$

Then the compatibility condition of a Lax pair

$$T_{q,z}(B)\, M = \tau_1(M)\, B, \quad B = s_1 s_0' s_1' \pi_2(S_0)\, s_0' s_1' \pi_2(S_1)\, s_1' \pi_2(S_0')\, \pi_2(S_1')\, \Pi_2,$$

implies a 8th order q-Painlevé system with parameters $a_1, \dots, a_5, b_1, b_2, b_1', b_2'$ and q.

Assume that

$$P_{[1,1]} = P^*_{[1,2]} = P_{[4,1]} = P^*_{[4,2]} = 0,$$

which contains a specialization between parameters

$$b'_1 (b'_2)^2 = a_1^2\, a_2\, a_4^2\, a_5\, b_1\, b_2^2.$$

Then we have 2 invariants

$$\tau_1(y_{[0,0]}\, y_{[0,2]}\, y_{[1,0]}) = y_{[0,0]}\, y_{[0,2]}\, y_{[1,0]}, \quad \tau_1(y_{[1,2]}\, y_{[2,0]}\, y_{[2,2]}) = y_{[1,2]}\, y_{[2,0]}\, y_{[2,2]},$$

and hence obtain 3rd order q-Riccati like system. Introduce variables $x_0, \dots, x_3$ such that

$$\frac{x_1}{x_0} = (1 + y_{[0,0]})\, y_{[0,2]}, \quad \frac{x_2}{x_0} = \frac{y_{[2,0]}}{y_{[3,2]}}, \quad \frac{x_3}{x_0} = (1 + y_{[3,0]})\, y_{[2,0]},$$

and assume that

$$a_1^2\, a_2\, a_4^2\, a_5\, b_1\, b_2^2 = q.$$

Proposition ([S 2021])

A vector of variables $(x_0, \dots, x_3)$ satisfies a system of linear q-difference equation, which reduces a rigid system of type $22, 211, 1111$ (EO_4) in a continuous limit $q \to 1$.

Remark

Another hypergeometric-type particular solution has been proposed by Park.

We have proposed an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$ in two ways.

- Cluster mutation for a quiver on a torus with $m^2 n$ vertices
- Lax form with $mn \times mn$ matrices

We have also investigated for $(m, n) = (3, 2)$ as an experiment and derived a "q-rigid" system as a particular solution.

There are some future problems.

- Particular solutions in terms of q-hypergeometric functions
- A classification theory of higher order q-Painlevé systems
- Formulations of higher order elliptic Painlevé systems

Thank you for your attention.