Genus two curves associated with the autonomous 4-dimensional Painlevé-type systems

Web-seminar on Painlevé Equations and related topics

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Outline

- The autonomous limit of the Painlevé-type equations
- Generic degeneration of spectral curves for 4-dimensional equations
- Families of Laurent series solutions and the Painlevé divisors
- Uniqueness (up to isomorphism) of the genus two curves in the Jacobians of our spectral curves
- Eric Rains, "Generalized Hitchin systems on rational surfaces" and generic degeneration of spectral curves

The linear equation of the Painlevé equations and autonomous limit

Consider the following system of linear equations

$$\delta \frac{\partial Y}{\partial x} = A(x,s)Y, \frac{\partial Y}{\partial t} = B(x,s)Y,$$

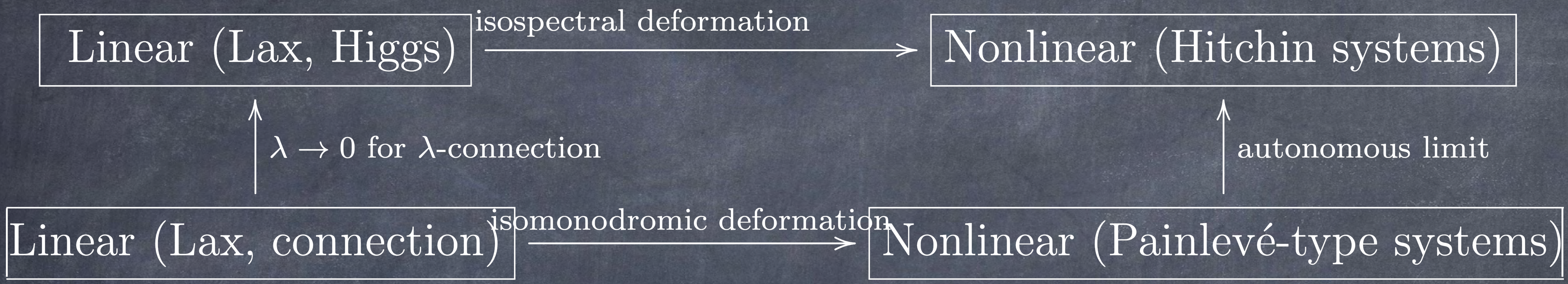
The integrability condition $\frac{\partial^2 Y}{\partial x \partial t} = \frac{\partial^2 Y}{\partial t \partial x}$ gives a nonlinear equation

$$\frac{\partial A}{\partial t} - \delta \frac{\partial B}{\partial x} + [A,B] = 0.$$

When $\delta = \frac{ds}{dt} = 1$, this gives the usual isomonodromic deformation. When $\delta = 0$, we have an isospectral deformation.

Linear and nonlinear problem

commutative



noncommutative

Example

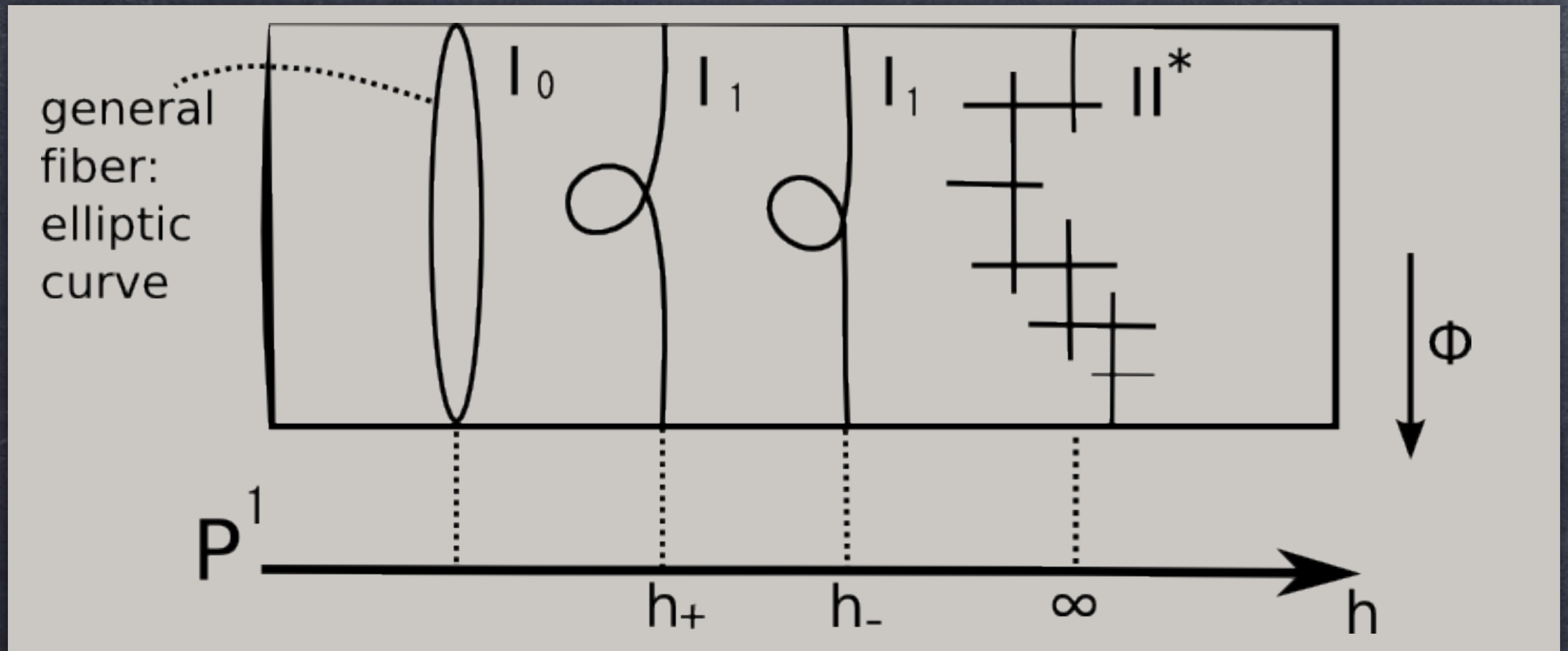
The autonomous first Painlevé equation has the following Lax pair.

$$A(x) = \begin{pmatrix} & -px^2 + qx + q^2 + s \\ x - q & p \end{pmatrix}.$$

The spectral curve is defined by $\det(yI_2 - A(x)) = 0$, which is equivalent to

$$y^2 = x^3 + sx + H_{\mathrm{I}}\,.$$

The family of spectral curves parametrized by $h = H_{\mathrm{I}} \in \mathbb{P}^1$ also defines an elliptic surface with $E_8^{(1)}$ singular fiber.



Kodaira's classification of singular fibers and Tate's algorithm

We can compute the type of singular fiber of an elliptic surface from the orders of discriminant and the j-invariant.

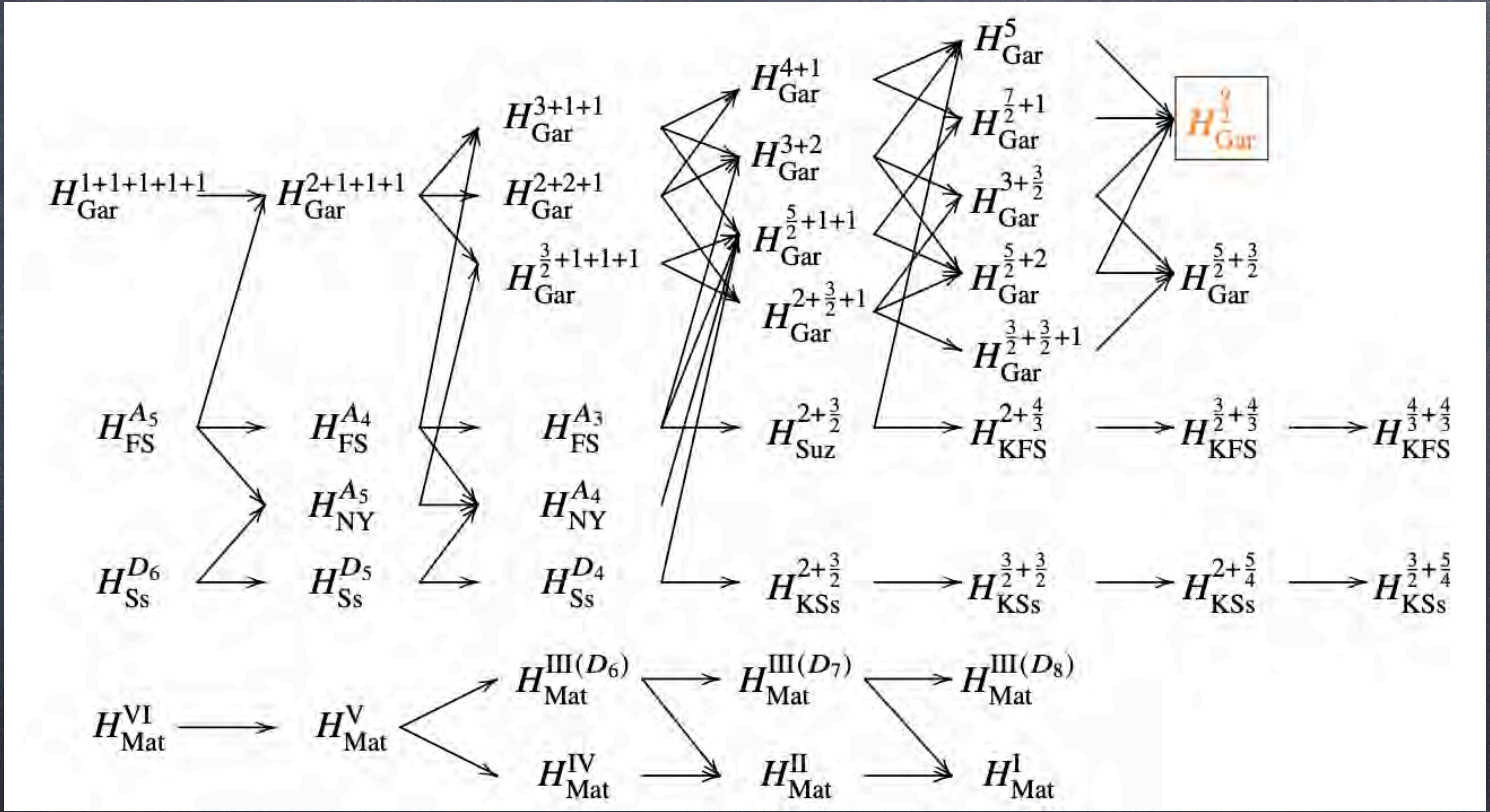
$$y^2 = x^3 + ax + b \qquad \Delta = 4a^3 + 27b^2, j = \frac{4a^3}{\Delta}.$$

For an affine equation $\tilde{y}^2 = \tilde{x}^3 + s\tilde{h}^4\tilde{x} + \tilde{h}^5$ around $h = \infty$ obtained by $h = 1/\tilde{h}$, $y = \tilde{y}/\tilde{h}^3, x = \tilde{x}/\tilde{h}^2$, Δ and j are

$$\Delta = 4(s\tilde{h}^4)^3 + 27(\tilde{h}^5)^2 = \tilde{h}^{10}(4s^3\tilde{h}^2 + 27), j = \frac{4(s\tilde{h}^4)^3}{\Delta} = \frac{4s^4\tilde{h}^2}{4s^3\tilde{h}^2 + 27}.$$

Kodaira	Dynkin	ord(Δ)	ord(j)	Kodaira	Dynkin	ord(Δ)	ord(j)
I_0	-	0	≥ 0	I_0^*	$D_4^{(1)}$	6	≥ 0
I_m	$A_{m-1}^{(1)}$	m	$-m$	I_m^*	$D_{4+m}^{(1)}$	$6+m$	$-m$
II	-	2	≥ 0	IV^*	$E_6^{(1)}$	8	≥ 0
III	$A_1^{(1)}$	3	≥ 0	III^*	$E_7^{(1)}$	9	≥ 0
IV	$A_2^{(1)}$	4	≥ 0	II^*	$E_8^{(1)}$	10	≥ 0

Degeneration scheme of the 4-dimensional Painlevé-type equations (Kimura, Kawamuko, Sakai, Sakai-Kawakami-N., Kawakami)



Example

The autonomous Garnier system of type $\dfrac{9}{2}$ is given by the Hamiltonian

$$H^{\frac{9}{2}}_{\mathrm{Gar},s_1} = p_1q_2^2 - p_1s_1 + p_2s_2 + p_1^4 + 3p_2p_1^2 + p_2^2 - 2q_1q_2,$$

$$H^{\frac{9}{2}}_{\mathrm{Gar},s_2} = p_1^2q_2^2 - 2p_1q_1q_2 + p_2q_2^2 + p_1^3s_2 + p_1s_2^2 + p_2p_1s_2 + p_2s_1 - p_2p_1^3 - 2p_2^2p$$
$$-q_2^2s_2 + q_1^2,$$

A regular level set (= the Liouville torus)

$$\cap_{i=1}\left\{H_i(q_1,p_1,q_2,p_2) = h_i\right\}$$

is an affine part of an abelian surface.

Example

The autonomous Garnier system of type $\frac{9}{2}$ has a Lax pair with

$$A(x) \;= A_0x^3 + A_1x^2 + A_2x + A_3,$$

and its spectral curve $\det(yI - A(x)) = 0$ is

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2\,.$$

This is a genus 2 hyperelliptic curve.

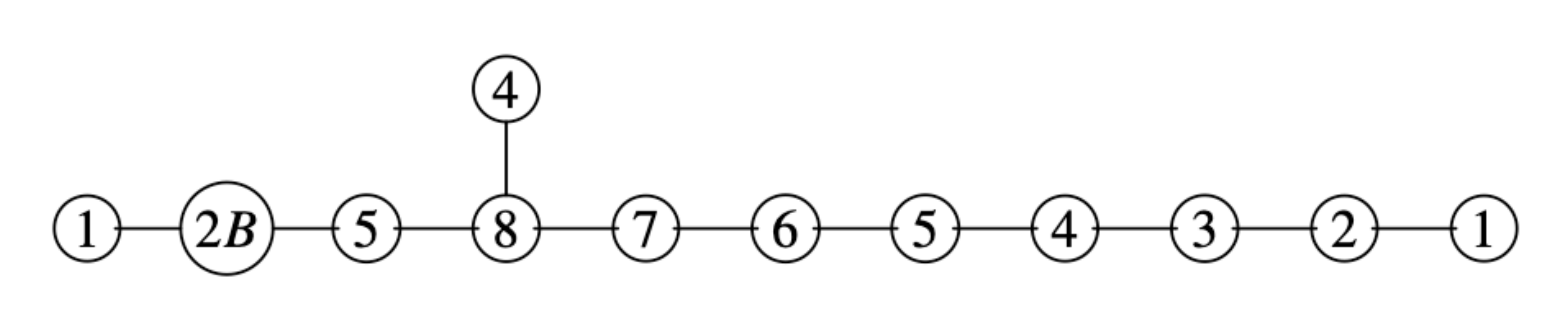
Comparison of genus 1 and 2 curves

genus	1	2
curve	elliptic curve	hyperelliptic curve
Jacobian	elliptic curve	abelian surface
possible types of singular fibers	Kodaira	Namikawa-Ueno
algorithm	Tate's algorithm	Liu's algorithm
normal form	$y^2 = x^3 + ax + b$	$y^2 = a_1 x^5 + \cdots + a_5$

Generic degeneration of the spectral curves

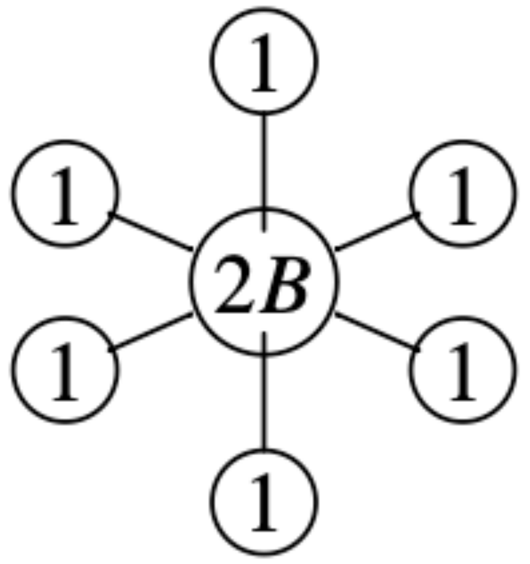
Our genus 2 spectral curves are parametrized by h_1, h_2. We shall consider the degeneration along a line $h_2 = ah_1 + b$, where a, b are generic constant in the base space.

Performing the genus counterpart of Tate's algorithm, we find Namikawa-Ueno's VII^* as the type of special fiber at $h_1 = \infty$.

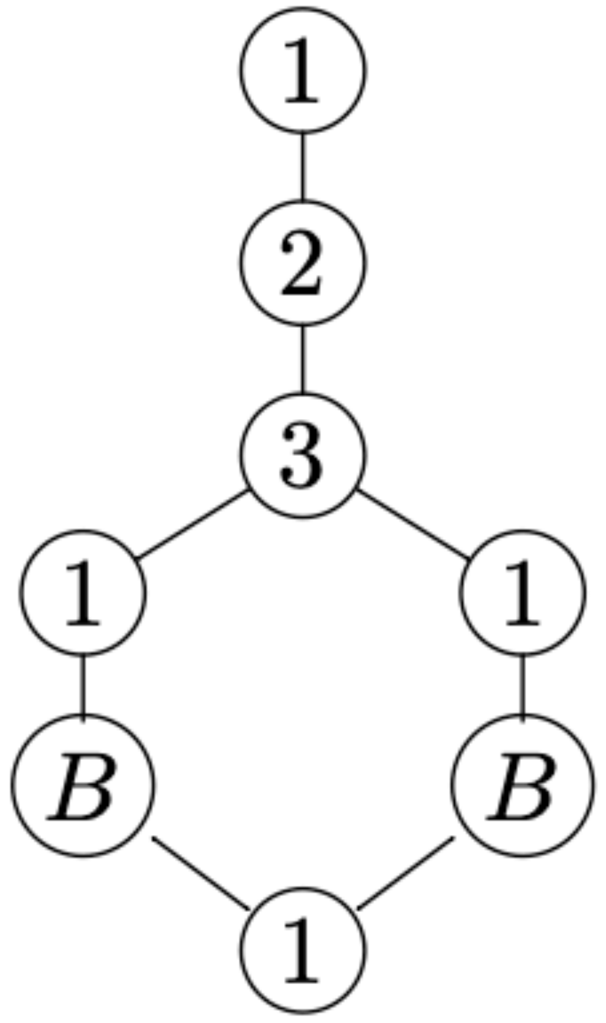


Other examples

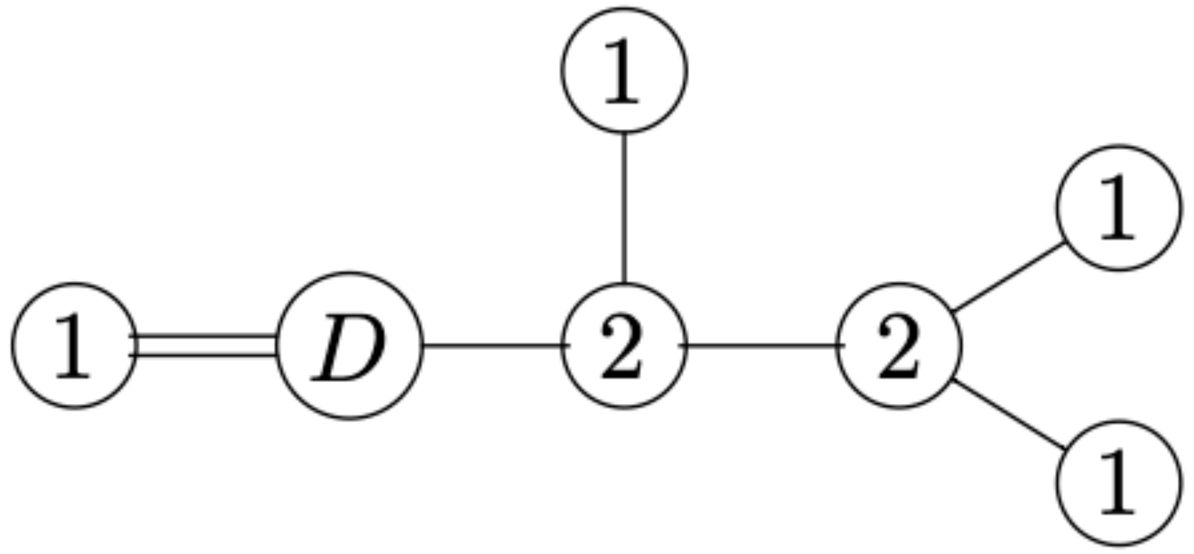
$\mathrm{I}^*_{0-0-0}: H^{1+1+1+1+1}_{\mathrm{Gar}}$



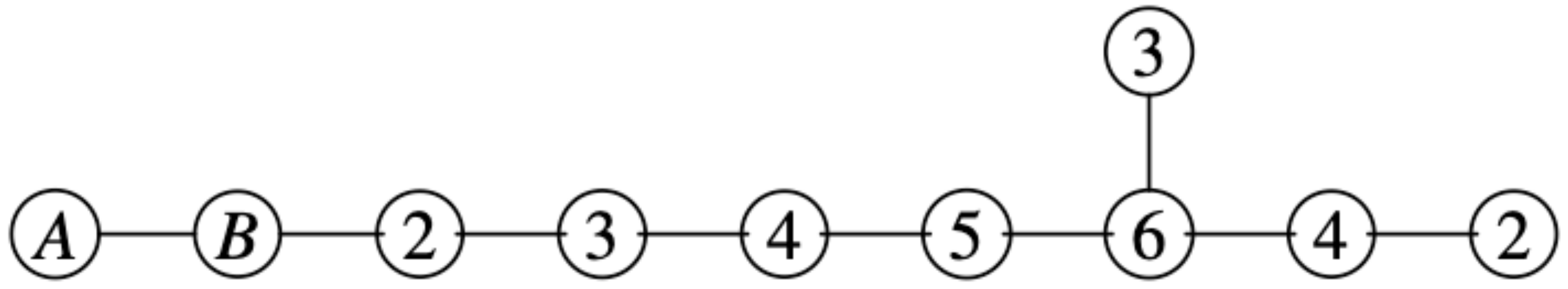
$\mathrm{IV}^* - \mathrm{II}_3: H^{A_4}_{\mathrm{NY}}$



$\mathrm{I}_2 - \mathrm{I}^*_1 - 0: H^{D_5}_{\mathrm{Ss}}$



$\mathrm{I}_0 - \mathrm{II}^* - 1: H^{\mathrm{Mat}}_{\mathrm{I}}$



Weight homogeneous vector field and the Laurent series solutions

Definition

A polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is a weight homogeneous polynomial of degree k if

$$f(t^{\nu_1}x_1, \cdots, t^{\nu_n}x_n) = t^k f(x_1, \cdots, x_n)\,.$$

A polynomial vector field

$$\dot{x}_i = f_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n)$$

is weight homogeneous if $f_1, \cdots, f_n$ are weight homogeneous with weight $\nu_i + 1 \ (i = 1, \cdots, n)$ respectively.

Theorem(Kowalevskaya, Yoshida, Adler-van Moerbeke)

Let $\dot{x}_i = f_i(x_1, \ldots, x_n)$, $(i = 1, \ldots, n)$ be a weight homogeneous vector field and suppose

$$x_i(t) = \frac{1}{t^{\nu_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k \ (i = 1, \cdots, n)$$

be its formal series solution. Then the leading coefficients $x_i^{(0)}$ satisfy

$$\nu_i x_i^{(0)} + f_i(x_1^{(0)}, \ldots, x_n^{(0)}) = 0 \ (i = 1, \cdots, n).$$

The subsequent terms $x_i^{(k)}$ for $k \geq 1$ satisfy

$$(kI_n - \mathscr{K}(x^{(0)}))x^{(k)} = R^{(k)}$$

where $R^{(k)}$ depends only on $x_1^{(l)}, \cdots, x_n^{(l)}$ with $0 \leq l < k$ and

$$\mathscr{K}_{i,j} = \frac{\partial f_i}{\partial x_j} + \nu_i \delta_{i,j}.$$

Example: The 2-dimensional first Painlevé equation

Let us consider the autonomous H_{I} given by the Hamiltonian

$$H_{\mathrm{I}}(q,p) = p^2 - q^3 - sq\,.$$

The Hamiltonian system is thus

$$\dot{q} = 2p, \quad \dot{p} = 3q^2 + s\,.$$

This is a weight-homogeneous system with the weights

$\deg(q,p)$	$\deg(H_1, s)$
$(2,3)$	$(6,4)$

We assume the following form of formal solutions

$$q(t) = \sum_{k=0}^{\infty} x_1^{(k)} t^{-2+k}, \quad p(t) = \sum_{k=0}^{\infty} x_2^{(k)} t^{-3+k}\,.$$

We will solve for the coefficient of the formal solution

$$q(t) = \sum_{k=0}^{\infty} x_1^{(k)} t^{-2+k}, \quad p(t) = \sum_{k=0}^{\infty} x_2^{(k)} t^{-3+k}$$

for

$$\dot{q} = 2p, \quad \dot{p} = 3q^2 + s.$$

The initial terms have to satisfy the following nonlinear equations

$$2x_1^{(0)} + 2x_2^{(0)} = 0,\ 3x_2^{(0)} + 3\left(x_1^{(0)}\right)^2 = 0.$$

These indicial equations have two solutions

$$\left(x_1^{(0)}, x_2^{(0)}\right) = (0,0) = m_1,\ (1,-1) = m_2\,.$$

The subsequent terms can be computed by solving linear equations

$$\left(kI_2 - \mathscr{K}(x^{(0)})\right)\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} = \begin{pmatrix} R_1^{(k)} \\ R_2^{(k)} \end{pmatrix},$$

where each $R_i^{(k)}$ is a polynomial which depends on the variables $x_1^{(l)}, x_2^{(l)}$ with $1 \leq l \leq k-1$.Also, the Kowalevskatya matrix $\mathscr{K}$ is

$$\mathcal{K} = \begin{pmatrix} \frac{\partial f_1}{\partial q} & \frac{\partial f_1}{\partial p} \\ \frac{\partial f_2}{\partial q} & \frac{\partial f_2}{\partial p} \end{pmatrix} + \begin{pmatrix} \nu_1 & \\ & \nu_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6q & 0 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}.$$

$f_1 = 2p, f_2 = 3q^2 + s$

When $(x_1^{(0)}, x_2^{(0)}) = (0,0) = m_1, \mathscr{K}(m_1) = \begin{pmatrix} 2 & 2 \\ 6x_1^{(0)} & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}$.

For $k \neq 2,3$ (the eigenvalues of $\mathscr{K}(m_1)$), $\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix}$ is uniquely determined by the previous terms from $\left(kI_2 - \mathscr{K}(m_1)\right) \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} = \begin{pmatrix} R_1^{(k)} \\ R_2^{(k)} \end{pmatrix}$.

We obtain a family of Laurent series solutions with free parameters α, β.

$$q(t; m_1) = \alpha + \beta t + t^2 \left(3\alpha^2 + s\right) + 2\alpha\beta t^3 + t^4 \left(3\alpha^3 + \frac{\beta^2}{2} + \alpha s \right) + O\left(t^5\right),$$

$$p(t; m_1) = \frac{\beta}{2} + t\left(3\alpha^2 + s\right) + 3\alpha\beta t^2 + t^3 \left(6\alpha^3 + \beta^2 + 2\alpha s\right) + O\left(t^4\right).$$

When $(x_1^{(0)}, x_2^{(0)}) = (1, -1) = m_2$, the Kowalevskaya matrix is now

$$\mathscr{K}(m_1) = \begin{pmatrix} 2 & 2 \\ 6x_1^{(0)} & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}. \sim \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$$

There terms $x^{(k)}$ for $k \neq 6$ (the only non-negative eigenvalue of $\mathscr{K}(m_2)$) are uniquely determined by

$$\left(kI_2 - \mathscr{K}(m_2)\right) \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} = \begin{pmatrix} R_1^{(k)} \\ R_2^{(k)} \end{pmatrix}.$$

Therefore, we obtain the following family of the Laurent series solution with a parameter γ.

$$q(t; m_2) = \frac{1}{t^2} - \frac{s}{5}t^2 + \gamma t^4 + \frac{s^2}{75}t^6 - \frac{3s\gamma}{55}t^8 + O\left(t^{10}\right),$$

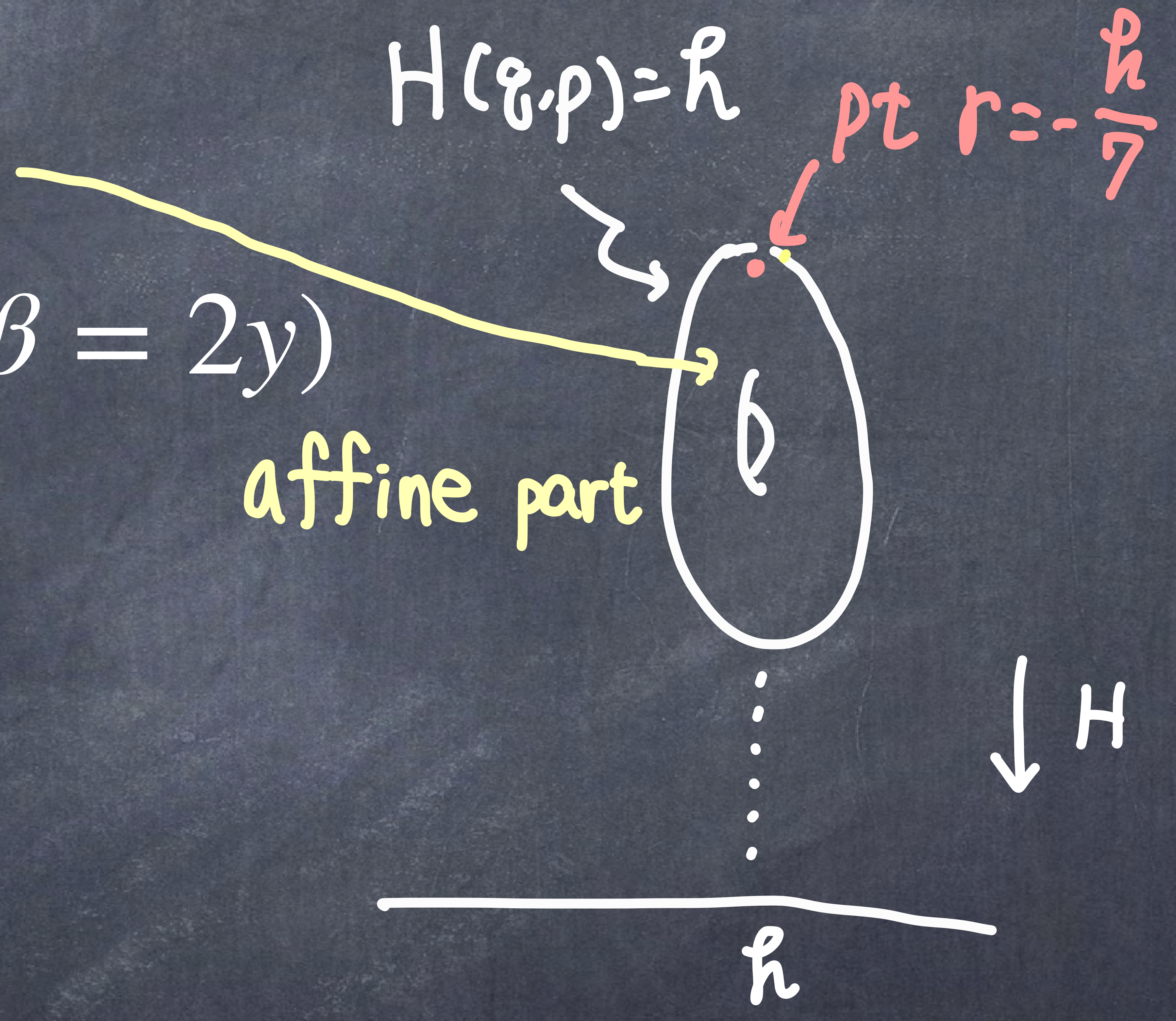
$$p(t; m_2) = -\frac{1}{t^3} - \frac{s}{5}t + 2\gamma t^3 + \frac{s^2}{25}t^5 - \frac{12s\gamma}{55}t^7 + O\left(t^9\right).$$

If we confine these Taylor/Laurent series solution to the level set $H(q,p)=h$, we have

$$H(q(t;m_1),p(t;m_2)) = -s\alpha - \alpha^3 + \frac{\beta^2}{4} = h\,.$$

$$(\leftrightarrow y^2 = x^3 + sx + h,\ \alpha = x, \beta = 2y)$$

$$H(q(t;m_2),p(t;m_2)) = -7\gamma = h\,.$$



indicial locus	K-exponents	# free para's	fiber (Liouville torus)
$m_1=(0,0)$	$(2,3)$	2	affine elliptic curve
$m_2=(1,-1)$	$(-1,6)$	1	point

The autonomous Garnier system of type 9/2 is a Hamiltonian system

$$\frac{dq_1}{dt} = 4p_1^3 + 6p_2p_1 + q_2^2 - s_1, \qquad \frac{dq_2}{dt} = 3p_1^2 + 2p_2 + s_2,$$

$$\frac{dp_1}{dt} = 2q_2, \qquad \frac{dp_2}{dt} = 2\left(q_1 - p_1q_2\right).$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s_1, s_2)$
$(5, 2, 3, 4)$	$(8, 10, 6, 4)$

There are 3 types of family of Laurent series solutions to $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$.

indicial locus	K-exponents	# free para's	fiber (Liouville torus)
$m_1 = (0,0,0,0)$	$(2,3,4,5)$	4	affine abelian surface
$m_2 = (-1,1,-1,0)$	$(-1,2,5,8)$	3	genus two curve
$m_3 = (9,3,-3,-9)$	$(-1,-3,8,10)$	2	point

The following families of Laurent series starting from $m_2 = (-1,1,-1,0)$ contains three free parameters, α, β, γ.

$$q_1(t;m_2) = -\frac{1}{t^5} + \frac{\alpha}{t^3} + \beta + t\left(-\frac{\alpha^3}{2} - \frac{9\alpha s_2}{35} + \frac{s_1}{7}\right) - \frac{15}{2}t^2(\alpha\beta) + \gamma t^3 + t^4\left(\frac{18\beta s_2}{7} - \frac{15\alpha^2\beta}{2}\right) + O\left(t^5\right),$$

$$p_q(t;m_2) = \frac{1}{t^2} + \frac{\alpha}{2} + t^2\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 4\beta t^3 + \frac{1}{28}t^4\left(-35\alpha^3 - 24\alpha s_2 + 4s_1\right) + O\left(t^5\right),$$

$$q_2(t;m_2) = -\frac{1}{t^3} + t\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 6\beta t^2 + \frac{1}{14}t^3\left(-35\alpha^3 - 24\alpha s_2 + 4s_1\right) - \frac{15}{2}t^4(\alpha\beta) + O\left(t^5\right),$$

$$p_2(t;m_2) = -\frac{3\alpha}{2t^2} + \left(\frac{3\alpha^2}{2} + s_2\right) + 6\beta t + t^2\left(\frac{9\alpha^3}{8} + \frac{9\alpha s_2}{10}\right) + \frac{3t^4\left(1925\alpha^4 + 1680\gamma - 120\alpha^2 s_2 - 400\alpha s_1 - 1008 s_2^2\right)}{12320} + O\left(t^5\right).$$

The level set of the moment map is

$$H_{s_1}(q_1(t;m_2),p_1(t;m_2),q_2(t;m_2),p_2(t;m_2)) = h_1,$$
$$H_{s_2}(q_1(t;m_2),p_1(t;m_2),q_2(t;m_2),p_2(t;m_2)) = h_2.$$

These are equivalent to the followings

$$\frac{405\alpha^4}{32} + \frac{81\gamma}{22} + \frac{648\alpha^2 s_2}{77} - \frac{150\alpha s_1}{77} - \frac{23 s_2^2}{110} = h_1,$$

$\gamma = \cdots$?

$$s_1\left(s_2 - \frac{207\alpha^2}{308}\right) + \frac{81\left(35\left(99\alpha^5 + 48\alpha\gamma + 704\beta^2\right) + 760\alpha^3 s_2 - 1008\alpha s_2^2\right)}{24640} = h_2.$$

This is equivalent to

$$-\frac{243\alpha^5}{32} + 81\beta^2 + \frac{3\alpha h_1}{2} - \frac{81\alpha^3 s_2}{8} + s_1\left(\frac{9\alpha^2}{4} + s_2\right) - 3\alpha s_2^2 = h_2.$$

By replacing $\alpha = \frac{2}{3}x, \beta = \frac{1}{9}y$, the equation reads

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

Painlevé divisor (the boundary divisor of the Liouville torus)

In this example of the autonomous Garnier system of type $\frac{9}{2}$, the spectral curve (linear) and the Painleve divisor (nonlinear) are difined by the same equation

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2 \, .$$

We will show in the following that it is not a mere coincidence.

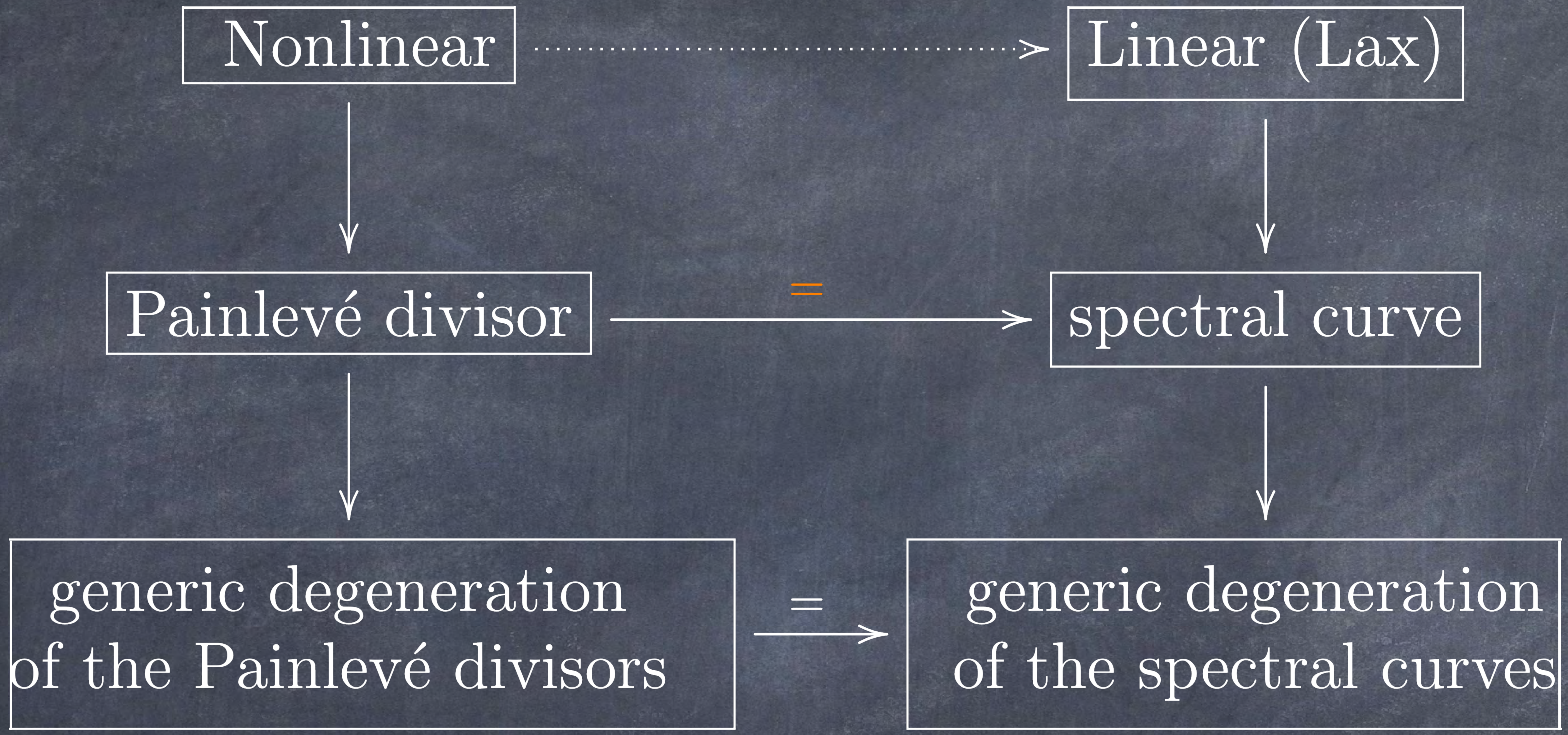
Theorem(N.-Rains)

For the 4-dimensional autonomous Painlevé-type equations, any genus 2 curve in the Jacobian of the generic spectral curve is isomorphic to the spectral curve.

Corollary

For the 4-dimensional autonomous Painlevé-type equations, any genus 2 component of the generic Painlevé divisor is isomorphic to the corresponding spectral curve. In particular, the generic degeneration of the spectral curve and the generic degeneration of any irreducible component of the Painlevé divisor are the same.

Nonlinear to linear



Sketch of the proof

- It is enough to prove the uniqueness of the principal polarization of the Jacobian $J(C)$ for our spectral curves.

(Then the classical Torelli theorem for curves assures the uniqueness of isomorphism class of curves.)

Theorem(Torelli)

Two Jacobians $(J(C), \Theta)$ and $(J(C'), \Theta')$ of smooth curves C and C' are isomorphic as polarized abelian varieties if and only if C and C' are isomorphic.

• It is enough to prove the Jacobian $A = J(C)$ has no nontrivial endomorphisms.

For a polarization $L \in NS(A) = \mathrm{Im}(c_1 : H^1(\mathcal{O}_A^*) \to H^2(A, \mathbb{Z}))$, we get

$$\phi_L : A \to \hat{A} = \mathrm{Pic}^0(A)$$

$$a \mapsto t_a^* L \otimes L^{-1}$$

where $t_a : A \to A$ is the translation by $a \in A$.

For a principal polarization $L_0 \in NS(A)$, we get an isomorphism of $NS(A)$ and the symmetric (w.r.t Rosati involution) endomorphisms.

$$NS(A) \simeq \mathrm{End}^s(A)$$

$$L \mapsto \phi_{L_0}^{-1} \phi_L$$

- It is enough to prove the triviality of the endomorphism rings for the most degenerated 6 cases.

When an equation P_A degenerates to P_B, the endomorphism ring for P_A injects in the endomorphism ring for P_B.

Under some mild assumptions for S, we have the following,

$$\mathrm{End}\left(A_{K(S)}\right) \simeq \mathrm{End}_S(A) \hookrightarrow \mathrm{End}_s(A_s)$$

where $s \in S$.

If we can prove that the endomorphism rings of the most degenerated equations are trivial, then all the other equations degenerating to one of these 6 equations also have trivial endomorphism rings.

- For the most degenerated 6 types, we can check that their endomorphism rings are trivial by direct computation.

1)Note that the Jacobian of a generic hyperelliptic curve has trivial endomorphism ring.

We can show that the family of spectral curves of a system of type $H^{\frac{9}{2}}_{\mathrm{Gar}}$, $H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar}}$, $H^{\mathrm{III}(D_8)}_{\mathrm{Mat}}$, $H^{\mathrm{I}}_{\mathrm{Mat}}$ dominates the moduli space of genus two curves, so that a typical curve in our family has no non-trivial endomorphisms.

We can use absolute Igusa invariants

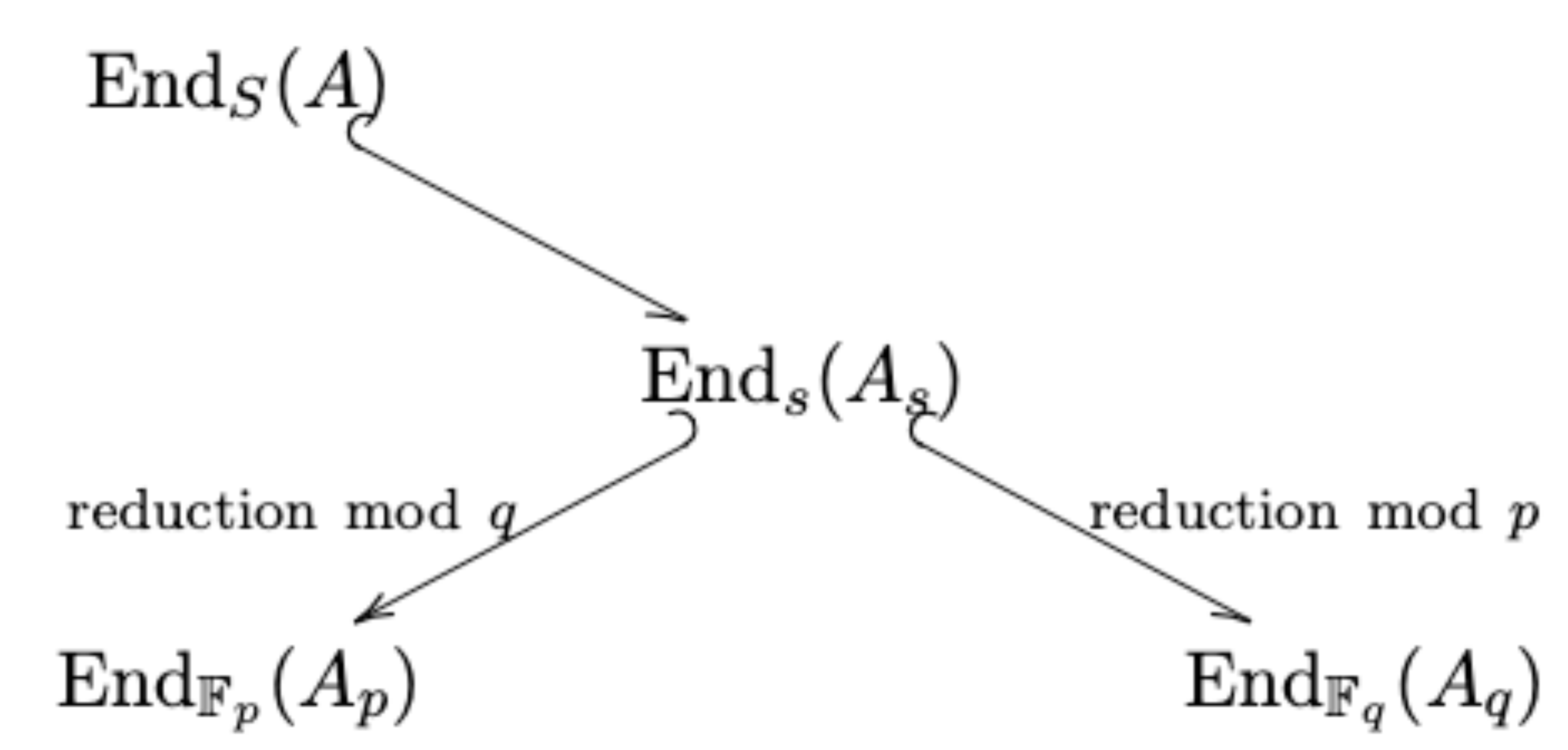
$$I_1 = \frac{J_4}{J_2^2}, I_2 = \frac{J_6}{J_2^3}, I_3 = \frac{J_{10}}{J_2^5}$$

as coordinates for the affine subset $\mathscr{M}_2 \backslash \{J_2 \neq 0\}$. Use Jacobian criterion to check that these are algebraically independent.

- For the most degenerated 6 types, we can check that their endomorphism rings are trivial by direct computation.

2)Use reduction modulo two different primes to reduce the problems to curves over finite field study the intersection.

semicontinuity of endomorphism rings



<u>Remark</u>

Approach 1) using the Igusa invariants does not work for $H_{\mathrm{K}FS}^{\frac{4}{3}+\frac{4}{3}}$ and $H_{\mathrm{K}Ss}^{\frac{3}{2}+\frac{5}{4}}$.

The endomorphism of an abelian variety A over a finite field $\mathbb{F}_p$ can be studied using the Frobenius endomorphism

$$\pi\colon A \longrightarrow A$$
$$(x_0 : \cdots : x_n) \mapsto (x_0^p : \cdots : x_n^p)\,.$$

If the characteristic polynomial of the Frobenius endomorphism has no multiple root, then $\mathrm{End}_{\mathbb{F}_p}(A) \otimes \mathbb{Q} = \mathbb{Q}[\pi]$.

For the Jacobian $A = J(C)$ the characteristic polynomial of π can be computed from the Zeta function of the curve C (Weil conjecture).

$$Z(C,s) = \exp\left(\sum_{m=1}^{\infty} \frac{\#C(\mathbb{F}_{p^m})}{m} p^{-ms}\right) = \frac{L(t)}{(1-t)(1-pt)},$$

for $t = p^{-s}$, and $P(t) = t^{2g} L\left(\frac{1}{t}\right)$ is the characteristic polynomial.

The spectral curve for $H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$ is
$y^2 = x^6 - 2x^5 + (2h_1 + 1)x^4 + 2(h_2 - h_1)x^3 + (h_1^2 - 2h_2)x^2 + 2h_1h_2x + h_2^2 - 4s\,.$
Consider an instance $h_1 = 12$, $h_2 = 17$, $s = 29$ and reduce this curve modulo $p = 37$.

$$C_1 : y^2 = x^6 + 35x^5 + 25x^4 + 10x^3 + 36x^2 + x + 25.$$

We can compute using Magma that

$$N_1 = \#C_1\left(\mathbb{F}_p\right) = 36, \quad N_2 = \#C_1\left(\mathbb{F}_{p^2}\right) = 1442.$$

The zeta function of this hyperelliptic curve is

$$Z_{C_1}(t) = \frac{37^2t^4 - 37 \cdot 2t^3 + 38t^2 - 2t + 1}{(1-t)(1-37t)} = \frac{L_1(t)}{(1-t)(1-37t)}\,.$$

The characteristic polynomial of Frobenius is

$$P_1(t) = t^4 L_1\left(\frac{1}{t}\right) = t^4 - 2t^3 + 38t^2 - 37 \cdot 2t + 37^2\,.$$

We have,

$$\mathrm{End}_{\mathbb{F}_p}(J(C_1)) \otimes \mathbb{Q} \simeq \mathbb{Q}(\alpha) = \mathbb{Q}[t]/P_1(t),$$

where α is a root of the characteristic polynomial

$$P_1(t) = t^4 - 2t^3 + 38t^2 - 37 \cdot 2t + 37^2$$
$$= \left(t^2 - (1+\sqrt{37})t + 37\right)\left(t^2 - (1-\sqrt{37})t + 37\right).$$

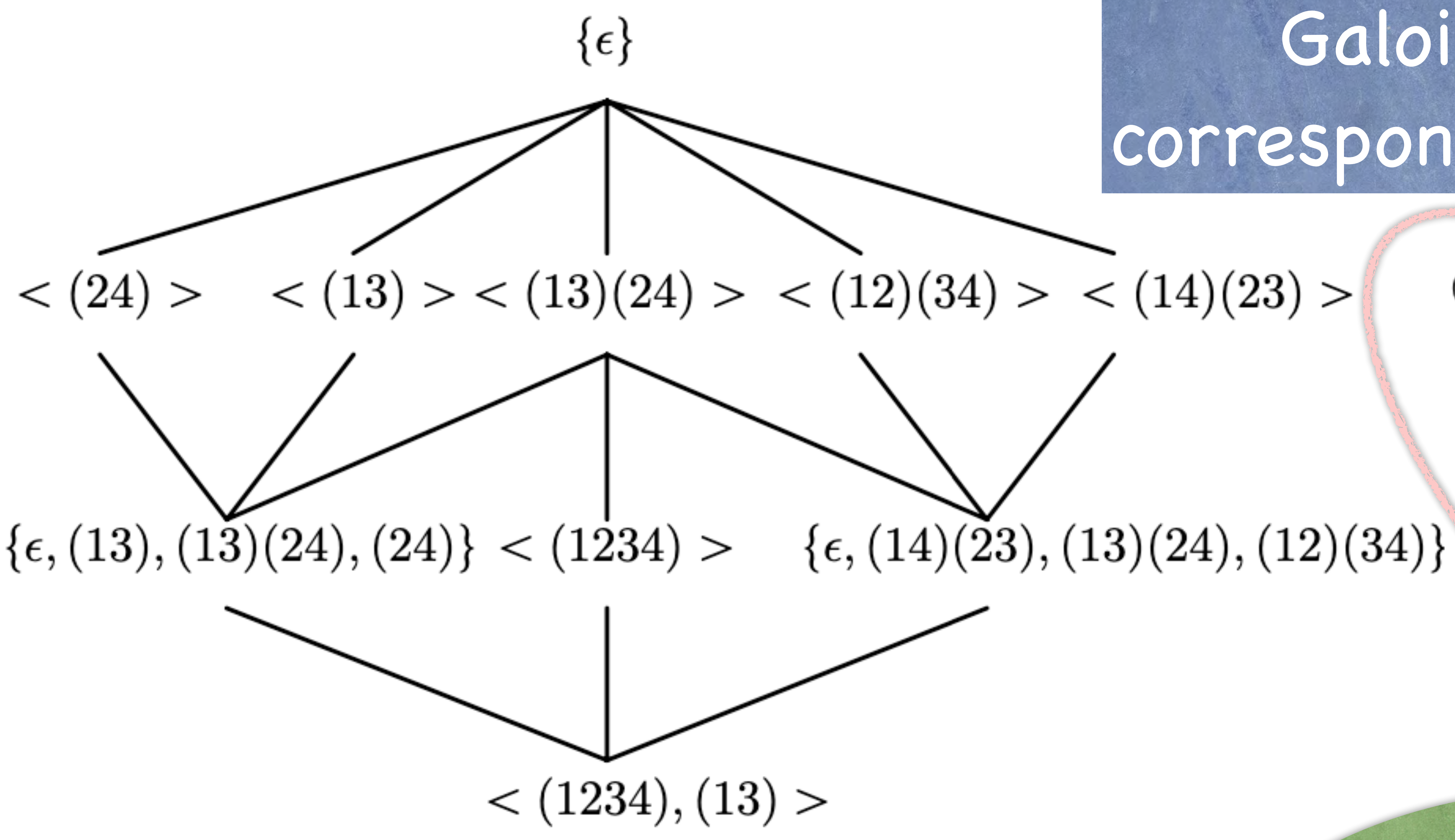
Note that $\mathbb{Q}(\alpha)$ contains the unique degree 2 subfield over $\mathbb{Q}$, since the Galois group of $P_1(t)$ is $D_4 = \langle \sigma = (1234), \tau = (13) \rangle$.

∵ • $P_1(t)$ is irreducible over $\mathbb{Q}$ (⇒The Galois group is transitive.)

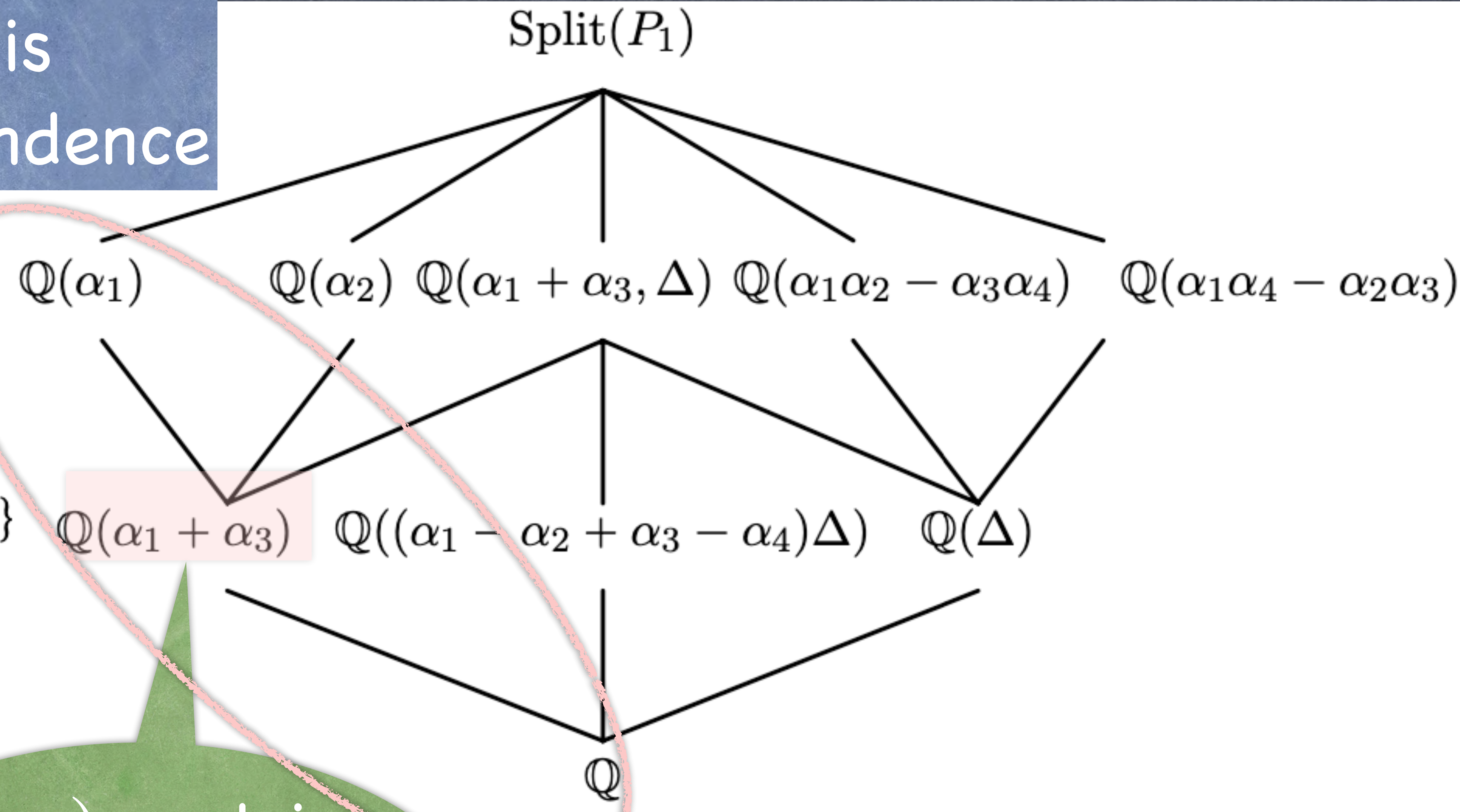
• The Galois group contains a transposition (⇒not a subgroup of A_4)

• The Galois group does not contain the whole group S_4

$$P_1(t) = = \left(t^2 - (1+\sqrt{37})t + 37\right)\left(t^2 - (1-\sqrt{37})t + 37\right).$$



Galois correspondence



The Galois group of $P_1(t)$ and its subgroups

$\mathbb{Q}(\alpha_1)$ contains the unique subfield of degree 2 over $\mathbb{Q}$

Subfields of the splitting field of $P_1(t)$

Consider the same curve over $\mathbb{Q}$, but reduce modulo a different prime $q = 53$ to obtain

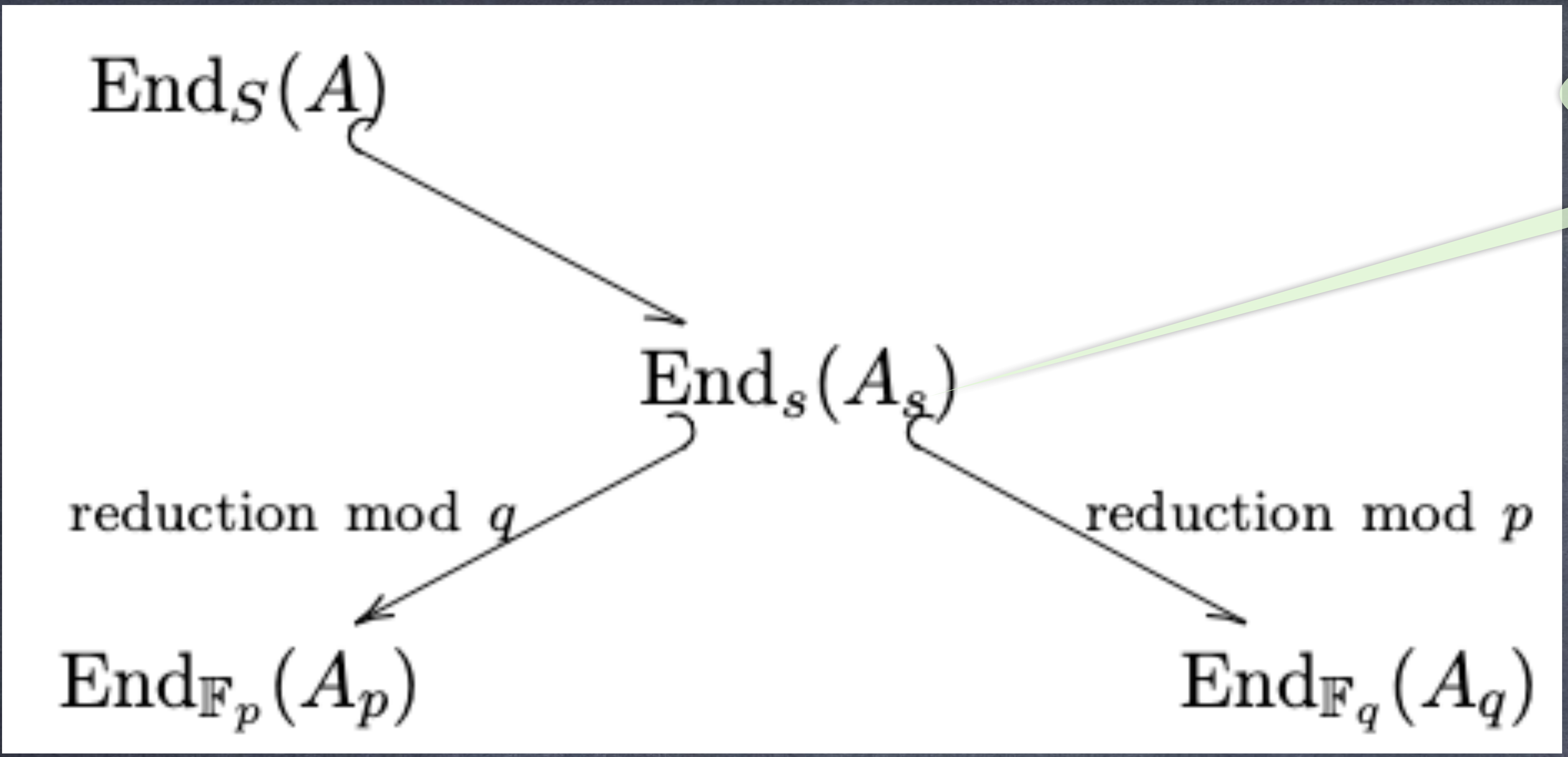
$$C_2 \colon y^2 = x^6 + 51x^5 + 25x^4 + 10x^3 + 4x^2 + 37x + 14.$$

We find that $\#C_2\left(\mathbb{F}_q\right) = 57$, $\#C_2\left(\mathbb{F}_{q^2}\right) = 3001$. The zeta function of this hyperelliptic curve is

$$Z_{C_2}(t) = \frac{53^2t^4 + 53 \cdot 3t^3 + 100t^2 + 3t + 1}{(1-t)(1-53t)}.$$

The characteristic polynomial of Frobenius is

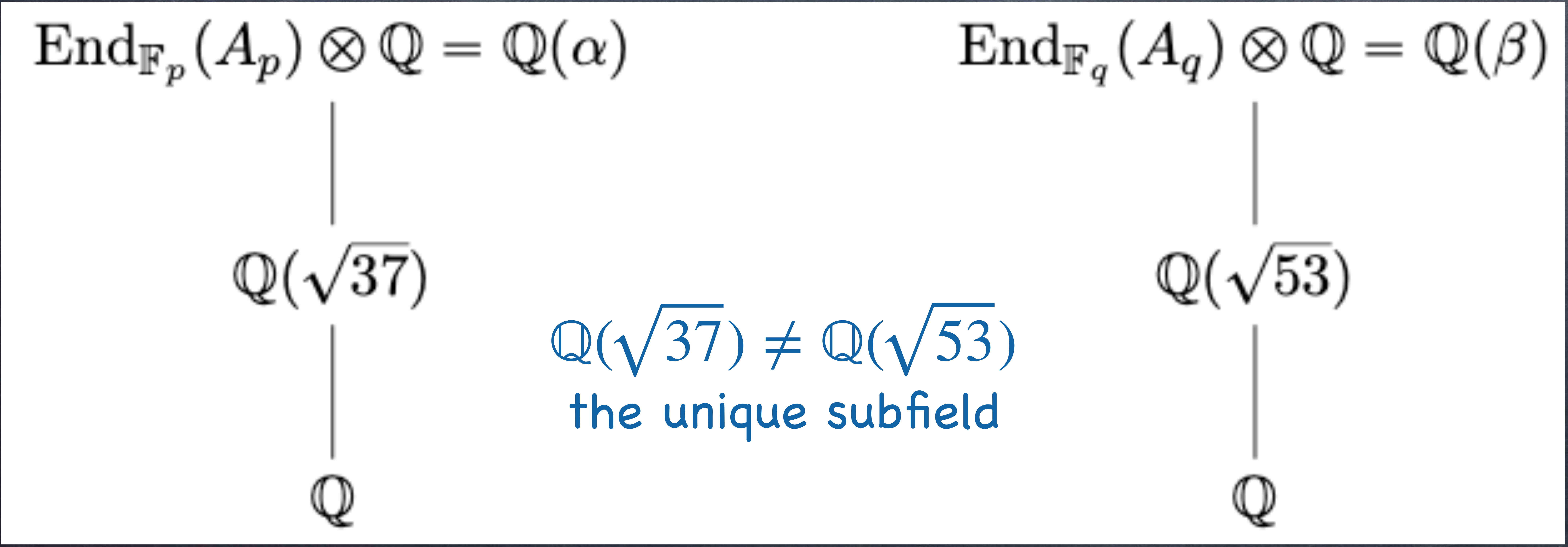
$$\begin{aligned} P_2(t) &= t^4 + 3t^3 + 100t^2 + 53 \cdot 3t + 53^2 \\ &= \left(t^2 + \frac{3+\sqrt{33}}{2}t + 53\right)\left(t^2 + \frac{3-\sqrt{33}}{2}t + 53\right). \end{aligned}$$



$(h_1, h_2, s) = (12,17,29)$

$\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$

$\therefore \mathrm{End}_s(A_s) = \mathbb{Z}$



Néron-Severi group (polarization)

symmetric endomorphisms

$$\mathrm{NS}\left(A_{K(S)}\right) \xrightarrow{\sim} \mathrm{End}^{\mathrm{sym}}_{K(S)}\left(A_{K(S)}\right)$$

$$\downarrow$$

specialize to a point $s \in S$

$$\mathrm{End}_{K(S)}\left(A_{K(S)}\right) \xleftarrow{\sim} \mathrm{End}_S(A) \hookrightarrow \mathrm{End}_s(A_s)$$

endomorphism ring of the generic fiber

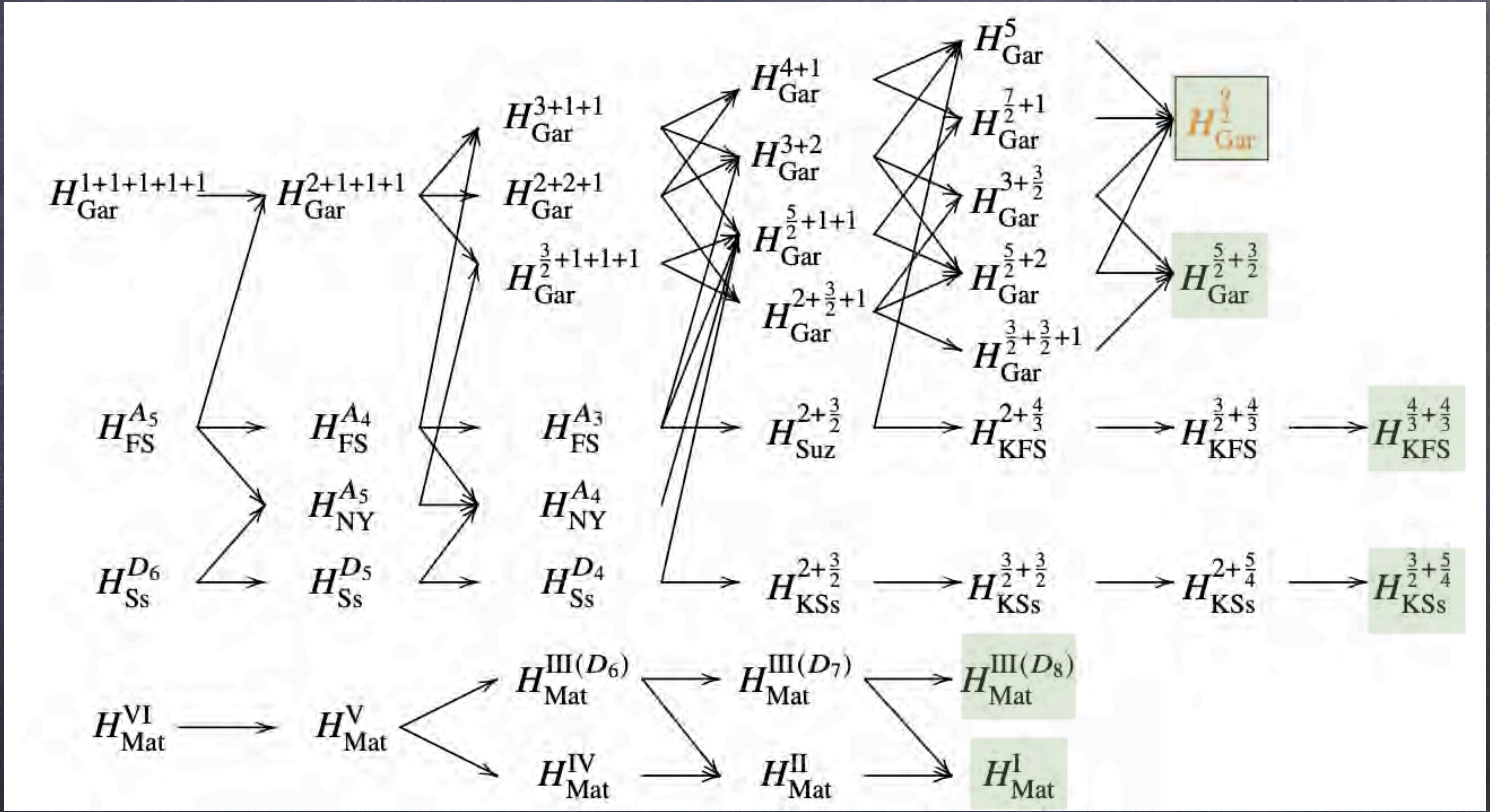
endomorphism on the generic fiber can be extended to the family

reduction mod p $\downarrow$

$$\mathrm{End}_{\mathbb{F}_p}(A_p)$$

$\mathrm{End}_{\mathbb{F}_p}(A_p) \otimes \mathbb{Q}$ is generated by the Frobenius (non-trivial, yet computable by counting points)

Degeneration scheme of the 4-dimensional Painlevé-type equations
(Kimura, Kawamuko, Sakai, Sakai-Kawakami-N., Kawakami)



Eric Rains, Generalized Hitchin systems on rational surfaces
arXiv:1307.4033, p.65

6 Moduli of sheaves on surfaces

Let (X, C_α) be an anticanonical rational surface. Say a coherent sheaf on X has *integral support* if its 0-th Fitting scheme is an integral curve on X, and it contains no 0-dimensional subsheaf.

Theorem 6.1. *Let (X, C_α) be an anticanonical rational surface over an algebraically closed field of characteristic p, let D be a divisor class with generic representative an integral curve disjoint from C_α, and let r be the largest integer such that $D \in r\operatorname{Pic}(X)$. Then the moduli problem of classifying sheaves M on X with integral support, $c_1(M) = D$, $\chi(M) = x$, and $M|_{C_\alpha} = 0$ is represented by a quasiprojective variety $\mathcal{I}rr_X(D, x)$ of dimension $D^2 + 2$, with a symplectic structure induced by any choice of nonzero holomorphic differential on C_α. Moreover, $\mathcal{I}rr_X(D, x)$ is unirational if the generic representative of D has no cusp, separably unirational if $p = 0$ or $\gcd(x, r, p) = 1$, and rational if $x \bmod r \in \{1, r-1\}$. Finally, if $\gcd(x, r) = 1$, then there exists a universal sheaf over $\mathcal{I}rr_X(D, x)$.*

Blow up the surface (F_2 etc.) to separate D (spectral curve) from the anticanonical curve C_α.

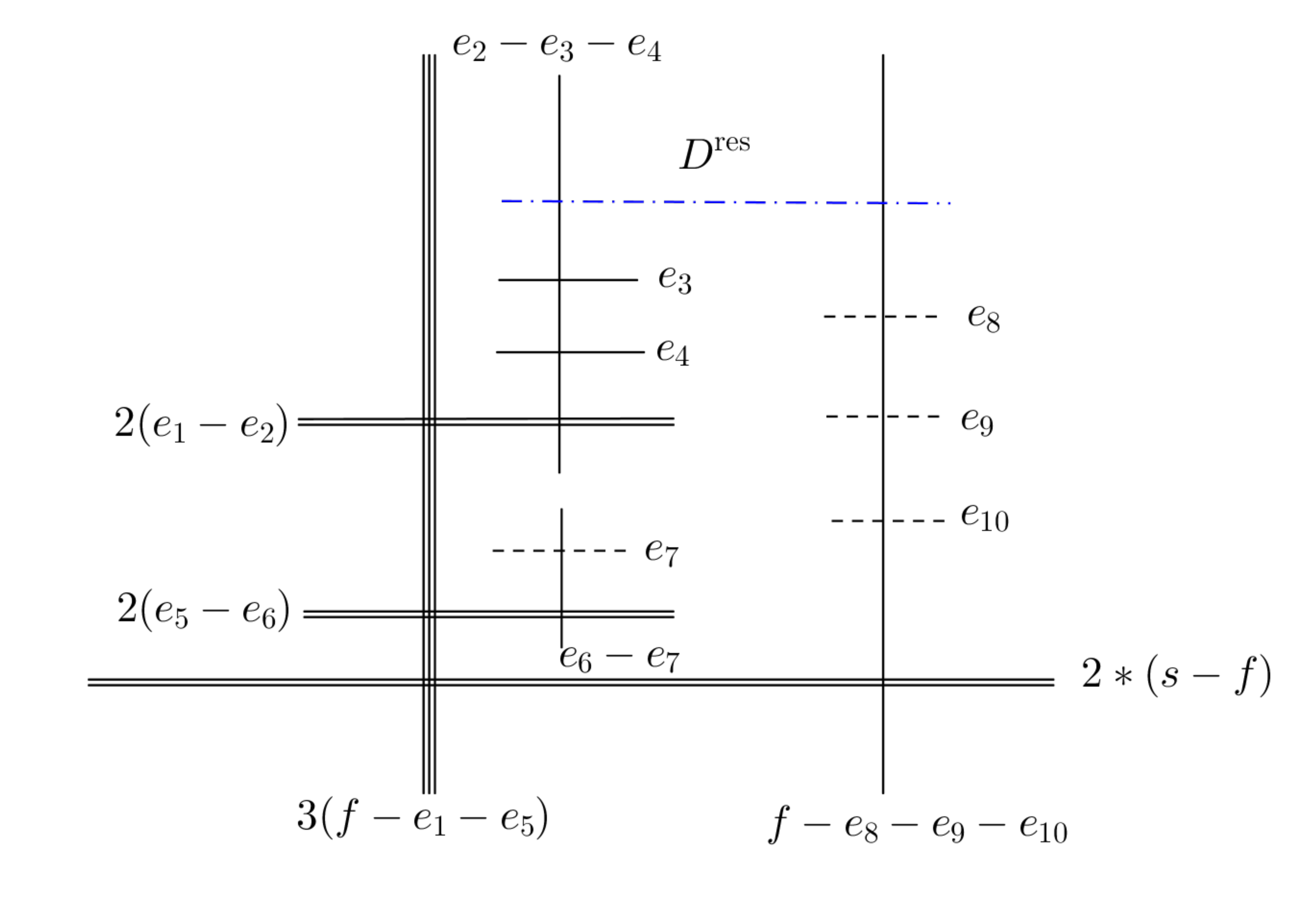
Example: The $A_4^{(1)}$-Noumi-Yamada System

nonsymmetric differece Garnier): $((11))((1)),111$

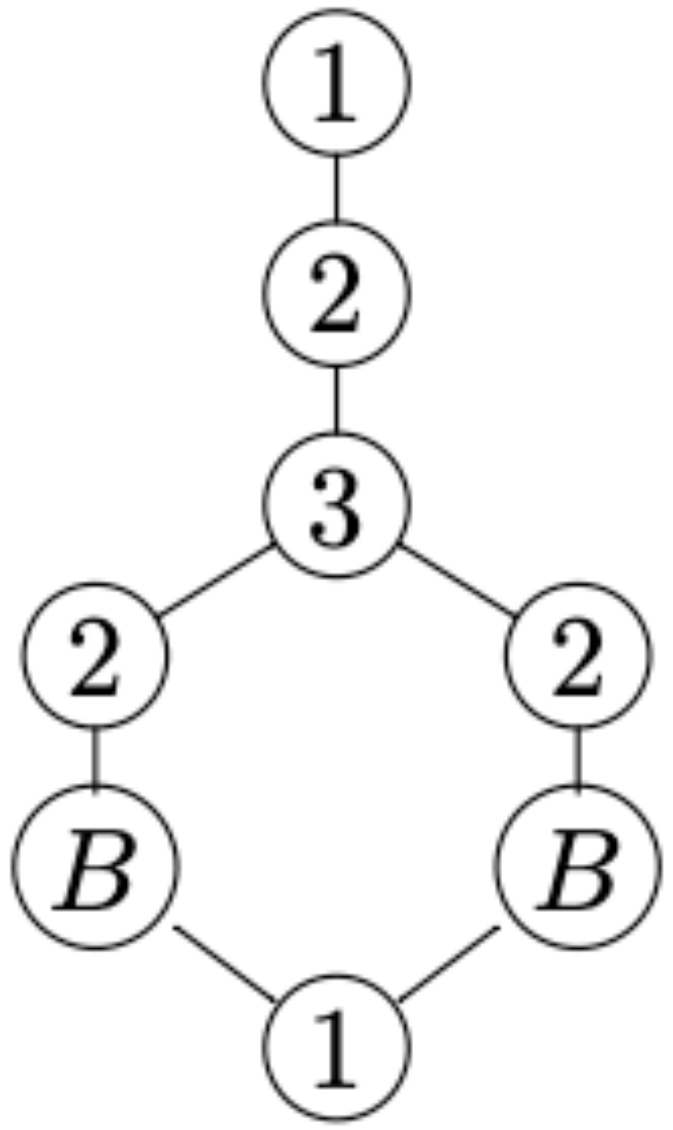
$$\begin{pmatrix} & \overbrace{}^{x=0} & & x=\infty \\ 0 & 0 & 0 & \theta_1^\infty \\ 0 & 0 & \theta_1^0 & \theta_2^\infty \\ 1 & -t & \theta_2^0 & \theta_3^\infty \end{pmatrix}$$

$$D = 3s + 3f - 2e_1 - 2e_2 - e_3 - e_4 - e_5$$
$$-e_6 - e_7 - e_8 - e_9 - e_{10}$$
$$= C_\alpha + D^{\mathrm{res}}$$

C_α :anticanonical $\quad D^{\mathrm{res}} = s + f - e_1 - e_2$



$\mathrm{IV}^* - \mathrm{II}_3 : H_{\mathrm{NY}}^{A_4}$



See you at the RIMS Review Seminar
“Generalized Hitchin Systems,
Non-commutative Geometry and Special Functions”,
with Prof. Eric Rains as a special guest.

This is a part of RIMS Research Project 2020 "Differential Geometry and Integrable Systems - Mathematics of Symmetry, Stability and Moduli -“, organized by Prof. Ohnita.

2020 May—>2021 May—>2021 November—>??

Thank you!

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