BOUTROUX ANSATZ FOR THE DEGENERATE THIRD PAINLEVÉ TRANSCENDENTS

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Sakai: Classified $\quad P_{\mathrm{III}}(D_6), \quad P_{\mathrm{III}}(D_7), \quad P_{\mathrm{III}}(D_8)$

$$P_{\mathrm{III}}(D_6) \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{4}{x}(\theta_0 y^2 + 1 - \theta_\infty) + 4y^3 - \frac{4}{y}$$

singular points: $\quad x = 0$ (regular), $\quad x = \infty$ (irregular)

$\mathcal{R}(\mathbb{C} \setminus \{0\})$: universal covering space of $\mathbb{C} \setminus \{0\}$

$$P_{\mathrm{III}}(D_7) \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{4}{x}(\theta_0 y^2 + 1 - \theta_\infty) - \frac{4}{y}$$

$$P_{\mathrm{III}'}(D_7) \quad v_{\xi\xi} = \frac{v_\xi^2}{v} - \frac{v_\xi}{\xi} - \frac{2v^2}{\xi^2} + \frac{a}{\xi} + \frac{1}{v}$$

Ohyama et al.: τ-function, irreducibility, space of initial values, algebraic solutions

$$\text{(dP3)} \qquad u_{\tau\tau} = \frac{u_\tau^2}{u} - \frac{u_\tau}{\tau} + \frac{1}{\tau}(-8\epsilon u^2 + 2ab) + \frac{b^2}{u}, \qquad \epsilon = \pm 1, \;\; a \in \mathbb{C}, \;\; b \in \mathbb{R}_{\neq 0}$$

Kitaev-Vartanian: asymptotics on $\mathbb{R}^\pm$, $i\mathbb{R}^\pm$ near $0, \infty$ $\quad$ isomonodromy deformation

$\epsilon\tau u = (x/3)^2 y$, $\epsilon b \tau^2 = 2(x/3)^3 \quad \Longleftrightarrow$

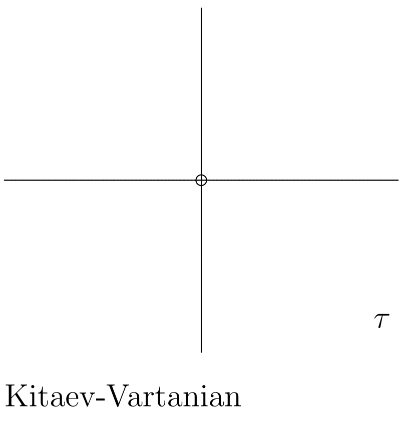
$$\text{(E)} \qquad y'' = \frac{(y')^2}{y} - \frac{y'}{x} - 2y^2 + \frac{3a}{x} + \frac{1}{y}$$

$\epsilon = 1$, $b > 0$

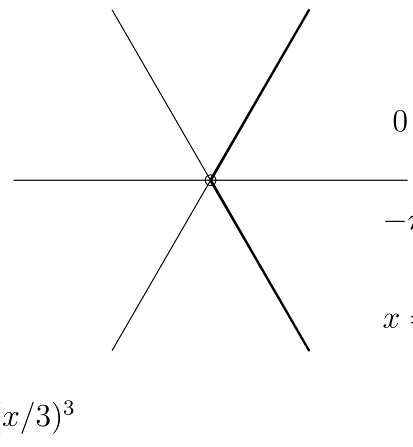
$$u(\tau) = \frac{\sqrt{b}}{3^{1/4}}\Big(\sqrt{\frac{\vartheta(\tau)}{12}} + \sqrt{\nu(G)}e^{\frac{3\pi i}{4}} \cosh(i\vartheta(\tau) + \nu(G)\ln\vartheta(\tau) + C(G) + o(\tau^{-\delta}))\Big)$$

as $\tau \to +\infty$ along $\mathbb{R}$, where $\vartheta(\tau) = 3\sqrt{3}b^{1/3}\tau^{2/3}$, $\nu(G)$, $C(G)$: constant depending on the monodromy data $G = (g_{ij}) \in SL_2(\mathbb{C})$ if $g_{11}g_{12}g_{21}g_{22} \neq 0$

$g_{12} = 0 \quad \Longrightarrow \quad u(\tau) = \frac{1}{2}b^{2/3}\tau^{1/3} + c(a,b)(s_0^0 - ie^{-\pi a})\exp(i(3\sqrt{3}b^{1/3}\tau^{2/3} + \frac{3}{4}\pi))(1 + o(1))$



Kitaev-Vartanian



$\epsilon b \tau^2 = 2(x/3)^3$

Boutroux Ansatz: representation of solutions by elliptic functions as $x \to \infty$

P. Boutroux: 1913 P_1, P_2 line of periods, Boutroux equations (determine the modulus of the elliptic function)

N. Joshi, M. D. Kruskal: 1988 P_1, P_2 multi-scale expansion method

V. Yu. Novokshenov: 1990 P_2 isomonodromy deformation, WKB

A. A. Kapaev: 1991 P_2 isomonodromy deformation, WKB

A. V. Kitaev: 1993 P_1, 1994 P_1, P_2 isomonodromy deformation, WKB

This study is basically along the lines of [Kapaev], [Kitaev].

A. R. Its, A. A. Kapaev: 2001 P_2 Riemann-Hilbert approach (Deift-Zhou method)

K. Iwaki: 2020 P_1 topological recursion, τ-function

P_3: Novokshenov: 2007 sine-Gordon type

P_4: Kapaev: 1996 announcement
P_4: V. L. Vereshchagin: 1997 $\beta = 0$
P_3, P_5: S.

Kitaev-Vartanian:

$$\text{(dP3)} \qquad u_{\tau\tau} = \frac{u_\tau^2}{u} - \frac{u_\tau}{\tau} + \frac{1}{\tau}(-8\epsilon u^2 + 2ab) + \frac{b^2}{u}, \qquad \epsilon = \pm 1,\ \ a \in \mathbb{C},\ \ b \in \mathbb{R}_{\neq 0}$$

governs the isomonodromy deformation for

$$\text{(U)} \quad \frac{dU}{d\mu} = \mathcal{U}(\mu,\tau)U,$$

$$\mathcal{U}(\mu,\tau) = -2i\tau\mu\sigma_3 + 2\tau \begin{pmatrix} 0 & 2i\epsilon e^{i\varphi} \\ -(\epsilon/4)e^{-i\varphi}(u^\tau/u - 1/\tau - i\varphi_\tau) & 0 \end{pmatrix}$$

$$- \frac{1}{\mu}\Big(ia + \frac{\tau}{2}(u^\tau/u - i\varphi_\tau)\Big)\sigma_3 + \frac{1}{\mu^2}\begin{pmatrix} 0 & 2\epsilon e^{i\varphi}(ia - i\tau\varphi_\tau/2) \\ -iue^{-i\varphi} & 0 \end{pmatrix}$$

$\varphi_\tau = (d/d\tau)\varphi = 2a/\tau + b/u$, that is, the monodromy data remain invariant under small change of τ if and only if $u^\tau = (d/d\tau)u$ holds and $u(\tau)$ solves (dP3)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\epsilon = 1$, $b > 0$

$$u(\tau) = \frac{\sqrt{b}}{3^{1/4}}\Big(\sqrt{\frac{\vartheta(\tau)}{12}} + \sqrt{\nu(G)}e^{\frac{3\pi i}{4}}\cosh(i\vartheta(\tau) + \nu(G)\ln\vartheta(\tau) + C(G) + o(\tau^{-\delta}))\Big)$$

as $\tau \to +\infty$ along $\mathbb{R}$, where $\vartheta(\tau) = 3\sqrt{3}b^{1/3}\tau^{2/3}$, $\nu(G)$, $C(G)$: constant depending on the monodromy data $G = (g_{ij}) \in SL_2(\mathbb{C})$ if $g_{11}g_{12}g_{21}g_{22} \neq 0$

$$g_{12} = 0 \quad \Longrightarrow \quad u(\tau) = \tfrac{1}{2}b^{2/3}\tau^{1/3} + c(a,b)(s_0^0 - ie^{-\pi a})\exp(i(3\sqrt{3}b^{1/3}\tau^{2/3} + \tfrac{3}{4}\pi))(1 + o(1))$$

Kitaev-Vartanian Monodromy data of (U)

(U): irregular singular points $\mu = 0, \infty$

$$Y_k^\infty(\mu) = (I + O(\mu^{-1}))\mu^{-(1/2+ia)\sigma_3}\exp(-i\tau\mu^2\sigma_3)$$

$$\mu \to \infty \text{ through } |\arg\mu + \arg\tau^{1/2} - \pi k/2| < \pi/2,$$

$$X_k^0(\mu) = (i/\sqrt{2})\Theta_0^{\sigma_3}(\sigma_1 + \sigma_3 + O(\mu))\exp(-i\sqrt{\tau\epsilon b}\mu^{-1}\sigma_3)$$

$$\mu \to 0 \text{ through } |\arg\mu - \arg(\tau\epsilon b)^{1/2} - \pi k| < \pi \quad (k \in \mathbb{Z})$$

S_j^∞, S_j^0, $G \in SL_2(\mathbb{C})$:

$$Y_{j+1}^\infty(\mu) = Y_j^\infty(\mu)S_j^\infty, \qquad X_{j+1}^0(\mu) = X_j^0(\mu)S_j^0, \qquad Y_0^\infty(\mu) = X_0^0(\mu)G$$

$$S_0^\infty = \begin{pmatrix} 1 & 0 \\ s_0^\infty & 1 \end{pmatrix}, \quad S_1^\infty = \begin{pmatrix} 1 & s_1^\infty \\ 0 & 1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 1 & s_0^0 \\ 0 & 1 \end{pmatrix}, \quad G = (g_{ij})$$

$\mathcal{M}:\quad GS_0^\infty S_1^\infty \sigma_3 e^{\pi(i/2-a)\sigma_3} = S_0^0\sigma_1 G$ manifold of monodromy data, $\dim\mathcal{M} = 3$

$\mathcal{M}/\sim\ \subset \mathbb{C}^3$: equivalence class (orbit) by **ac**,

nonsingular affine cubic surface (vd Put, Saito)

$g_{11}g_{22} \neq 0 \Longrightarrow s_0^\infty,\ s_1^\infty,\ s_0^0$: expressed by g_{ij}'s

coordinates: $(g_{11}g_{22},\ g_{12}/g_{22},\ g_{21}/g_{11})$: invariant under **ac**

$\Longrightarrow$ $u(\tau)$, $y(x)$: labelled by G

$\forall c \in \mathbb{C}\setminus\{0\}$, $(Y_j^\infty(\mu),\ X_j^0(\mu)) \sim (c^{\sigma_3/2}Y_j^\infty(\mu)c^{-\sigma_3/2},\ c^{\sigma_3/2}X_j^0(\mu))$

$Y_0^\infty(\mu) = X_0^0(\mu)G \quad G \sim Gc^{-\sigma_3/2}$,

$\Longrightarrow$

$$\mathbf{ac}: (S_0^\infty, S_1^\infty, S_0^0, G) \mapsto (c^{\sigma_3/2}S_0^\infty c^{-\sigma_3/2}, c^{\sigma_3/2}S_1^\infty c^{-\sigma_3/2}, S_0^0, Gc^{-\sigma_3/2})$$

$(g_{11}, g_{22}) \neq (0,0)$

$g_{11} = 0 \quad \Longrightarrow (s_0^0, g_{22})$

(E) $\quad y'' = \dfrac{(y')^2}{y} - \dfrac{y'}{x} - 2y^2 + \dfrac{3a}{x} + \dfrac{1}{y}$

(E) with $x = e^{i\phi}t$ governs isomonodromy deformation of the linear system $t(>1)$: not necessarily $\in \mathbb{R}_+$, but allowed to be in a strip containing $\mathbb{R}_+$.

$$\text{(LE)} \quad \frac{d\Psi}{d\lambda} = \frac{t}{3}\mathcal{B}(\lambda,t)\Psi,$$

$$\begin{aligned}\mathcal{B}(\lambda,t) =& -ie^{i\phi}\lambda\sigma_3 + \begin{pmatrix} 0 & -2ie^{i\phi}y \\ \Gamma_0(t,y,y^t)/y & 0 \end{pmatrix} \\ & - (\Gamma_0(t,y,y^t) + 3(1/2+ia)t^{-1})\lambda^{-1}\sigma_3 + 2e^{i\phi}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\lambda^{-2},\end{aligned}$$

$$\Gamma_0(t,y,y^t) = \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - (1+3ia)t^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

y, y^t: arbitrary complex parameters

(LE): isomonodromy property $\iff$ $y^t = dy/dt,\ y = y(te^{i\phi}) = y(x)$ solves (E) system (LE) admits the matrix solutions

$\hat{Y}_j^{\infty}(\lambda) = (I + O(\lambda^{-1}))\lambda^{-(1/2+ia)\sigma_3} \exp(-(i/6)e^{i\phi}t\lambda^2\sigma_3)$

as $\lambda \to \infty$ through $|\arg\lambda + \phi/2 - j\pi/2| < \pi/2$,

$\hat{Y}_j^{0}(\lambda) = (i/\sqrt{2})(\sigma_1 + \sigma_3 + O(\lambda)) \exp(-(2i/3)e^{i\phi}t\lambda^{-1}\sigma_3)$

as $\lambda \to 0$ through $|\arg\lambda - \phi - j\pi| < \pi$.

$$\hat{Y}_{j+1}^{\infty}(\lambda) = \hat{Y}_j^{\infty}(\lambda)\hat{S}_j^{\infty}, \quad \hat{Y}_{j+1}^{0}(\lambda) = \hat{Y}_j^{0}(\lambda)\hat{S}_j^{0}, \quad \hat{Y}_0^{\infty}(\lambda) = \hat{Y}_0^{0}(\lambda)\hat{G},$$

$\Longrightarrow$

$$\hat{G}\hat{S}_0^{\infty}\hat{S}_1^{\infty}\sigma_3 e^{\pi(i/2-a)\sigma_3} = \hat{S}_0^{0}\sigma_1\hat{G},$$

$$\hat{S}_0^{\infty} = \begin{pmatrix} 1 & 0 \\ \hat{s}_0^{\infty} & 1 \end{pmatrix}, \quad \hat{S}_1^{\infty} = \begin{pmatrix} 1 & \hat{s}_1^{\infty} \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_0^{0} = \begin{pmatrix} 1 & \hat{s}_0^{0} \\ 0 & 1 \end{pmatrix}.$$

$$\hat{S}_{k+2}^{\infty} = \sigma_3 e^{-\pi(a-i/2)\sigma_3}\hat{S}_k^{\infty} e^{\pi(a-i/2)\sigma_3}\sigma_3, \quad \hat{S}_k^{0} = \sigma_1\hat{S}_{k+1}^{0}\sigma_1$$

$\hat{G} = G\Theta_{0,*}^{\sigma_3}, \quad \hat{S}_j^{\infty} = \Theta_{0,*}^{-\sigma_3}S_j^{\infty}\Theta_{0,*}^{\sigma_3}, \quad \hat{S}_j^{0} = S_j^{0}$

$\Theta_{0,*} = (\epsilon b)^{1/4}\tau^{1/4}(-ue^{-i\varphi})^{-1/2}((3/2)(\epsilon b)^{1/2}x^{-1})^{1/2+ia}$

$(g_{11}g_{22},\, g_{12}/g_{22},\, g_{21}/g_{11}) = (\hat{g}_{11}\hat{g}_{22},\, \hat{g}_{12}/\hat{g}_{22},\, \hat{g}_{21}/\hat{g}_{11}), \quad \hat{G} = (\hat{g}_{ij})$

Π_A: $w(A,z)^2 = 4z^3 - Az^2 + 1$ elliptic curve

$\Pi_A = \Pi_+ \cup \Pi_-$: two-sheeted Riemann surface glued along the cuts $[\infty, z_2] \cup [z_0, z_1]$

$\Pi_\pm$: $P^1(\mathbb{C}) \setminus ([\infty, z_2] \cup [z_0, z_1])$

z_j: roots of $w(A,z)$ such that $z_0 = z_1 = 2^{-1/3}$, $z_2 = -4^{-2/3}$ when $A = 3 \cdot 2^{2/3}$

branch of $\sqrt{4z^3 - Az^2 + 1} := 2\sqrt{z - z_0}\sqrt{z - z_1}\sqrt{z - z_2}$:
Re $\sqrt{z - z_j} \to +\infty$ as $z \to \infty$ along $\mathbb{R}^+ \subset \Pi_+$

Number z_0, z_1 such that $\operatorname{Im} z_0 \le \operatorname{Im} z_1$ if $\phi > 0$, $\operatorname{Im} z_1 \le \operatorname{Im} z_0$ if $\phi < 0$ when $A_\phi \sim 3 \cdot 2^{2/3}$

$\forall \phi \in \mathbb{R}$, $\exists A_\phi \in \mathbb{C}$ such that $\forall$ cycle $\mathbf{c} \subset \Pi_{A_\phi}$ $\quad \operatorname{Im} e^{i\phi} \int_{\mathbf{c}} \frac{w(A_\phi, z)}{z^2} dz = 0$

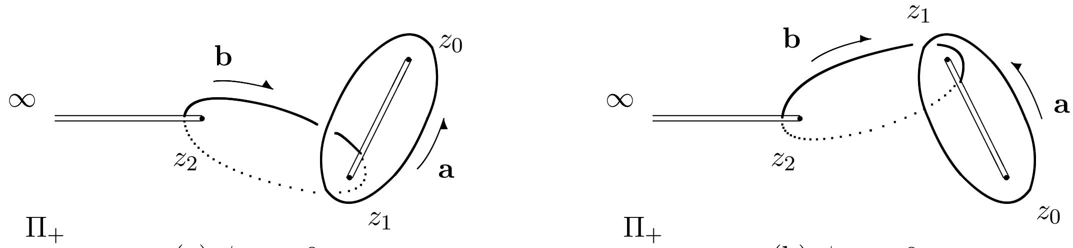
In other words, Boutroux equations.

(1) for every ϕ, A_ϕ is uniquely determined;
(2) A_ϕ: continuous in $\phi \in \mathbb{R}$, smooth in $\phi \in \mathbb{R} \setminus \{k\pi/3 \,|\, k \in \mathbb{Z}\}$;
(3) $A_{\phi \pm 2\pi/3} = e^{\pm 2\pi i/3} A_\phi$, $A_{\phi+\pi} = A_\phi$, $A_{-\phi} = \overline{A_\phi}$;
(4) Π_{A_ϕ} degenerates iff $\phi = k\pi/3$ then
$A_0 = A_{\pm\pi} = 3 \cdot 2^{2/3}, \quad A_{\pm\pi/3} = e^{\mp 2\pi i/3} A_0, \quad A_{\pm 2\pi/3} = e^{\pm 2\pi i/3} A_0,$

In other words, Boutroux equations.

Boutroux equations

$0 < |\phi| < \pi/3$ basic cycles $\mathbf{a}$ and $\mathbf{b}$ on Π_{A_ϕ}



(a) $\phi \to -0$ (b) $\phi \to +0$

$$w(A,z)^2 = 4z^3 - Az^2 + 1$$

Boutroux equations

$$\text{(BE)} \quad \operatorname{Im} e^{i\phi} \int_{\mathbf{a}} \frac{w(A_\phi, z)}{z^2} dz = 0, \quad \operatorname{Im} e^{i\phi} \int_{\mathbf{b}} \frac{w(A_\phi, z)}{z^2} dz = 0$$

admit a unique solution A_ϕ

(1) for every ϕ, A_ϕ is uniquely determined;

(2) A_ϕ: continuous in $\phi \in \mathbb{R}$, smooth in $\phi \in \mathbb{R} \setminus \{k\pi/3 \,|\, k \in \mathbb{Z}\}$;

(3) $A_{\phi \pm 2\pi/3} = e^{\pm 2\pi i/3} A_\phi$, $A_{\phi+\pi} = A_\phi$, $A_{-\phi} = \overline{A_\phi}$;

(4) Π_{A_ϕ} degenerates iff $\phi = k\pi/3$ then

$$A_0 = A_{\pm\pi} = 3 \cdot 2^{2/3}, \quad A_{\pm\pi/3} = e^{\mp 2\pi i/3} A_0, \quad A_{\pm 2\pi/3} = e^{\pm 2\pi i/3} A_0,$$

periods on Π_{A_ϕ}:

$$\Omega^\phi_{\mathbf{a}} = \Omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(A_\phi, z)}, \quad \Omega^\phi_{\mathbf{b}} = \Omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(A_\phi, z)}, \qquad \operatorname{Im} \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}} > 0$$

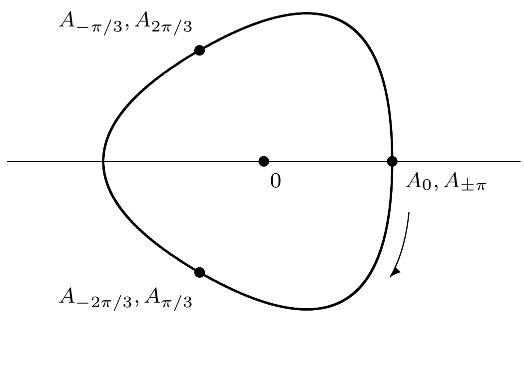


Figure 1. Trajectory of A_ϕ for $|\phi| \le \pi$

Results

$y(x) = y(G, x)$: solution of (E) labelled by $G = (g_{ij}) \in SL_2(\mathbb{C})$.

Theorem 1. Suppose $0 < \phi < \pi/3$ and $g_{11}g_{12}g_{22} \neq 0$.

$\Longrightarrow$

$$y(x) = \wp\big(i(x - x_0^+) + O(x^{-\delta}); g_2(A_\phi), g_3(A_\phi)\big) + \frac{A_\phi}{12}$$

as $x = te^{i\phi} \to \infty$ through the cheese-like strip

$$S(\phi, t_\infty, \kappa_0, \delta_0) = \{x = te^{i\phi} \,|\, \mathrm{Re}\, t > t_\infty,\ |\mathrm{Im}\, t| < \kappa_0\} \setminus \bigcup_{\sigma \in \mathcal{P}(x_0^+)} \{|x - \sigma| < \delta_0\}$$

$$\mathcal{P}(x_0^+) = \{\sigma \,|\, \wp(i(\sigma - x_0^+); g_2(A_\phi), g_3(A_\phi)) = \infty\} = \{x_0^+ - i\Omega_{\mathbf{a}}\mathbb{Z} - i\Omega_{\mathbf{b}}\mathbb{Z}\}$$

$\exists \delta > 0$ κ_0, δ_0: given, $\exists t_\infty = t_\infty(\kappa_0, \delta_0)$

$$g_2(A_\phi) = \frac{A_\phi^2}{12}, \quad g_3(A_\phi) = \frac{A_\phi^3}{216} - 1,$$

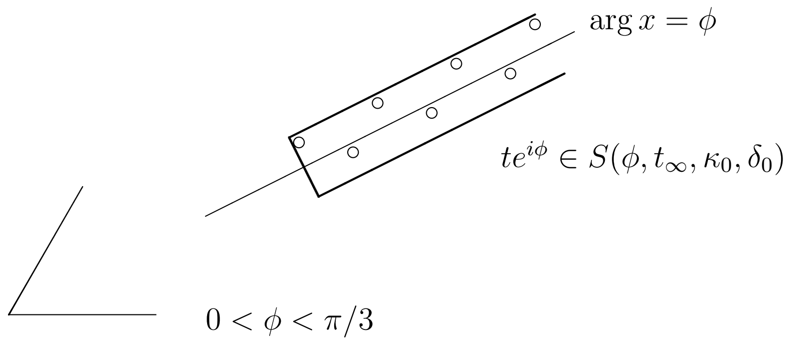
$$-ix_0^+ \equiv \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}} \log \frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}(\log(g_{11}g_{22}) - \pi i)\Big) - ia\Omega_0 \qquad \mathrm{mod}\ \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}$$

with

$$\Omega_0 = \int_\infty^{0^+} \frac{dz}{w(A_\phi, z)}, \quad \Pi_+ \supset [\infty, 0^+] \supset \overline{-\infty, z_2} \text{ along the upper shore of } [\infty, z_2]$$

Theorem 2. Suppose $-\pi/3 < \phi < 0$ and $g_{11}g_{21}g_{22} \neq 0$. the phase shift of $y(G, x)$:

$$-ix_0^- \equiv \frac{-i}{2\pi}\Big(\Omega_{\mathbf{a}} \log \frac{g_{21}}{g_{11}} + \Omega_{\mathbf{b}}(\log(g_{11}g_{22}) - \pi i)\Big) - ia\Omega_0 \qquad \mathrm{mod}\ \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}.$$



The phase shifts above are represented by $g_{11}g_{22}$, g_{21}/g_{11} and g_{12}/g_{22}, which are invariants under an action on G

$\Omega_{\mathbf{a},\mathbf{b}}$, Ω_0: depend on A_ϕ, for $0<|\phi|<\pi/3$

$$y(x)=P(A_\phi, x_0(G,\Omega^\phi_{\mathbf{a}},\Omega^\phi_{\mathbf{b}},\Omega^\phi_0);x)$$

For $|\phi-2m\pi/3|<\pi/3$ $(m\in\mathbb{Z})$, set $\Omega^\phi_{\mathbf{a},\mathbf{b}}=e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_{\mathbf{a},\mathbf{b}}$, $\Omega^\phi_0=e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_0$.

analytic continuation of $y(x)$ beyond $|\phi|<\pi/3$:

Theorem 3. Suppose $0<\phi-2m\pi/3<\pi/3$ (respectively, $-\pi/3<\phi-2m\pi/3<0$) $\Longrightarrow$

$$y(x)=y(G,x)=P(A_\phi,x_0(G^{(m)},\Omega^\phi_{\mathbf{a}},\Omega^\phi_{\mathbf{b}},\Omega^\phi_0);x)$$

as $x=te^{i\phi}\to\infty$ through $S(\phi,t_\infty,\kappa_0,\delta_0)$ with $\mathcal{P}(x_0(G^{(m)},\Omega^\phi_{\mathbf{a}},\Omega^\phi_{\mathbf{b}},\Omega^\phi_0))$, if $g^{(m)}_{11}g^{(m)}_{12}g^{(m)}_{22}\neq 0$ (respectively, $g^{(m)}_{11}g^{(m)}_{21}g^{(m)}_{22}\neq 0$), where

$$G^{(m)}=\begin{cases}(S^0_0\sigma_1)^m G\sigma_3^m e^{(m\pi/3)(a-i/2)\sigma_3} & \text{if } m\ge 1;\\ (\sigma_1 S^0_0)^n G\sigma_3^n e^{(n\pi/3)(i/2-a)\sigma_3} & \text{if } m=-n\le -1.\end{cases}$$

The period, say, $\Omega^\phi_{\mathbf{a}}$ may be expressed by the integral on Π_+

$$\begin{aligned}\Omega^\phi_{\mathbf{a}}=&\int_{e^{2m\pi i/3}\mathbf{a}}\frac{dz}{w(A_\phi,z)}=\int_{e^{2m\pi i/3}\mathbf{a}}\frac{dz}{w(e^{2m\pi i/3}A_{\phi-2m\pi/3},z)}\\ =&e^{2m\pi i/3}\int_{\mathbf{a}}\frac{d\zeta}{w(A_{\phi-2m\pi/3},\zeta)}=e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_{\mathbf{a}}\quad (z=e^{2m\pi i/3}\zeta).\end{aligned}$$

The matrix $G^{(m)}$ has another expression of the form

$$G^{(m)}=\begin{cases}G(S^\infty_0 S^\infty_1\sigma_3 e^{\pi(i/2-a)\sigma_3})^m\sigma_3^m e^{(m\pi/3)(a-i/2)\sigma_3} & \text{if } m\ge 1;\\ G(\sigma_3 e^{\pi(a-i/2)\sigma_3}S^\infty_1 S^\infty_0)^n\sigma_3^n e^{(n\pi/3)(i/2-a)\sigma_3} & \text{if } m=-n\le -1.\end{cases}$$

Outline of our argument

(1) For system (LE) with (y, y^t), under a constraint, **solve the direct monodromy problem** by using WKB analysis, that is, calculate the connection matrix $\hat{G}$.

(2) For a given monodromy data $\hat{G}$ (or G), **find** an asymptotic form $(y_{\rm as}(x, \hat{G}), (B_\phi)_{\rm as}(x, \hat{G}))$ (an approximate solution to the inverse monodromy problem).

(3) For $\hat{G}$ **solve the inverse monodromy problem** with the justification by Kitaev (rigorous proof of the unique existence: **Brouwer's fixed point theorem**).

Our isomonodromy system

$$\text{(LE)}\quad \frac{d\Psi}{d\lambda} = \frac{t}{3}\mathcal{B}(\lambda,t)\Psi, \qquad (x = e^{i\phi}t)$$

$$\begin{aligned}\mathcal{B}(\lambda,t) =& -ie^{i\phi}\lambda\sigma_3 + \begin{pmatrix} 0 & -2ie^{i\phi}y \\ \Gamma_0(t,y,y^t)/y & 0 \end{pmatrix} \\ &- (\Gamma_0(t,y,y^t) + 3(1/2+ia)t^{-1})\lambda^{-1}\sigma_3 + 2e^{i\phi}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\lambda^{-2}, \\ \Gamma_0(t,y,y^t) =& \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - (1+3ia)t^{-1},\end{aligned}$$

characteristic roots of $\mathcal{B}(t,\lambda)$: $\pm\mu$

$$\begin{aligned}\mu(t,\lambda) &= ie^{i\phi}\lambda^{-2}\sqrt{4 - a_\phi\lambda^2 + \lambda^6 - 3ie^{-i\phi}(1+2ia)\lambda^4 t^{-1}}, \\ a_\phi =& a_\phi(t) = e^{-2i\phi}\Big(\frac{y^t}{y} + \frac{1}{2t}\Big)^2 + 4y + \frac{1}{y^2} - 3ie^{-i\phi}(1+2ia)\frac{1}{ty}\end{aligned}$$

First let $\phi = 0$. If $a_0 = a_{\phi=0} = A_0 = 3 \cdot 2^{2/3}$, then

$$\mu(\infty, \lambda)^2|_{\phi=0} = -\lambda^2 + a_0\lambda^{-2} - 4\lambda^{-4} = -\lambda^{-4}(\lambda^2 - 2^{1/3})^2(\lambda^2 + 4^{2/3}).$$

$|\phi| \le \pi/3$:

6 turning points $\mu(t,\lambda)$: $\lambda_0,\ \lambda_1,\ \lambda_0',\ \lambda_1',\ \lambda_2,\ \lambda_2'$ singular points $\lambda = 0,\ \infty$

(coalescing $\lambda_0,\ \lambda_1 \to 2^{1/6}$; $\lambda_0',\ \lambda_1' \to -2^{1/6}$; $\lambda_2\ \lambda_2' \to \pm 2^{2/3} i$ $(\phi \to 0,\ t = \infty)$)

limit **turning points**: $\lambda_j = \lambda_j(t = \infty)$, $\lambda_j' = \lambda_j'(t=\infty)$ $(j = 0, 1, 2)$:

simple as long as they do not coalesce, three lines issue

limit **Stokes curve**: $\displaystyle\int_{\lambda_*}^{\lambda} \operatorname{Re} \mu(\infty, \lambda) d\lambda = 0$

limit **Stokes graph**: on $\mathcal{R}_0 = \mathcal{R}_0^+ \cup \mathcal{R}_0^-$: two-sheeted Riemann surface
$\mathcal{R}_0^\pm$: glued along the cuts $[\lambda_0, \lambda_1], [\lambda_0', \lambda_1'], [\lambda_2, ie^{-i\phi/2}0] \cup [\lambda_2', -e^{-i\phi/2}0]$

along the Stokes curve both canonical solutions are oscillatory, which is suitable for calculation of the connection matrix

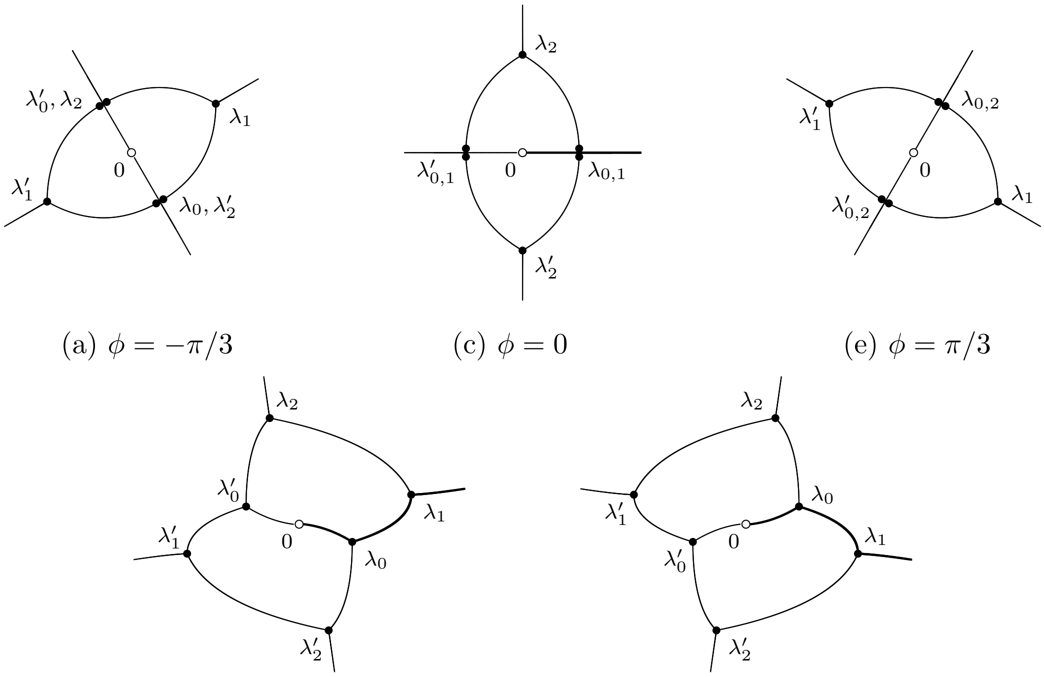
$$\begin{aligned}
-ie^{-i\phi}\lambda^2\mu(\infty,\lambda) =& 2\sqrt{(1-\lambda_{0,\infty}^{-2}\lambda^2)(1-\lambda_{1,\infty}^{-2}\lambda^2)(1-\lambda_{2,\infty}^{-2}\lambda^2)} \\
=& 2\sqrt{1-\lambda_{0,\infty}^{-2}\lambda^2}\sqrt{1-\lambda_{1,\infty}^{-2}\lambda^2}\sqrt{1-\lambda_{2,\infty}^{-2}\lambda^2} \\
\lambda_{j,\infty} =& \lambda_j(\infty), \quad \lambda_{0,\infty}^2\lambda_{1,\infty}^2\lambda_{2,\infty}^2 = -4,
\end{aligned}$$

$$\sqrt{1-\lambda_{j,\infty}^{-2}\lambda^2} \to 1 \ \text{ as } \lambda \to 0 \text{ on } \mathcal{R}_0^+$$

$$\Longrightarrow \quad \mu(t,\lambda) \to -ie^{i\phi}\lambda + O(1) \ \ (\mathcal{R}_0^+ \ni \lambda \to \infty)$$

$$\mu(t,\lambda) \to 2ie^{i\phi}\lambda^{-2} + O(1) \ \ (\mathcal{R}_0^+ \ni \lambda \to 0)$$

Stokes graphs



(a) $\phi = -\pi/3$ (c) $\phi = 0$ (e) $\phi = \pi/3$

(b) $-\pi/3 < \phi < 0$ (d) $0 < \phi < \pi/3$

FIGURE 2. Limit Stokes graphs for $|\phi| \leq \pi/3$

Stokes graph with $\phi = 0$ is used by Kitaev-Vartanian in calculating asymptotics on $\mathbb{R}$

WKB solution (globally asymptotic)

In the canonical domain whose interior contains a Stokes curve issuing from the turning point λ_0 or λ_1, system (LE) admits

$$\Psi_{\mathrm{WKB}}(\lambda) = T(I + O(t^{-\delta}))\exp\left(\int_{\tilde{\lambda}_*}^{\lambda} \Lambda(\tau)d\tau\right)$$

$$\Lambda(\lambda) = \frac{t}{3}\mu(t,\lambda)\sigma_3 - \mathrm{diag}T^{-1}T_\lambda, \quad T = \begin{pmatrix} 1 & \frac{b_3-\mu}{b_1+ib_2} \\ \frac{\mu-b_3}{b_1-ib_2} & 1 \end{pmatrix}$$

outside suitable neighbourhoods of zeros of $b_1 \pm ib_2$ as long as $|\lambda - \lambda_\iota| \gg t^{-2/3+(2/3)\delta}$ ($\iota = 0, 1$), $0 < \delta < 1$ being arbitrary, $\tilde{\lambda}_*$ a base point near λ_0 or λ_1

We use the WKB solution along Stokes curves

$$\frac{d\Psi}{d\lambda} = \frac{t}{3}\mathcal{B}(\lambda,t)\Psi,$$

$$\mathcal{B}(\lambda,t) = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3,$$

$$b_1 = -(i/2)(2e^{i\phi}y + i\Gamma_0(t,y,y^t)y^{-1}) + 2ie^{i\phi}\lambda^{-2},$$

$$b_2 = (1/2)(2e^{i\phi}y - i\Gamma_0(t,y,y^t)y^{-1}),$$

$$b_3 = -ie^{i\phi}\lambda - (\Gamma_0(t,y,y^t) + 3(1/2+ia)t^{-1})\lambda^{-1},$$

$$\Gamma_0(t,y,y^t) = \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - \frac{1+3ia}{t}.$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mu(\lambda,t)^2 = b_1^2 + b_2^2 + b_3^2$$

canonical domain $\mathcal{D} \subset \mathcal{R}_0 \iff$ for $\forall\lambda \in \mathcal{D}$, $\exists$ contours $\mathcal{C}_\pm(\lambda) = (\lambda_\pm, \lambda)^\sim \subset \mathcal{D}$, such that $\mathrm{Re}\int_{\lambda_-}^{\lambda}\mu(\tau)d\tau \to -\infty$, $\mathrm{Re}\int_{\lambda_+}^{\lambda}\mu(\tau)d\tau \to +\infty$ as $\lambda_\pm \to \infty$

around a turning point

λ_ι ($\iota = 0, 1, 2$): turning points, $\quad c_k = b_k(\lambda_\iota)$, $c'_k = (b_k)_\lambda(\lambda_\iota)$ $(k = 1, 2, 3)$

suppose that c_k, c'_k are bounded and $c_1 \pm ic_2 \neq 0$

Then (LE) admits the matrix solution

$$\Phi_\iota(\lambda) = T_\iota(I + O(t^{-\delta'}))\begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix} W(\zeta), \quad T_\iota = \begin{pmatrix} 1 & -\frac{c_3}{c_1 + ic_2} \\ -\frac{c_3}{c_1 - ic_2} & 1 \end{pmatrix}$$

as long as $|\zeta| \ll t^{(2/3-\delta')/3}$, $0 < \delta' < 2/3$ being arbitrary. Here

(1) $\lambda - \lambda_\iota = (2\kappa)^{-1/3}(t/3)^{-2/3}(\zeta + \zeta_0)$ with $\kappa = c_1 c'_1 + c_2 c'_2 + c_3 c'_3$, $|\zeta_0| \ll t^{-1/3}$;

(2) $\hat{t} = 2(2\kappa)^{-1/3}(c_1 - ic_2)(t/3)^{1/3}$;

(3) $W(\zeta)$ solves

$$\frac{dW}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} W, \qquad \begin{pmatrix} \mathrm{Bi}(\zeta) & \mathrm{Ai}(\zeta) \\ \mathrm{Bi}_\zeta(\zeta) & \mathrm{Ai}_\zeta(\zeta) \end{pmatrix}, \quad \mathrm{Bi}(\zeta) = e^{-\pi i/6}\mathrm{Ai}(e^{-2\pi i/3}\zeta),$$

which admits canonical solutions

$$W_\nu(\zeta) = \zeta^{-(1/4)\sigma_3}(\sigma_3 + \sigma_1)(I + O(\zeta^{-3/2}))\exp\bigl(\tfrac{2}{3}\zeta^{3/2}\sigma_3\bigr)$$

($\nu \in \mathbb{Z}$) as $\zeta \to \infty$ in $\Sigma_\nu : |\arg\zeta - (2\nu - 1)\pi/3| < 2\pi/3$, and $W_{\nu+1}(\zeta) = W_\nu(\zeta)S_\nu$ with

$$S_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad S_{\nu+1} = \sigma_1 S_\nu \sigma_1.$$

Direct monodromy problem

Calculate $\hat{G}$ for (LE).

Recall

$$a_\phi = a_\phi(t) = e^{-2i\phi}\Big(\frac{y^t}{y} + \frac{1}{2t}\Big)^2 + 4y + \frac{1}{y^2} - 3ie^{-i\phi}(1+2ia)\frac{1}{ty}$$

$$\begin{aligned} \mu(t,\lambda) &= ie^{i\phi}\lambda^{-2}\sqrt{4 - a_\phi\lambda^2 + \lambda^6 - 3ie^{-i\phi}(1+2ia)\lambda^4 t^{-1}}, \\ &= ie^{i\phi}\lambda\sqrt{4z^3 - a_\phi z^2 + 1 + O(\lambda^{-2}t^{-1})} \quad (\lambda^{-2} = z) \end{aligned}$$

$w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1$

Constraint: A_ϕ a solution of the Boutroux equations,

$$\text{(C)} \quad a_\phi(t) = A_\phi + \frac{B_\phi(t)}{t}, \quad B_\phi(t) \ll 1$$

for $t \in S_\phi(t'_\infty, \kappa_1, \delta_1)$,

$$S_\phi(t'_\infty, \kappa_1, \delta_1) = \{t \,|\, \mathrm{Re}\, t > t'_\infty,\ \ |\mathrm{Im}\, t| < \kappa_1,\ \ |y(t)| + |y^t(t)| + |y(t)|^{-1} < \delta_1^{-1}\}$$

$f \ll g$ or $g \gg f$ means $f = O(|g|)$

$f \asymp g$ means $f \ll g \ll f$

Let $0 < \phi < \pi/3$

$\mathbf{c}_\infty = (\infty, \lambda_1)^\sim, \qquad \mathbf{c}_1 = (\lambda_1, \lambda_0)^\sim, \qquad \mathbf{c}_0 = (\lambda_0, 0)^\sim$

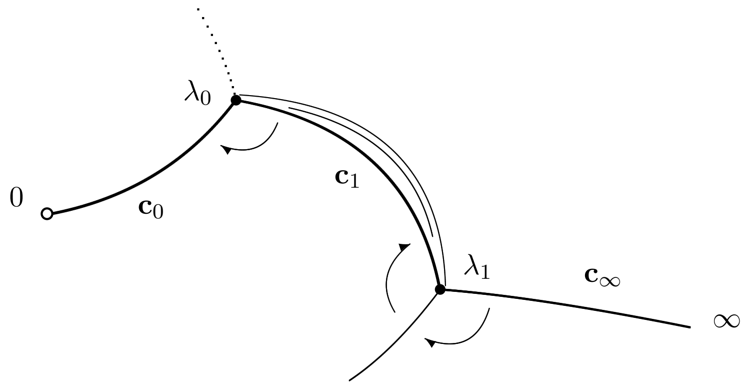


FIGURE 3. Stokes curve for $0 < \phi < \pi/3$

Matching

(1) $\Psi_\infty(\lambda)$: WKB solution along $\mathbf{c}_\infty = (\infty, \lambda_1)^\sim$,

$\hat{Y}_0^\infty(\lambda)$: canonical solution of (LE)

Set $\hat{Y}_0^\infty(\lambda) = \Psi_\infty(\lambda)\Gamma_\infty$

$$\begin{aligned}\Gamma_\infty =& \Psi_\infty(\lambda)^{-1}\hat{Y}_0^\infty(\lambda) = \Psi_\infty(\lambda)^{-1}Y_0^{\infty,*}(\lambda)\Theta_{0,*}^{\sigma_3}\\ =& \exp\Big(-\int_{\tilde{\lambda}_1}^{\lambda}\Lambda(\tau)d\tau\Big)T^{-1}(I+O(t^{-\delta}+|\lambda|^{-1}))\\ &\times\exp\Big(-\frac{1}{6}(ie^{i\phi}t\lambda^2+3(1+2ia)\log\lambda)\sigma_3\Big),\\ =& C_3(\tilde{\lambda}_1)c_I(\tilde{\lambda}_1)(I+O(t^{-\delta}))\\ &\times\exp\Big(-\lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_\infty}}\Big(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)d\tau+\frac{1}{6}(ie^{i\phi}t\lambda^2+3(1+2ia)\log\lambda)\sigma_3\Big)\Big),\end{aligned}$$

in which $C_3(\tilde{\lambda}_1) = \exp(\int_{\lambda_1}^{\tilde{\lambda}_1}\Lambda_3(\tau)d\tau)$, $c_I(\tilde{\lambda}_1) = \exp(-\int_{\tilde{\lambda}_1}^{\infty}\Lambda_I(\tau)d\tau)$,
$\tilde{\lambda}_1 \in \mathbf{c}_\infty$, $\tilde{\lambda}_1 - \lambda_1 \asymp t^{-1}$.

$$\begin{aligned}\Lambda(\lambda) &= \Lambda_3(\lambda)+\Lambda_I(\lambda),\\ \Lambda_3(\lambda) &= \frac{t}{3}\mu(t,\lambda)\sigma_3 - \mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}\sigma_3, \quad \Lambda_I(\lambda) = -\mathrm{diag}T^{-1}T_\lambda|_I I\end{aligned}$$

(2) $\Psi_\infty(\lambda)$: WKB solution above

$\Phi_1^+(\lambda)$: solution in a sector containing $\mathbf{c}_\infty$ near the turning point λ_1

set $\Psi_\infty(\lambda) = \Phi_1^+(\lambda)\Gamma_{1+}$ along $\mathbf{c}_\infty$

(in the annulus $\mathcal{A}_\varepsilon^1$ around λ_1, where both solution is defined)

$$\begin{aligned}\Gamma_{1+} =& \Phi_1^+(\lambda)^{-1}\Psi_\infty(\lambda)\\ =& W(\zeta)^{-1}\begin{pmatrix}1 & 0\\ 0 & (t/3)^{-1/3}K\end{pmatrix}^{-1}(I+O(t^{-\delta}))\begin{pmatrix}1 & -\frac{d_3}{d_1+id_2}\\ -\frac{d_3}{d_1-id_2} & 1\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}1 & \frac{b_3-\mu}{b_1+ib_2}\\ \frac{\mu-b_3}{b_1-ib_2} & 1\end{pmatrix}(I+O(t^{-\delta}))\exp\Big(\int_{\tilde{\lambda}_1}^{\lambda}\Lambda(\tau)d\tau\Big)\end{aligned}$$

$\Longrightarrow$

$$\Gamma_{1+} = (\tilde{\zeta}_1)^{1/4}(I+O(t^{-\delta}))C_3(\tilde{\lambda}_1)^{-1}\begin{pmatrix}1 & 0\\ 0 & -\frac{d_1-id_2}{d_3}\end{pmatrix}$$

with suitably chosen $\tilde{\zeta}_1 \asymp \tilde{\lambda}_1 - \lambda_1$.

$c_k = b_k(\lambda_0)$, $d_k = b_k(\lambda_1)$ $(k=1,2,3)$

....

$\mathcal{A}_\varepsilon^1 : t^{-2/3+(2/3)(2/9-\varepsilon)} \ll |\lambda-\lambda_1| \ll t^{-2/3+(2/3)(2/9+\varepsilon/2)}$

Solution to the direct monodromy problem

$\Longrightarrow$ Repeating matchings we get the connection matrix

$0 < \phi < \pi/3$**:**

$$
\begin{aligned}
\hat{G} =& G\Theta_{0,*}^{\sigma_3} = \hat{Y}_0^0(\lambda)^{-1} Y_0^{\infty,*}(\lambda)\Theta_{0,*}^{\sigma_3} = \hat{Y}_0^0(\lambda)^{-1}\hat{Y}_0^{\infty}(\lambda) \\
=& \Gamma_0\Gamma_{0-}\Gamma_{0*}\Gamma_{0+}\Gamma_{01}\Gamma_{1-}\Gamma_{1*}\Gamma_{1+}\Gamma_{\infty} \\
=& \epsilon_+ i(\sigma_3 + O(t^{-\delta}))\exp(J_0\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -c_0^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -c_0\end{pmatrix} \\
&\times \exp(-J_1\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -d_0^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -d_0\end{pmatrix}\exp(-J_\infty\sigma_3) \\
=& \epsilon_+(I + O(t^{-\delta})) \\
&\times \begin{pmatrix} i\exp(J_0 - J_1 - J_\infty) & -d_0\exp(J_0 - J_1 + J_\infty) \\ (c_0^{-1}\exp(-J_1) + d_0^{-1}\exp(J_1))\exp(-J_0 - J_\infty) & ic_0^{-1}d_0\exp(-J_0 - J_1 + J_\infty)\end{pmatrix}
\end{aligned}
$$

Here $\epsilon_+^2 = 1$, $c_0 = (c_1 - ic_2)/c_3$, $d_0 = (d_1 - id_2)/d_3$,

$$
J_0\sigma_3 = \lim_{\substack{\lambda\to 0\\ \lambda\in\mathbf{c}_0}}\Big(\int_{\lambda_0}^{\lambda}\Lambda_3(\tau)d\tau + \frac{2i}{3}e^{i\phi}t\lambda^{-1}\sigma_3\Big), \qquad J_1\sigma_3 = \int_{\lambda_0}^{\lambda_1}\Lambda_3(\tau)d\tau \quad (\text{along } \mathbf{c}_1),
$$

$$
J_\infty\sigma_3 = \lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_\infty}}\Big(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)d\tau + \frac{1}{6}(ie^{i\phi}t\lambda^2 + 3(1+2ia)\log\lambda)\sigma_3\Big)
$$

$c_k = b_k(\lambda_0)$, $d_k = b_k(\lambda_1)$ $(k = 1, 2, 3)$

Proposition. If $0 < \phi < \pi/3$, then

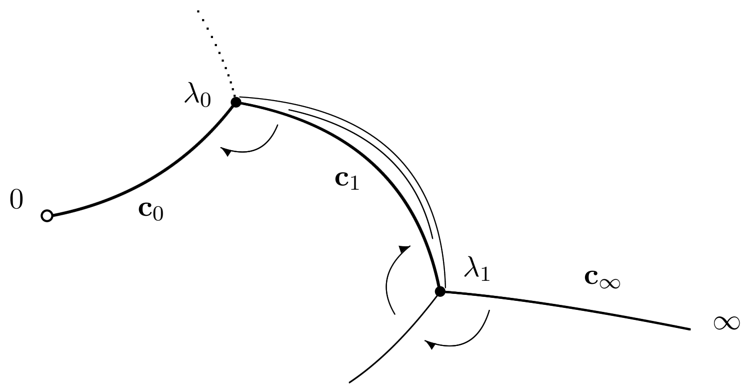
$$\hat{G} = (\hat{g}_{ij}) = \epsilon_+(I + O(t^{-\delta})) \times \begin{pmatrix} i\exp(J_0 - J_1 - J_\infty) & -d_0\exp(J_0 - J_1 + J_\infty) \\ (c_0^{-1}\exp(-J_1) + d_0^{-1}\exp(J_1))\exp(-J_0 - J_\infty) & ic_0^{-1}d_0\exp(-J_0 - J_1 + J_\infty) \end{pmatrix}.$$

$$\hat{G} = G\Theta_{0,*}^{\sigma_3},\ G = (g_{ij})$$

Corollary. If $0 < \phi < \pi/3$ and $g_{11}g_{12}g_{22} \neq 0$, then

$$g_{11}g_{12} = -c_0^{-1}d_0(1 + O(t^{-\delta}))\exp(-2J_1), \quad \frac{g_{12}}{g_{22}} = ic_0(1 + O(t^{-\delta}))\exp(2J_0).$$

$$J_1\sigma_3 = \int_{\lambda_0}^{\lambda_1} \Lambda_3(\tau)d\tau \quad (\text{along } \mathbf{c}_1), \qquad \Lambda_3(\lambda) = \frac{t}{3}\mu(t,\lambda)\sigma_3 - \mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}\sigma_3$$



Expressions of J_0 and J_1

set $\lambda^{-2} = z \quad \Longrightarrow$

$$\begin{aligned}\mu(t,\lambda)d\lambda =& \Big(-\frac{e^{2i\phi}}{z} + e^{2i\phi}a_\phi z - 4e^{2i\phi}z^2 + 3ie^{i\phi}(1+2ia)t^{-1}\Big)^{1/2}\frac{(-z^{-3/2})}{2}dz \\ =& \Big(-\frac{i}{2}e^{i\phi}\frac{w(z)}{z^2} - \frac{3}{4}(1+2ia)t^{-1}\frac{1}{zw(z)} + O(t^{-2}w(z)^{-3})\Big)dz\end{aligned}$$

turning points: $\lambda_0,\ \lambda_1,\ \lambda_2,\ 0 \in \mathcal{R}_\phi \quad \Longrightarrow \quad z_0 = \lambda_0^{-2},\ z_1 = \lambda_1^{-2},\ z_2 = \lambda_2^{-2},\ \infty \in \Pi_{a_\phi}$

Π_{a_ϕ} : elliptic curve for $\quad w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1$

cycles: $\mathbf{a}, \mathbf{b} \subset \Pi_{a_\phi} \qquad a_\phi(t) = A_\phi + B_\phi(t)t^{-1} = A_\phi + O(t^{-1})$

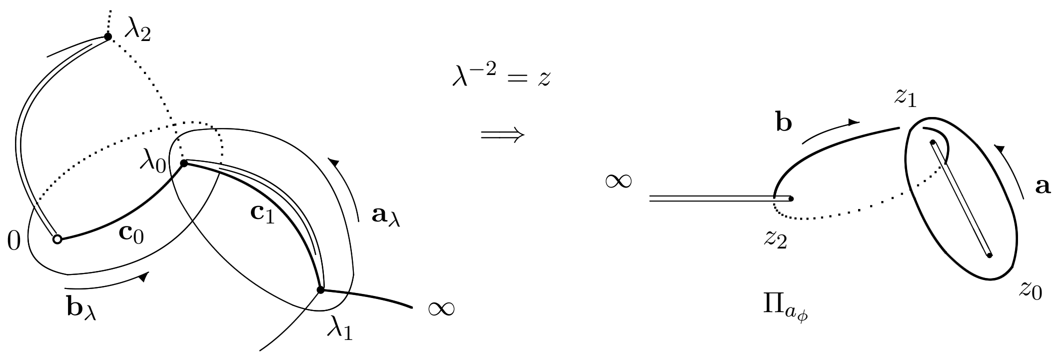


FIGURE 4. Correspondence of the cycles under the map $z = \lambda^{-2}$

$$J_1\sigma_3 = \int_{\lambda_0}^{\lambda_1} \Lambda_3(\tau)d\tau \quad (\text{along } \mathbf{c}_1),$$

$$\Lambda_3(\lambda) = \frac{t}{3}\mu(t,\lambda)\sigma_3 - \mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}\sigma_3, \quad \mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}\sigma_3 = \frac{1}{4}\Big(1-\frac{b_3}{\mu}\Big)\frac{\partial}{\partial\lambda}\log\frac{b_1+ib_2}{b_1-ib_2}$$

$$\begin{aligned}\int_{\lambda_0\,(\mathbf{c}_1)}^{\lambda_1} \mu(t,\tau)d\tau =& \frac{i}{4}e^{i\phi}a_\phi \int_{\mathbf{a}}\frac{dz}{w(z)} - \frac{3i}{4}e^{i\phi}\int_{\mathbf{a}}\frac{dz}{z^2w(z)} - \frac{3}{8}(1+2ia)t^{-1}\int_{\mathbf{a}}\frac{dz}{zw(z)} + O(t^{-2})\\ =& -\frac{i}{4}e^{i\phi}\int_{\mathbf{a}}\frac{w(z)}{z^2}dz - \frac{3}{8}(1+2ia)t^{-1}\int_{\mathbf{a}}\frac{dz}{zw(z)} + O(t^{-2}),\end{aligned}$$

$$\begin{aligned}&z_+ = y, \quad z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t,y,y^t),\\ &\qquad b_1 - ib_2 = 2ie^{i\phi}(z-z_+), \quad b_1+ib_2 = 2ie^{i\phi}(z-z_-),\\ &\qquad \Gamma_0(t,y,y^t) = y^ty^{-1} - ie^{i\phi}y^{-1} - (1+3ia)t^{-1}\end{aligned}$$

$$\Longrightarrow$$

$$-\int_{\lambda_0\ (\mathbf{c}_1)}^{\lambda_1} \mathrm{diag}T^{-1}T_\lambda|_{\sigma_3}d\lambda + \frac{1}{2}\log(c_0d_0^{-1}) = \frac{1}{8}\int_{\mathbf{a}}\Big(\frac{w(z_+)}{z-z_+} - \frac{w(z_-)}{z-z_-}\Big)\frac{dz}{w(z)} + O(t^{-1})$$

Proposition. Let

$$W(z)=\Big(\frac{w(z_+)}{z-z_+}-\frac{w(z_-)}{z-z_-}\Big)\frac{1}{w(z)}.$$

Suppose that $g_{11}g_{22}\neq 0$, $g_{12}/g_{22}\neq 0$. For $0<\phi<\pi/3$,

$$\log\frac{g_{12}}{g_{22}}=\frac{ie^{i\phi}t}{6}\int_{\mathbf{b}}\frac{w(z)}{z^2}dz-\frac{1}{4}\int_{\mathbf{b}}W(z)dz+\frac{1}{4}(1+2ia)\int_{\mathbf{b}}\frac{dz}{zw(z)}+O(t^{-\delta}),$$

$$\log(g_{11}g_{22})=\frac{ie^{i\phi}t}{6}\int_{\mathbf{a}}\frac{w(z)}{z^2}dz-\frac{1}{4}\int_{\mathbf{a}}W(z)dz+\frac{1}{4}(1+2ia)\int_{\mathbf{a}}\frac{dz}{zw(z)}+\pi i+O(t^{-\delta}).$$

$$z_+=y,\quad z_-=(i/2)e^{-i\phi}y^{-1}\Gamma_0(t,y,y^t),$$

$$w(z)^2=w(a_\phi,z)^2=4z^3-a_\phi z^2+1$$

a, **b**: the cycles on $\Pi_{a_\phi} = \Pi_+ \cup \Pi_-$

$$\omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(a_\phi, z)}, \quad \omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(a_\phi, z)}, \quad \operatorname{Im}\tau = \frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}} > 0,$$

in our calculation we use the ϑ-function

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2 + 2\pi i z n}, \qquad \nu = \frac{1+\tau}{2}$$

$z,\ \tilde{z} \in \Pi_{a_\phi} = \Pi_+ \cup \Pi_-$, let

$$F(\tilde{z}, z) = \frac{1}{\omega_{\mathbf{a}}} \int_{\tilde{z}}^{z} \frac{dz}{w(z)} = \frac{1}{\omega_{\mathbf{a}}} \int_{\infty}^{z} \frac{dz}{w(z)} - \frac{1}{\omega_{\mathbf{a}}} \int_{\infty}^{\tilde{z}} \frac{dz}{w(z)}$$

projection of $z_0 \in \Pi_{a_\phi}$ on the respective sheets:
$z_0^+ = (z_0, w(z_0)) = (z_0, w(z_0^+))$, $z_0^- = (z_0, -w(z_0)) = (z_0, -w(z_0^+))$

$$\frac{dz}{(z-z_0)w(z)} = \frac{1}{w(z_0^+)} d\log \frac{\vartheta(F(z_0^+, z)+\nu,\tau)}{\vartheta(F(z_0^-, z)+\nu,\tau)} - g_0(z_0)\frac{dz}{w(z)},$$

$$g_0(z_0) = \frac{w'(z_0^+)}{2w(z_0^+)} - \frac{1}{\omega_{\mathbf{a}}} \frac{1}{w(z_0^+)} \Big(\pi i + \frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+)+\nu,\tau)\Big)$$

$$(dz/du)^2 = w(a_\phi, z)^2, \quad w(z)^2 = w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1$$

$$\Longrightarrow \quad z = \wp(u; g_2, g_3) + \frac{a_\phi}{12}, \quad g_2 = \frac{a_\phi^2}{12}, \quad g_3 = -1 + \frac{a_\phi^3}{216}$$

$$\log(g_{11}g_{22}) = \frac{ie^{i\phi}t}{6}\int_{\mathbf{a}}\frac{w(z)}{z^2}dz - \frac{1}{4}\int_{\mathbf{a}}W(z)dz + \frac{1}{4}(1+2ia)\int_{\mathbf{a}}\frac{dz}{zw(z)} + \pi i + O(t^{-\delta})$$

$$\log\frac{g_{12}}{g_{22}} = \frac{ie^{i\phi}t}{6}\int_{\mathbf{b}}\frac{w(z)}{z^2}dz - \frac{1}{4}\int_{\mathbf{b}}W(z)dz + \frac{1}{4}(1+2ia)\int_{\mathbf{b}}\frac{dz}{zw(z)} + O(t^{-\delta})$$

$$\begin{aligned}\frac{1}{z^2}(w(a_\phi,z)-w(A_\phi,z)) &= \frac{1}{z^2}(\sqrt{4z^3-a_\phi z^2+1}-\sqrt{4z^3-A_\phi z^2+1})\\ &= -\frac{t^{-1}B_\phi(t)}{2w(A_\phi,z)}(1+O(t^{-1}B_\phi(t))).\end{aligned}$$

$$W(z) = \Big(\frac{w(z_+)}{z-z_+} - \frac{w(z_-)}{z-z_-}\Big)\frac{1}{w(z)}, \qquad F(z_\pm^-, z_\pm^+) = \frac{1}{\omega_{\mathbf{a}}}\int_{z_\pm^-}^{z_\pm^+}\frac{dz}{w(a_\phi,z)} = \frac{2}{\omega_{\mathbf{a}}}\int_{\infty}^{z_\pm^+}\frac{dz}{w(a_\phi,z)}$$

$$\begin{aligned}&\int_{\mathbf{a}}W(z)dz = 2\Big(\frac{\vartheta'}{\vartheta}(\tfrac{1}{2}F(z_+^-,z_+^+)+\nu,\tau) - \frac{\vartheta'}{\vartheta}(\tfrac{1}{2}F(z_-^-,z_-^+)+\nu,\tau)\Big)\\ &\Big(\int_{\mathbf{b}} - \tau\int_{\mathbf{a}}\Big)W(z)dz = 2\pi i(F(z_+^-,z_+^+) - F(z_-^-,z_-^+)),\\ &\Big(\int_{\mathbf{b}} - \tau\int_{\mathbf{a}}\Big)\frac{dz}{zw(z)} = -2\pi i F(0^-,0^+),\\ &\Big(\int_{\mathbf{b}} - \tau\int_{\mathbf{a}}\Big)\frac{dz}{z^2w(z)} = \frac{4\pi i}{\omega_{\mathbf{a}}}.\end{aligned}$$

Boutroux equations are necessary

Let $0 < \phi < \pi/3$.

$$ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{a}} - \frac{\Omega_{\mathbf{a}}}{2}B_\phi(t)\Big)$$
$$=\frac{3}{2}\int_{\mathbf{a}} W(z)dz + \frac{3}{2}(1+2ia)g_0(0^+)\omega_{\mathbf{a}} + 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta})$$
$$=3\Big(\frac{\vartheta'}{\vartheta}(\tfrac{1}{2}F(z_+^-, z_+^+) + \nu, \tau) - \frac{\vartheta'}{\vartheta}(\tfrac{1}{2}F(z_-^-, z_-^+) + \nu, \tau)\Big) + \frac{3}{2}(1+2ia)g_0(0^+)\omega_{\mathbf{a}}$$
$$+ 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta})$$

$$\Omega_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} \frac{dz}{w(A_\phi, z)}, \quad \mathcal{J}_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} \frac{w(A_\phi, z)}{z^2}dz, \quad g_0(0^+) = \frac{1}{\omega_{\mathbf{a}}}\Big(\pi i + \frac{\vartheta'}{\vartheta}(F(0^-, 0^+) + \nu, \tau)\Big)$$

$\mathrm{Re}\,(\vartheta'/\vartheta)(\tfrac{1}{2}F(z_\pm^-, z_\pm^+) + \nu, \tau)$:bounded in $S_\phi(t'_\infty, \kappa_1, \delta_1)$

$\Longrightarrow$ $\mathrm{Re}\int_{\mathbf{a}} W(z)dz$: bounded in $S_\phi(t'_\infty, \kappa_1, \delta_1)$

(Note that A_ϕ satisfies **Boutroux equations** $\iff$ $\mathrm{Im}\, e^{i\phi}\mathcal{J}_{\mathbf{a},\mathbf{b}} = 0$)

- $|\log(g_{11}g_{22})| \ll 1$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$ $\iff$ $\mathrm{Im}\, e^{i\phi}\mathcal{J}_{\mathbf{a}} = 0$

$$ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{b}} - \frac{\Omega_{\mathbf{b}}}{2}B_\phi(t)\Big)$$
$$=\frac{3}{2}\int_{\mathbf{b}} W(z)dz + \frac{3}{2}(1+2ia)(2\pi i F(0^-, 0^+) + g_0(0^+)\omega_{\mathbf{b}}) + 6\log\frac{g_{12}}{g_{22}} + O(t^{-\delta}),$$

- $|\log(g_{12}/g_{22})| \ll 1$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$ $\iff$ $\mathrm{Im}\, e^{i\phi}\mathcal{J}_{\mathbf{b}} = 0$

Constraint: A_ϕ a solution of the Boutroux equations,

$$\text{(C)}\quad a_\phi(t) = A_\phi + \frac{B_\phi(t)}{t}, \quad B_\phi(t) \ll 1$$

for $t \in S_\phi(t'_\infty, \kappa_1, \delta_1)$,

$$S_\phi(t'_\infty, \kappa_1, \delta_1) = \{t \,|\, \mathrm{Re}\, t > t'_\infty,\ \ |\mathrm{Im}\, t| < \kappa_1,\ \ |y(t)| + |y^t(t)| + |y(t)|^{-1} < \delta_1^{-1}\}$$

$$F(z_\pm^-, z_\pm^+) = \frac{1}{\omega_{\mathbf{a}}}\int_{z_\pm^-}^{z_\pm^+} \frac{dz}{w(a_\phi, z)} = \frac{2}{\omega_{\mathbf{a}}}\int_{\infty}^{z_\pm^+} \frac{dz}{w(a_\phi, z)}$$

Inverse monodromy problem

Suppose that $0 < \phi < \pi/3$ Given $G = (g_{ij}) \in SL_2(\mathbb{C})$ such that $g_{11}g_{12}g_{22} \neq 0$

$$\log(g_{12}/g_{22}) - \tau \log(g_{11}g_{22})$$
$$= \Big(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\Big)\Big(\frac{ie^{i\phi}t}{6} \cdot \frac{w(a_\phi, z)}{z^2} - \frac{1}{4}W(z) + \frac{1+2ia}{4zw(a_\phi, z)}\Big)dz - \tau\pi i + O(t^{-\delta})$$
$$= -\frac{2\pi e^{i\phi}t}{\omega_{\mathbf{a}}} - \frac{\pi i}{2}(F(z_+^-, z_+^+) - F(z_-^-, z_-^+)) - \frac{\pi i}{2}(1+2ia)F(0^-, 0^+) - \tau\pi i + O(t^{-\delta})$$

$a_\phi \to A_\phi$ (ok)

- $$\log(g_{12}/g_{22}) - \tau \log(g_{11}g_{22}) = -\frac{2\pi e^{i\phi}t}{\Omega_{\mathbf{a}}}$$
$$-\frac{\pi i}{\Omega_{\mathbf{a}}}\Big(\int_\infty^{z_+^+} - \int_\infty^{z_-^+}\Big)\frac{dz}{w(A_\phi, z)} - \frac{\pi i}{2}(1+2ia)F_{A_\phi}(0^-, 0^+) - \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}\pi i + O(t^{-\delta})$$

Here

$$F(z_\pm^-, z_\pm^+)|_{a_\phi \to A_\phi} = F_{A_\phi}(z_\pm^-, z_\pm^+) = 2F_{A_\phi}(\infty, z_\pm^+) = \frac{2}{\Omega_{\mathbf{a}}}\int_\infty^{z_\pm^+}\frac{dz}{w(A_\phi, z)}$$
$$z_+ = y, \quad z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t, y, y^t) = y^t y^{-1} - ie^{i\phi}y^{-1} - (1+3ia)t^{-1}$$

$$(dz/du)^2 = w(A_\phi, z)^2, \quad w(z)^2 = w(A_\phi, z)^2 = 4z^3 - A_\phi z^2 + 1$$
$$\Longrightarrow \quad z = \wp(u; g_2, g_3) + \frac{A_\phi}{12}, \quad g_2 = \frac{A_\phi^2}{12}, \quad g_3 = -1 + \frac{A_\phi^3}{216}$$

Set $\quad u_+ = 2\displaystyle\int_\infty^{z_+^+} \frac{dz}{w(A_\phi, z)}, \quad u_- = 2\int_\infty^{z_-^+} \frac{dz}{w(A_\phi, z)}, \quad$ i.e. $z_\pm^+ = \wp(u_\pm) + \dfrac{A_\phi}{12}$

with $\wp(u) = \wp(u; g_2, g_3)$, $g_2 = \frac{1}{12}A_\phi^2$, $g_3 = \frac{1}{216}A_\phi^3 - 1$ to write

- $$u_+ - u_- = 2ie^{i\phi}t + \frac{i}{\pi}\Big(\Omega_{\mathbf{a}} \log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) - \Omega_{\mathbf{b}} - (1+2ia)\Omega_{\mathbf{a}} F_{A_\phi}(\infty, 0^+) + O(t^{-\delta}).$$

$$\begin{aligned}\wp(u_+ + u_-) &= -\wp(u_+) - \wp(u_-) + \frac{1}{4}\Big(\frac{\wp'(u_+) - \wp'(u_-)}{\wp(u_+) - \wp(u_-)}\Big)^2 \\ &= -z_+^+ - z_-^+ + \frac{A_\phi}{6} + \frac{1}{4}\Big(\frac{w(z_+^+) - w(z_-^+)}{z_+^+ - z_-^+}\Big)^2 \qquad \text{(the addition theorem)}\end{aligned}$$

$z_+ = y$ and $z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t, y, y^t) \Longrightarrow$ RHS$= -\frac{A_\phi}{12} + O(t^{-1})$, which implies

- $$u_+ + u_- = \int_\infty^{0^+} \frac{dz}{w(A_\phi, z)} + O(t^{-1}) = \Omega_{\mathbf{a}} F_{A_\phi}(\infty, 0^+) + O(t^{-1}).$$

Then we have

$$\begin{aligned}u_+ &= \int_\infty^{z_+^+} \frac{dz}{w(A_\phi, z)} \\ &= ie^{i\phi}t + \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}}\log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) - \frac{\Omega_{\mathbf{b}}}{2} - ia\Omega_0 + O(t^{-\delta}),\end{aligned}$$

$\Longrightarrow$ expression of Theorem 1

$$\begin{aligned}u_- &= \int_\infty^{z_-^+} \frac{dz}{w(A_\phi, z)} \\ &= -ie^{i\phi}t - \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}}\log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) + \frac{\Omega_{\mathbf{b}}}{2} + (1+ia)\Omega_0 + O(t^{-\delta})\end{aligned}$$

Proposition. In $S_\phi(t'_\infty, \kappa_1, \delta_1)$,

$$\begin{aligned}ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{a}} - \frac{\Omega_{\mathbf{a}}}{2}B_\phi(t)\Big) = &3\Big(\frac{\vartheta'}{\vartheta}(\Omega_{\mathbf{a}}^{-1}i(x - x_0^+) + \nu, \tau_\Omega) + \frac{\vartheta'}{\vartheta}(\Omega_{\mathbf{a}}^{-1}(i(x - x_0^+) - \Omega_0) + \nu, \tau_\Omega)\Big) \\ &+ \tfrac{3}{2}(1+2ia)g_0(0^+)\Omega_{\mathbf{a}} + 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta})\end{aligned}$$

with $x = e^{i\phi}t$, $\tau_\Omega = \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}$.

Justification. The justification by the lines of Kitaev.

$\mathcal{G}=(g_{12}/g_{22}, g_{11}g_{22})$: given such that $g_{11}g_{12}g_{22}\neq 0$ on monodromy manifold for (LE)

Set $\quad y_{\rm as}=y_{\rm as}(\mathcal{G},t)=\wp(i(e^{i\phi}t-x_0^+);g_2(A_\phi),g_3(A_\phi))+\dfrac{A_\phi}{12},\quad x_0^+=x_0^+(g_{12}/g_{22},g_{11}g_{22}),$

$(B_\phi)_{\rm as}=(B_\phi)_{\rm as}(\mathcal{G},t) \qquad$ (leading parts without $O(t^{-\delta})$)

$$y^t_{\rm as}:=-\frac{y_{\rm as}}{2}t^{-1}+ie^{i\phi}\sqrt{4y_{\rm as}^3-A_\phi y_{\rm as}^2+1-(3ie^{-i\phi}(1+2ia)+(B_\phi)_{\rm as}y_{\rm as})y_{\rm as}t^{-1}}$$

with the branch of $y^t_{\rm as}$: compatible with $(\partial/\partial t)y_{\rm as}$

$\Longrightarrow$

$(y_{\rm as},y^t_{\rm as})=(y_{\rm as}(\mathcal{G},t),y^t_{\rm as}(\mathcal{G},t))$ fulfils (C) with $B_\phi(t)=(B_\phi)_{\rm as}(\mathcal{G},t)$

in $\{t\,|\,\mathrm{Re}\,t>t_\infty,\,|\mathrm{Im}\,t|<\kappa_0\}\setminus\bigcup_{i\sigma\in Z_0}\{|t-e^{-i\phi}\sigma|<\delta_2\}$, Z_0 : poles and zeros of $y_{\rm as}$

$\mathcal{G}_{\rm as}(t)$: the monodromy data for system (LE) with $\mathcal{B}(\lambda,t)$ containing $(y_{\rm as},y^t_{\rm as})$

the direct monodromy problem by WKB analysis $\quad\Longrightarrow$

$\|\mathcal{G}_{\rm as}(t)-\mathcal{G}\|\le C|t|^{-\delta}$ for $|t|\ge t_\infty$, $\qquad C, t_\infty$: uniform in a neighbourhood of $\mathcal{G}$

$\Longrightarrow$ the justification scheme of Kitaev applies to our case

$\Longrightarrow$ the existence and uniqueness of the solution to the inverse monodromy problem

$$\text{(C)}\quad a_\phi(t)=A_\phi+\frac{B_\phi(t)}{t},\quad B_\phi(t)\ll 1$$

$$a_\phi=a_\phi(t)=e^{-2i\phi}\Big(\frac{y^t}{y}+\frac{1}{2t}\Big)^2+4y+\frac{1}{y^2}-3ie^{-i\phi}(1+2ia)\frac{1}{ty}$$

$$Z_0=\{ix_0^++\Omega_{\mathbf{a}}\mathbb{Z}+\Omega_{\mathbf{b}}\mathbb{Z}\}\cup\{ix_0^++\Omega_0+\Omega_{\mathbf{a}}\mathbb{Z}+\Omega_{\mathbf{b}}\mathbb{Z}\}\cup\{ix_0^++\xi_0\,|\,\wp(\xi_0)=-A_\phi/12\}.$$

$\mathcal{M}_0$: given, $K(\varepsilon_0)$: $\|\mathcal{M} - \mathcal{M}_0\| \le \varepsilon_0$ (compact in $\mathbb{C}^2$): given

$\Longrightarrow$ $\exists C_0, T_\infty$ such that, for $\forall \mathcal{M} \in K(\epsilon_0)$, $\|\mathcal{M} - (\mathcal{M})_{\rm as}(t)\| \le C_0 |t|^{-\delta}$ for $|t| \ge T_\infty$

Let $f(t, \mathcal{M}) := \mathcal{M}_0 - (\mathcal{M})_{\rm as}(t) + \mathcal{M}$

$\Longrightarrow$ f: $K(\varepsilon_0) \to K(\varepsilon_0)$ if $\|\mathcal{M} - (\mathcal{M})_{\rm as}(t)\| \le \varepsilon_0$, i.e. if $|t|^\delta \ge C_0/\varepsilon_0$

Brouwer's fixed point theorem:

$\exists \mathcal{M}_* \in K(\varepsilon_0)$ such that $f(t, \mathcal{M}_*) = \mathcal{M}_*$ for each $|t| \ge \max\{T_\infty, (C_0/\varepsilon_0)^{1/\delta}\}$

i.e. $(\mathcal{M}_*)_{\rm as}(t) = \mathcal{M}_0$ (as shown later $\mathcal{M}_*$ is unique)

$\Longrightarrow$ $\|\mathcal{M}_* - \mathcal{M}_0\| = \|\mathcal{M}_* - (\mathcal{M}_*)_{\rm as}(t)\| \le C_0 |t|^{-\delta}$

$\Longrightarrow$ $\mathcal{M}_* = \mathcal{M}_0 + O(t^{-\delta})$ for $|t| \ge \max\{T_\infty, (C_0/\varepsilon_0)^{1/\delta}\}$

Then $(\mathcal{M}_*)_{\rm as}(t) = \mathcal{M}_0$ implies

$$\begin{aligned}(y_{\rm as}(\mathcal{M}_*, t), (B_\phi)_{\rm as}(\mathcal{M}_*, t)) &= (y_{\rm as}(\mathcal{M}_0 + O(t^{-\delta}), t), (B_\phi)_{\rm as}(\mathcal{M}_0 + O(t^{-\delta}), t)) \\ &= (y_{\rm as}(\mathcal{M}_0, t + O(t^{-\delta})), (B_\phi)_{\rm as}(\mathcal{M}_0, t + O(t^{-\delta}))),\end{aligned}$$

which realises isomonodromy deformation with the invariant monodromy data $\mathcal{M}_0$, and then $y^*_{\rm as} = (d/dt)y_{\rm as}$

Uniqueness: suppose that

$$\mathcal{M}_0 = (\mathcal{M}_{*1})_{\rm as}(t) = (\mathcal{M}_{*2})_{\rm as}(t), \quad y_1 = y_{\rm as}(\mathcal{M}_{*1}, t), \quad y_2 = y_{\rm as}(\mathcal{M}_{*2}, t),$$

Let $Y_1(\lambda)$ $Y_2(\lambda)$: solutions solving (LE) with $\mathcal{B}(t, \lambda)$ containing (y_1, y_1'), (y_2, y_2'). Then $Y_1(\lambda)^{-1}Y_2(\lambda) \equiv \Gamma_0$, implying $(dY_1/d\lambda)Y_1^{-1} = (dY_2/d\lambda)Y_2^{-1}$, and hence $y_1 = y_2$

$$\mathrm{P_{III}}(D_6): \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{4}{x}(\theta_0 y^2 + 1 - \theta_\infty) + 4y^3 - \frac{4}{y}$$

$dU/d\lambda = \mathcal{A}(\lambda, x)U \quad \lambda = 0,\ \infty$: irregular singular points

$$U_k^\infty(\lambda) = (I + O(\lambda^{-1}))\exp(\tfrac{1}{2}ix\lambda\sigma_3)\lambda^{-\frac{1}{2}\theta_\infty\sigma_3} \quad |\arg(\lambda x) - k\pi| < \pi \quad (\lambda \to \infty)$$
$$U_k^0(\lambda) = \Delta_0(I + O(\lambda))\exp(-\tfrac{1}{2}ix\lambda^{-1}\sigma_3)\lambda^{\frac{1}{2}\theta_0\sigma_3} \quad |\arg(\lambda/x) - k\pi| < \pi \quad (\lambda \to 0)$$
$$U_0^\infty(\lambda) = U_0^0(\lambda)G, \quad U_1^\infty(\lambda) = U_1^0(\lambda)\hat{G}$$

$$w(A,\lambda)^2 = \lambda^4 - A\lambda^2 + 1$$

$$\mathrm{Im}\, e^{i\phi}\int_{\mathbf{a}} \frac{w(A_\phi,\lambda)}{\lambda^2}d\lambda = \mathrm{Im}\, e^{i\phi}\int_{\mathbf{b}} \frac{w(A_\phi,\lambda)}{\lambda^2}d\lambda = 0 \quad \text{Boutroux equations}$$

$0 < \phi < \pi/2$

$$y(x)^{-1} = i\lambda_1 \mathrm{sn}(2i\lambda_2(x - x_0^+) + O(x^{-\delta}); \lambda_1/\lambda_2)$$
$$\lambda_1 = \sqrt{A_\phi/2 - \sqrt{A_\phi^2/4 - 1}}, \quad \lambda_2 = \sqrt{A_\phi/2 + \sqrt{A_\phi^2/4 - 1}},$$
$$2ix_0^+ \equiv \frac{1}{2\pi i}\Big(\Omega_{\mathbf{a}}\log(g_{11}g_{22}) + \Omega_{\mathbf{b}}\log\frac{g_{12}\hat{g}_{21}}{g_{22}\hat{g}_{11}}\Big) - \frac{\Omega_{\mathbf{a}}}{4}(\theta_0 - \theta_\infty + 2) - \frac{\Omega_{\mathbf{b}}}{2} \quad \mathrm{mod}\ \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}$$

$$\mathrm{P_V}: \quad y'' = \Big(\frac{1}{2y}+\frac{1}{y-1}\Big)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{2x^2}\Big(a^2y-\frac{b^2}{y}\Big) + c\frac{y}{x} - \frac{y(y+1)}{2(y-1)}$$

$a=\frac{1}{2}(\theta_0-\theta_1+\theta_\infty)$, $b=\frac{1}{2}(\theta_0-\theta_1-\theta_\infty)$, $c=1-\theta_0-\theta_1$

$d\Xi/d\xi = \mathcal{A}(\xi,x)\Xi$, $\quad \xi=0,1$: regular, $\xi=\infty$: irregular singular points

$\Xi_2^\infty(\xi) = (I+O(\xi^{-1}))\exp(\frac{1}{2}(x\xi-\theta_\infty\log\xi)\sigma_3)$ in $|\arg(\xi x)-\pi/2|<\pi$ $\quad(\xi\to\infty)$
$M^0=(m^0_{ij})$, $M^1=(m^1_{ij})$: monodromy matrices with respect to $\Xi_2(\xi)$ around $\xi=0,1$

$w(A,z)^2=(1-z^2)(A-z^2)$

$$\mathrm{Re}\, e^{i\phi}\int_{\mathbf{a}}\sqrt{\frac{A_\phi-z^2}{1-z^2}}dz = \mathrm{Re}\, e^{i\phi}\int_{\mathbf{b}}\sqrt{\frac{A_\phi-z^2}{1-z^2}}dz = 0 \qquad \text{Boutroux equations}$$

$$\Omega_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}}\frac{dz}{w(A_\phi,z)}$$

Let $0<|\phi|<\pi/2$.

$$\frac{y(x)+1}{y(x)-1} = A_\phi^{1/2}\mathrm{sn}(\tfrac{1}{2}(x-x_0)+O(x^{-\delta}); A_\phi^{1/2})$$

$$x_0 \equiv \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}}\log(m^0_{21}m^1_{12}) + \Omega_{\mathbf{a}}\log\mathfrak{m}_\phi\Big) - \Big(\frac{\Omega_{\mathbf{a}}}{2}+\Omega_{\mathbf{b}}\Big)(\theta_\infty+1) - \frac{\Omega_{\mathbf{a}}}{2}$$

$$\equiv \frac{-1}{\pi i}\Big(\Omega_{\mathbf{b}}\log(e^{\pi i\theta_\infty}m^0_{21}m^1_{12}) + \Omega_{\mathbf{a}}\log(e^{\pi i\theta_\infty/2}\mathfrak{m}_\phi)\Big) - \Omega_{\mathbf{a}} - \Omega_{\mathbf{b}} \qquad \mathrm{mod}\ 2\Omega_{\mathbf{a}}\mathbb{Z}+2\Omega_{\mathbf{b}}\mathbb{Z}$$

$\mathfrak{m}_\phi = m^0_{11}$ if $-\pi/2<\phi<0$, and $=e^{-\pi i\theta_\infty}(m^1_{11})^{-1}$ if $0<\phi<\pi/2$

• error bounds: is it possible to replace $O(x^{-\delta})$ with e.g. $O(x^{-1})$ or $O(x^{-1/2})$? explicit formula of the error term ?

• higher-order Painlevé equations, hyper-elliptic cases

• In asymptotic expressions what happens as $\phi \to 0$?

• τ-functions

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Thank you for your kind attention!