Moduli space of rank three logarithmic connections on the projective line with three poles

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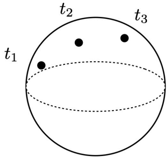
2023/11/08

Web-seminar on Painlevé Equations and related topics

Contents

Parabolic logarithmic connections



- $\boldsymbol{t} = \{t_1, t_2, t_3\} \subset \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$
- $D(\boldsymbol{t}) = t_1 + t_2 + t_3$
- $\boldsymbol{\nu} = (\nu_{i,j})^{1\leq i\leq 3}_{0\leq j\leq 2} \in \mathbb{C}^9$
- $(E, \nabla, l_* = \{l_{i,*}\}_{i=1}^3)$: a $\boldsymbol{\nu}$-parabolic logarithmic connection over $(\mathbb{P}^1, \boldsymbol{t})$
  - E : a vector bundle of rank 3 and degree d,
  - $\nabla \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$: a logarithmic connection
  - ▷ ∇ is locally written by

$$\nabla = d + \textstyle\sum_{i=1}^3 \frac{A_i}{z-t_i} dz + \text{holomorphic}, \ A_i \in M_3(\mathbb{C}).$$

  - $l_{i,*}$: a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$ such that $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j}\mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for $1 \leq i \leq 3, 0 \leq j \leq 2$.
- $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu}) := \{(E, \nabla, l_*) \mid \boldsymbol{\alpha}\text{-stable}\}/\sim$

Description of moduli space of logarithmic connections

Rank two case

	(1)	(2)
method	• apparent singularities • dual parameters	• apparent singularities • parabolic bundles
compactification	ϕ-connections	λ-connections
previous research	• Arinkin-Lysenko $(g = 0, n = 4)$ • Inaba-Iwasaki-Saito $(g = 0, n = 4)$ • Oblezin $(g = 0, n \geq 4)$ • Komyo-Saito $(g = 0, n \geq 4)$	• Loray-Saito $(g = 0)$ • Fassarella-Loray $(g = 1, n = 2)$ • Fassarella-Loray-Muniz $(g = 1)$ • M $(g \geq 1)$

We investigate the moduli space of rank three logarithmic connections on the projective line with three poles in the above two ways.

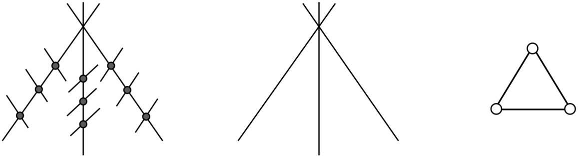
Difference Painlevé equations and moduli of connections

- **Sakai '01** characterized the spaces of initial conditions for the Painlevé equations as certain surfaces and classified them according to some affine root systems. Each such surface is the space of initial conditions for a class of discrete Painlevé equations.

- **Boalch '09** considered the difference Painelvé equations arising as symmetries of logarithmic connections on the trivial bundle.

surface type	$D_4^{(1)}$	$A_2^{(1)*}$	$A_1^{(1)*}$	$A_0^{(1)**}$
symmetry	$D_4^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
spectral type	$11, 11, 11, 11$	$111, 111, 111$	$22, 1111, 1111$	$33, 222, 111111$

$A_2^{(1)*}$-surface



$A_2^{(1)*}$-surfaces and moduli of connections

- **Arinkin and Borodin '06**

$$S(A_2^{(1)*}) \setminus -K_{S(A_2^{(1)*})} \cong \boxed{\begin{array}{l}\text{moduli of a certain type of}\\ \text{“difference connections”}\end{array}} \overset{\text{Mellin transform}}{\cong} M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$$

  for generic $\boldsymbol{\nu}$.

- **Dzhamay and Takenawa '15** gave a coordinate on a Zariski open subset of the moduli space $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ of rank 3 logarithmic connections on $\mathbb{P}^1$ with 3 poles by giving a normal form of connections over the trivial bundle.

Goal: We gives the natural compactification $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ of $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ and prove $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ is isomorphic to $S(A_2^{(1)*})$ by using the apparent singularity and its dual parameter.

Moduli of parabolic bundles and parabolic connections

Rank two case

- $(C, \boldsymbol{t})$: a n-pointed smooth irreducible projective curve over $\mathbb{C}$
- (L, ∇_L): a $\mathrm{tr}(\boldsymbol{\nu})$-connection over $(C, \boldsymbol{t})$ of rank one and degree $2g-1$
- $M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) := \{(E, \nabla, l_*) \mid \boldsymbol{\alpha}\text{-stable}, (\det E, \mathrm{tr}\nabla) \cong (L, \nabla_L)\}/\sim$
- $P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L) := \{(E, l_*) \mid \boldsymbol{\alpha}\text{-stable}, \det E \cong L\}/\sim$
- $N := \frac{1}{2}\dim M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) = \dim P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L)$

Theorem (Loray-Saito, Fassarella-Loray, Fassarella-Loray-Muniz, M)

For a suitable $\boldsymbol{\alpha}$, the rational map

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots\to \mathbb{P}^N \times P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L)$$

is birational.

Goal: We construct the compactification $\overline{M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0}$ of a Zariski open subset $M^w_3(\boldsymbol{t}, \boldsymbol{\nu})^0$ of $M^w_3(\boldsymbol{t}, \boldsymbol{\nu})$ and investigate $\mathrm{App} \times \mathrm{Bun}$.

Contents

Parabolic ϕ-connections

Definition

$(E_1, E_2, \phi, \nabla, l_*^{(1)} = \{l_{i,*}^{(1)}\}_{i=1}^3, l_*^{(2)} = \{l_{i,*}^{(2)}\}_{i=1}^3)$:
a $\boldsymbol{\nu}$-parabolic ϕ-connection of rank 3 and degree d

(1) E_1 and E_2 are vector bundles on $\mathbb{P}^1$ of rank 3 and degree d,

(2) $\phi \colon E_1 \to E_2$ is a homomorphism,

(3) $\nabla \colon E_1 \to E_2 \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$ is a logarithmic ϕ-connection, i.e. $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$, $f \in \mathcal{O}_{\mathbb{P}^1}, a \in E_1$, and

(4) $l_{i,*}^{(k)}$ is a filtration $E_k|_{t_i} = l_{i,0}^{(k)} \supsetneq l_{i,1}^{(k)} \supsetneq l_{i,2}^{(k)} \supsetneq l_{i,3}^{(k)} = 0$ satisfying $\phi_{t_i}(l_{i,j}^{(1)}) \subset l_{i,j}^{(2)}$, $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j}\phi_{t_i})(l_{i,j}^{(1)}) \subset l_{i,j+1}^{(2)}$ for $1 \leq i \leq 3, 1 \leq j \leq 3$.

- For (E, ∇, l_*), $(E, E, \mathrm{id}, \nabla, l_*, l_*)$ is a $\boldsymbol{\nu}$-parabolic ϕ-connection.
- $\wedge^3 \phi \neq 0 \Longleftrightarrow (E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \cong (E, E, \mathrm{id}, \nabla, l_*, l_*)$ for some (E, ∇, l_*)
- Rank 2 ϕ-connections were introduced by Inaba-Iwasaki-Saito. Their definition is a little different from the above definition.

Moduli of parabolic ϕ-connections

- $T_3 := \{(t_i)_{1\leq i\leq 3} \in (\mathbb{P}^1)^3 \mid t_i \neq t_j, i \neq j\}$
- $\mathcal{N}_d := \left\{ (\nu_{i,j})_{0\leq j\leq 2}^{1\leq i\leq 3} \in \mathbb{C}^9 \,\middle|\, \sum_{i=1}^{3} \sum_{j=0}^{2} \nu_{i,j} = -d \right\}, d \in \mathbb{Z}$

Proposition

(1) There exists a relative fine moduli scheme

$$\overline{M_3^{\boldsymbol{\alpha}}}(d) \longrightarrow T_3 \times \mathcal{N}_d$$

of $\boldsymbol{\alpha}$-stable parabolic ϕ-connections of rank 3 and degree d. If $\boldsymbol{\alpha}$ is generic, then the morphism becomes projective, and in particular, the fiber $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ over $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}_d$ is projective.

(2) Put

$$U_{\mathrm{isom}} := \left\{ (E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \in \overline{M_3^{\boldsymbol{\alpha}}}(d) \mid \wedge^3 \phi \neq 0 \right\}.$$

Then U_{isom} is an open subscheme of $\overline{M_3^{\boldsymbol{\alpha}}}(d)$ and $U_{\mathrm{isom}} \cong M_3^{\boldsymbol{\alpha}}(d)$.

The above proposition follows from the same arguments as the construction of moduli space of parabolic connections by Inaba-Iwasaki-Saito.

Main theorem

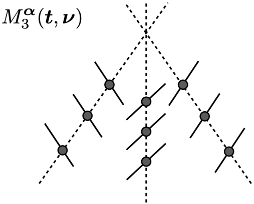
- $\mathcal{N}(0,0,2) := \left\{ (\nu_{i,j})^{1\le i\le 3}_{0\le j\le 2} \in \mathbb{C}^9 \,\middle|\, \sum_{j=0}^{2} \nu_{i,j} = 2\mathrm{res}_{t_i}\left(\frac{dz}{z-t_3}\right), 1 \le i \le 3 \right\} \subset \mathcal{N}_{-2}$
- $Y := \{\wedge^3 \phi = 0\} \subset \overline{M_3^{\boldsymbol{\alpha}}}(-2) \times_{(T_3 \times \mathcal{N}_d)} (T_3 \times \mathcal{N}(0,0,2))$

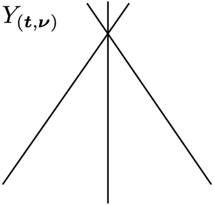
Theorem

Take $\boldsymbol{\alpha} = (\alpha_{i,j})_{1\le i,j\le 3}$ such that $0 < \alpha_{i,j} \ll 1$ for any $1 \le i,j \le 3$. Then for each $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}(0,0,2)$,

(1) $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ is isomorphic to an $A_2^{(1)*}$-surface, and

(2) the fiber $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ is the effective anti-canonical divisor of $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$.





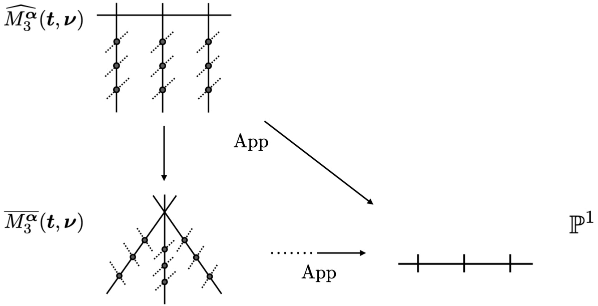
Outline of proof

Step 1: Show that $E_1 \cong E_2 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Step 2: Construct the apparent map $\mathrm{App}\colon \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \mathbb{P}^1$.

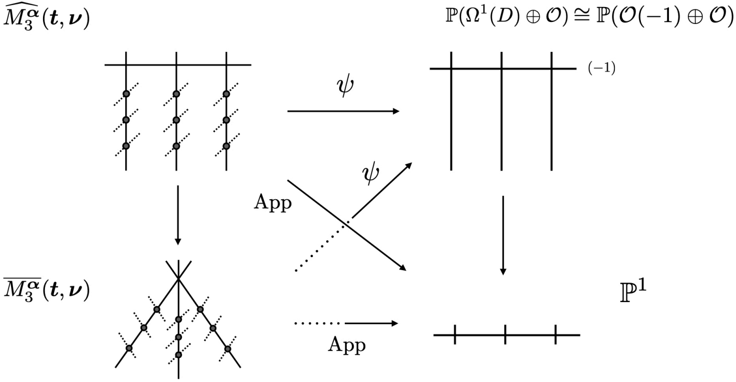
- The apparent map on $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})$ is defined by Saito and Szabó.
- We can extend it on the locus defined by $\operatorname{rank}\phi = 2$ on $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$, but cannot on the locus defined by $\operatorname{rank}\phi = 1$.

$\rightsquigarrow$ Construct the apparent map on moduli space $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ of pairs of a parabolic ϕ-connection and a certain subbundle of E_1.



Step 3: Construct a morphism ψ from $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ to $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\oplus\mathcal{O}_{\mathbb{P}^1})$ and show that ψ is a blow-up of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\oplus\mathcal{O}_{\mathbb{P}^1})$ at 9 points.

Step 4: Prove that the forgetful map from $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ to $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ is the blow-up along the locus defined by $\operatorname{rank}\phi=1$.



The apparent map App

$(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \in \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$.

Proposition

There exists a filtration $E_k = F_0^{(k)} \supsetneq F_1^{(k)} \supsetneq F_2^{(k)} \supsetneq F_3^{(k)} = 0$ for $k = 1, 2$ such that

$$F_1^{(1)} \cong F_1^{(2)} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1),\ F_2^{(1)} \cong F_2^{(2)} \cong \mathcal{O}_{\mathbb{P}^1},$$

and

$$\phi(F_i^{(1)}) \subset F_i^{(2)},\ \nabla(F_{i+1}^{(1)}) \subset F_i^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$$

for any $0 \leq i \leq 2$. $F_2^{(1)}, F_1^{(2)}, F_2^{(2)}$ are uniquely determined. If $\mathrm{rank}\,\phi = 2$ and 3, then $F_1^{(1)}$ is also unique. If $\mathrm{rank}\,\phi = 1$, then there is a one-to-one correspondence between the set of all such $F_1^{(1)}$ and $\mathbb{P}^1$.

The apparent map

Let $\mathrm{App}(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$ be the zero of the composite

$$\mathcal{O}_{\mathbb{P}^1}(-1) \cong F_1^{(1)}/F_2^{(1)} \to E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \to E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D) \cong \mathcal{O}_{\mathbb{P}^1}.$$

A normal form of connections

$\mathbf{rank}\, \phi = 3$

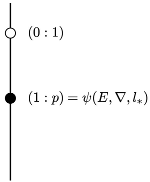
$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \nabla = d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & (z-t_1)(z-t_2)-p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)},$$

, where $a_{12}(z), a_{13}(z)$ are quadratic polynomials satisfying

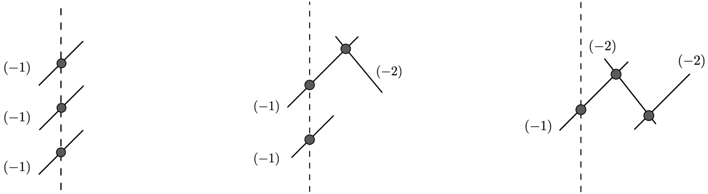
$$a_{12}(t_i) = -h'(t_i)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0}) - p^2,$$

$$(t_i - q)a_{13}(t_i) = h'(t_i)^3 \prod_{j=0}^{2} (\nu_{i,j} + \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}) - h'(t_i)^{-1}p)$$

for any $i = 1, 2, 3$ and $h(z) = (z-t_1)(z-t_2)(z-t_3)$.



the fiber of $\mathbf{App}$ at t_i



$\nu_{i,0} \neq \nu_{i,1} \neq \nu_{i,2} \neq \nu_{i,0}$ $\qquad$ $\nu_{i,0} = \nu_{i,1} \neq \nu_{i,2}$ $\qquad$ $\nu_{i,0} = \nu_{i,1} = \nu_{i,2}$

Normal forms and loci of ϕ-connections

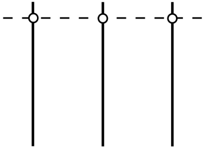
$\mathbf{rank}\,\phi = 2$

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \nabla = \phi \otimes d + \begin{pmatrix} 0 & 0 & \prod_{j \neq i}(z - t_j) \\ 1 & 0 & 0 \\ 0 & z - t_i & (z - t_1)(z - t_2) + p \end{pmatrix} \frac{dz}{h(z)}$$

$\mathbf{rank}\,\phi = 1$

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \nabla = \phi \otimes d + \begin{pmatrix} 0 & \prod_{j \neq i}(z - t_j) & 0 \\ 1 & 0 & 0 \\ 0 & z - q & z - t_i \end{pmatrix} \frac{dz}{h(z)}$$

$\mathrm{rank}\phi = 2$



$\mathrm{rank}\phi = 1$

$\mathrm{rank}\,\phi = 0$

- Under the assumption, parabolic ϕ-connections becomes $\boldsymbol{\alpha}$-unstable.
- Parabolic ϕ-connections of $\mathrm{rank}\,1$ and $\mathrm{rank}\,0$ are infinitesimally close.

$$\phi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \nabla := d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & (z-t_1)(z-t_2)-p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}$$

<u>$\mathrm{rank}\,\phi = 1$</u>

$$\begin{pmatrix} p^{-2} & 0 & 0 \\ 0 & p^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} (\phi, \nabla) \begin{pmatrix} p^{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix} \xrightarrow{p \to \infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & * & * \\ 1 & 0 & 0 \\ 0 & z-q & 1 \end{pmatrix} \frac{dz}{h(z)}$$

<u>$\mathrm{rank}\,\phi = 0$</u>

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p^{-1} & 0 \\ 0 & 0 & p^{-2} \end{pmatrix} (\phi, \nabla) \begin{pmatrix} p^{-1} & 0 & 0 \\ 0 & p^{-2} & 0 \\ 0 & 0 & p^{-3} \end{pmatrix} \xrightarrow{p \to \infty} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & * & * \\ 1 & -1 & 0 \\ 0 & z-q & 1 \end{pmatrix} \frac{dz}{h(z)}$$

Contents

Parabolic bundles and w-stability

Definition

$(E, l_* = \{l_{i,*}\}_{i=1}^{3})$: a parabolic bundle over $(\mathbb{P}^1, \boldsymbol{t} = (t_i)_{1 \le i \le 3})$

- E : a vector bundle of rank 3 and degree d, and
- $l_{i,*}$: a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$.

$\boldsymbol{\alpha} = (\alpha_{i,j})_{1 \le i,j \le 3}$: $0 < \alpha_{i,1} < \alpha_{i,2} < \alpha_{i,3} < 1$ for each $i = 1, 2, 3$

Assumption

For any $1 \le i \le 3, j = 1, 2,$

$$\alpha_{i,j+1} - \alpha_{i,j} = \text{constant} =: w$$

- $0 < w < 1/2$
- (E, l_*) is w-stable $\Longrightarrow E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$
- $P^w(-2) := \{(E, l_*) \mid w\text{-stable}, \deg E = -2\}/\sim$

Moduli space of parabolic bundles

Proposition

(1) If $0 < w < 2/9, 4/9 < w < 1/2$, then $P^w(-2) = \emptyset$.

(2) If $2/9 < w < 1/3$, then a w-stable parabolic bundle (E, l_*) fits into a nonsplit exact sequence

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^1}, \emptyset) \longrightarrow (E, l_*) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_* = \{l'_i\}_{i=1}^3) \longrightarrow 0,$$

where $n(l'_*) := \max_{\mathcal{O}_{\mathbb{P}^1}(-1) \cong F \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}} \#\{i \mid F|_{t_i} = l'_i\} = 1$. In particular, $P^w(-2) \cong \mathbb{P}^1$.

$\boldsymbol{2/9 < w < 1/3}$: $P^w(-2) \cong \mathbb{P}^1_{(a:b)}$

$$l_{1,2} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix},\ l_{1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}, l_{2,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{2,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$l_{3,2} = \mathbb{C}\begin{pmatrix}a+b\\1\\1\end{pmatrix},\ l_{3,1} = \mathbb{C}\begin{pmatrix}a\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}b\\0\\1\end{pmatrix}.$$

Proposition

(3) If $1/3 < w < 4/9$, then a w-stable parabolic bundle (E, l_*) is either type of the following:

- (i) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ holds.
- (ii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 1$, $n(l'_*) = 1$, and the condition $(*)$ holds.

In particular, $P^w(-2) \cong \mathbb{P}^1$.

$(*)$ There is no subbundle $F \subset E$ such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l_{i,2} \subset F|_{t_i}$ and $F|_{t_j} = l_{j,1}$ for some i and any $j \neq i$.

$P^w(-2) \cong \mathbb{P}^1$

$2/9 < w < 1/3$

$1/3 < w < 4/9$

Compactification by λ-connections

Definition

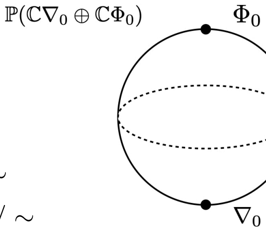
$\nabla \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(t_1 + t_2 + t_3)$: a $\boldsymbol{\nu}$-parabolic λ-connection over (E, l_*) $(\lambda \in \mathbb{C})$

- $\nabla(fa) = \lambda df \otimes a + f\nabla(a)$ for $a \in E$, $f \in \mathcal{O}_{\mathbb{P}^1}$, and
- $(\mathrm{res}_{t_i}\nabla - \nu_{i,j}\mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for $1 \leq i \leq 3, 0 \leq j \leq 2$.

When $\lambda = 0, \boldsymbol{\nu} = 0$, ∇ is called a parabolic Higgs field.

- ∇_0: a $\boldsymbol{\nu}$-parabolic connection over (E, l_*)
- $\Phi_0 \neq 0$: a parabolic Higgs field over (E, l_*)

$$\nabla_0 + \mathbb{C}\Phi_0 \underset{\text{open}}{\subset} \mathbb{P}(\mathbb{C}\nabla_0 \oplus \mathbb{C}\Phi_0)$$



- $M_3^w(\boldsymbol{t}, \boldsymbol{\nu})^0 := \{(E, \nabla, l_*) \mid (E, l_*) \in P^w(-2)\}/\sim$
- $\overline{M_3^w(\boldsymbol{t}, \boldsymbol{\nu})^0} := \{(\lambda, E, \nabla, l_*) \mid (E, l_*) \in P^w(-2)\}/\sim$
- $\mathrm{Bun} \colon \overline{M_3^w(\boldsymbol{t}, \boldsymbol{\nu})^0} \to P^w(-2), (\lambda, E, \nabla, l_*) \mapsto (E, l_*)$

Description of $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$

Theorem

Assume $2/9 < w < 1/3$. Then we have

$$\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \nu_{1,0}+\nu_{2,0}+\nu_{3,0} \neq 0 \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) & \nu_{1,0}+\nu_{2,0}+\nu_{3,0} = 0. \end{cases}$$

Outline of proof

Take coordinates $(\mathbb{C}, a), (\mathbb{C}, b)$ of $P^w(-2) \cong \mathbb{P}^1$ such that $ab = 1$.

$(\nabla_0(a), \Phi_0(a))$: a connection and a Higgs field over $(E, l_{a,*})$

$(\nabla_\infty(b), \Phi_\infty(b))$: a connection and a Higgs field over $(E, l_{b,*})$

$$\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} = \mathbb{P}(\mathbb{C}\nabla_0 \oplus \mathbb{C}\Phi_0) \cup \mathbb{P}(\mathbb{C}\nabla_\infty \oplus \mathbb{C}\Phi_\infty)$$

$$(\nabla_\infty, \Phi_\infty) \cong (\nabla_0, \Phi_0) \begin{pmatrix} 1 & 0 \\ -(\nu_{1,0}+\nu_{2,0}+\nu_{3,0})a^{-1} & a^{-2} \end{pmatrix}$$

$$h(z) = (z - t_1)(z - t_2)(z - t_3)$$

$$\nabla_0(a) := d + \begin{pmatrix} c_{11}(z) & c^0_{12}(a)(z-t_1)(z-t_2) & c^0_{13}(a)(z-t_1)(z-t_2) \\ 0 & (z-t_1)(z-t_2) + c_{22}(z) & c^0_{23}(t_3-t_1)(z-t_2) \\ c^0_{31}h'(t_3) & c^0_{32}(a)(t_3-t_2)(z-t_1) & (z-t_1)(z-t_2) + c_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

$$\Phi_0(a) := \begin{pmatrix} 0 & a(a+1)(z-t_1)(z-t_2) & -a(a+1)(z-t_1)(z-t_2) \\ h'(t_3) & 0 & -(a+1)(t_3-t_1)(z-t_2) \\ -ah'(t_3) & a(a+1)(t_3-t_2)(z-t_1) & 0 \end{pmatrix} \frac{dz}{h(z)},$$

$$\nabla_\infty(b) := d + \begin{pmatrix} c_{11}(z) & c^\infty_{12}(b)(z-t_1)(z-t_2) & c^\infty_{13}(b)(z-t_1)(z-t_2) \\ c^\infty_{21}h'(t_3) & (z-t_1)(z-t_2) + c_{22}(z) & c^\infty_{23}(b)(t_3-t_1)(z-t_2) \\ 0 & c^\infty_{32}(t_3-t_2)(z-t_1) & (z-t_1)(z-t_2) + c_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

$$\Phi_\infty(b) := \begin{pmatrix} 0 & b(1+b)(z-t_1)(z-t_2) & -b(1+b)(z-t_1)(z-t_2) \\ bh'(t_3) & 0 & -b(1+b)(t_3-t_1)(z-t_2) \\ -h'(t_3) & (1+b)(t_3-t_2)(z-t_1) & 0 \end{pmatrix} \frac{dz}{h(z)}.$$

$\mathrm{App} \times \mathrm{Bun}$

- $V_0 := P^w(-2) \setminus \{(1:0),(1:1),(0:1)\}$
- $\mathrm{Bun}\colon \overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \to P^w(-2), (\lambda, E, \nabla, l_*) \mapsto (E, l_*)$

Proposition

Assume $2/9 < w < 1/3$ and $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$. Then the morphism

$$\mathrm{App} \times \mathrm{Bun}\colon \mathrm{Bun}^{-1}(V_0) \longrightarrow \mathbb{P}^1 \times V_0$$

is finite and its generic fiber consists of three points.

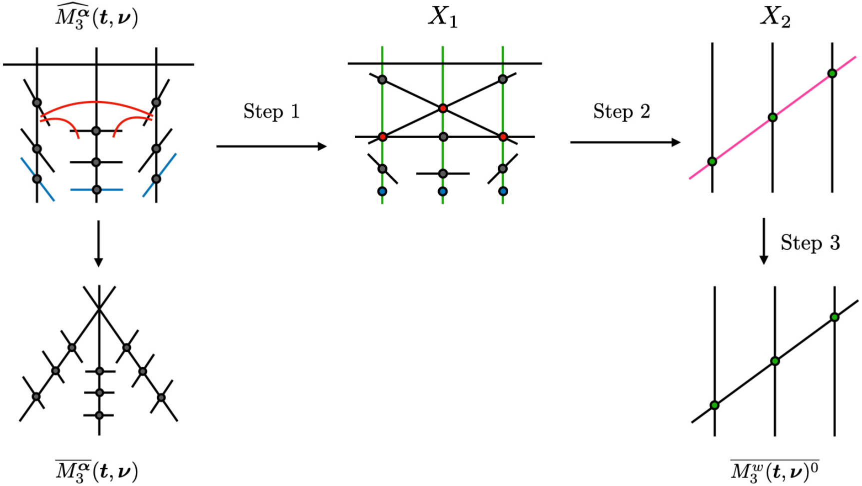
Sketch of proof

$\mathrm{Bun}^{-1}((E, l_{a,*})) = \mathbb{P}(\mathbb{C}\nabla_0(a) \oplus \mathbb{C}\Phi_0(a))$

$\mathrm{App}(\lambda\nabla_0(a) + \mu\Phi_0(a)) = (f_1(a;\lambda,\mu) + f_2(a;\lambda,\mu) : t_1 f_1(a;\lambda,\mu) + t_2 f_2(a;\lambda,\mu)),$

where $f_1(a;\lambda,\mu), f_2(a;\lambda,\mu)$ are homogeneous polynomials of degree 3 in variables λ, μ.

Relation between $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ and $\overline{M_3^{w}(\boldsymbol{t},\boldsymbol{\nu})}^0$



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