Flat structures on solutions to the sixth Painlevé equation

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Contents

§1 Triplet (M, D_M, Δ_M)

In this talk, triplet (M, D_M, Δ_M) introduced in the following definition plays a fundamental role.

Definition 1. A *triplet* (M, D_M, Δ_M) consists of

- M: an n-dimensional complex manifold,
- D_M: a holomorphic vector field on M (called *primitive vector field*),
- Δ_M: a free divisor on M (i.e., $\mathcal{D}er(-\log \Delta_M)$ is a locally free sheaf),

which enjoy the following properties (i)-(iv):

(i) $\exists$ a projection $\pi : M \to M'$ for an $(n-1)$-dimensional complex manifold M' such that $\pi^{-1}(U'_\alpha) \cong U'_\alpha \times \mathbb{C}$ for an open cover $\{U'_\alpha\}_{\alpha \in I}$ of M'.

(ii) Let $\boldsymbol{x}'^\alpha = (x_1, \ldots, x_{n-1})$ be a local coordinate system on U'_α and $\boldsymbol{x}^\alpha = (\boldsymbol{x}'^\alpha, x_n)$ be one on $\pi^{-1}(U'_\alpha)$. Then the coordinate transformation on $\pi^{-1}(U'_\alpha) \cap \pi^{-1}(U'_\beta)$ has the form

$$\boldsymbol{x}'^\alpha = \boldsymbol{f}(\boldsymbol{x}'^\beta), \quad x_n^\alpha = c x_n^\beta + g(\boldsymbol{x}'^\beta)$$

where c is a non-zero constant.

(iii) The primitive vector field D_M is represented as $D_M = d_\alpha \partial_{x_n^\alpha}$ ($d_\alpha \in \mathbb{C}^\times$) on $\pi^{-1}(U'_\alpha)$.

(iv) A defining function h_α of the divisor Δ_M on $\pi^{-1}(U'_\alpha)$ is a monic polynomial of degree n w.r.t. x_n^α:

$$h_\alpha(\boldsymbol{x}^\alpha) = (x_n^\alpha)^n - s_1^\alpha (x_n^\alpha)^{n-1} + \cdots + (-1)^n s_n^\alpha, \quad s_i^\alpha = s_i^\alpha(\boldsymbol{x}'^\alpha) \in \mathcal{O}_{M'}(U'_\alpha)$$

and satisfies

$$(1) \qquad \begin{vmatrix} \frac{\partial s_1^\alpha}{\partial x_1^\alpha} & \frac{\partial s_2^\alpha}{\partial x_1^\alpha} & \cdots & \frac{\partial s_n^\alpha}{\partial x_1^\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial s_1^\alpha}{\partial x_{n-1}^\alpha} & \frac{\partial s_2^\alpha}{\partial x_{n-1}^\alpha} & \cdots & \frac{\partial s_n^\alpha}{\partial x_{n-1}^\alpha} \\ -n & -(n-1)s_1^\alpha & \cdots & -s_{n-1}^\alpha \end{vmatrix} \neq 0 \in \mathcal{O}_{M'}(U'_\alpha)$$

Remark 1. By the conditions (i) and (ii), $\pi : M \to M'$ is a fiber bundle whose fibers are isomorphic to an affine line $\mathbb{C}$. This fiber bundle structure $\pi : M \to M'$ can be specified by the primitive vector field D_M as

$$\pi^{-1}\mathcal{O}_{M'} = \{f \in \mathcal{O}_M \mid D_M f = 0\}$$

So, we call (M, D_M) an *affine line bunlde with a primitive vector field.*

Remark 2. By the conditions (i)-(iv), we have

$$\forall\theta \in \mathcal{D}er_M(-\log \Delta_M) \text{ s.t. } [D_M, \theta] = 0 \implies \theta = 0. \tag{2}$$

Lemma 1. *Let a triplet* (M, D_M, Δ_M) *be given. Then,* $\varphi_{\Delta_M} \in \mathrm{End}_{\mathcal{O}_M}(\mathcal{D}er_M)$ *satisfying the conditions*

(i) $\mathrm{Im}\, \varphi_\Delta \subset \mathcal{D}er_M(-\log \Delta_M)$,

(ii) $(\det \varphi_{\Delta_M}) = \Delta_M$,

(iii) $\mathrm{Lie}_{D_M} \varphi_{\Delta_M} = 1_{\mathcal{D}er_M}$

(if it exists) is unique for (M, D_M, Δ_M) *and induces an isomorphism*

$$\varphi_{\Delta_M} : \mathcal{D}er_M \overset{\cong}{\to} \mathcal{D}er_M(-\log \Delta_M).$$

φ_{Δ_M} *is called the* <u>*canonical isomorphism*</u> *associated with the triplet* (M, D_M, Δ_M).

Proof. The lemma follows from (2) and the Saito criterion for free divisors. □

Definition 2. Assume that the canonical isomorphism φ_{Δ_M} associated with a triplet (M, D_M, Δ_M) exists. Then, $E_{\Delta_M} := \varphi_{\Delta_M}(D_M)$ is a (global) holomorphic vector field on M. E_{Δ_M} is called the *Euler vector field associated with the triplet* (M, D_M, Δ_M).

The following proposition provides a typical situation where the canonical isomorphism φ_{Δ_M} exists.

Proposition 1. *Consider the case of* $M = \mathbb{C}^n = \{\boldsymbol{x} = (\boldsymbol{x}', x_n)\}$ *and* $D_M = \partial_{x_n}$. *(In this case,* $M' = \mathbb{C}^{n-1} = \{\boldsymbol{x}' = (x_1, \ldots, x_{n-1})\}$.*) Let the defining polynomial of* Δ_M

$$h(\boldsymbol{x}) = x_n^n - s_1(\boldsymbol{x}')x_n^{n-1} + \cdots + (-1)^n s_n(\boldsymbol{x}') \in \mathbb{C}[\boldsymbol{x}]$$

be weighted homogenious w.r.t. the (usual) Euler operator

$$E = \sum_{i=1}^{n} w_i x_i \partial_{x_i}$$

with $w_i \in \mathbb{R}_{>0}$ $(1 \leq i \leq n)$ *in order of*

$$0 < w_1 \leq w_2 \leq \cdots \leq w_{n-1} < w_n = 1,$$

i.e., $Eh(\boldsymbol{x}) = nh(\boldsymbol{x})$.

Then, there exists the canonical isomorphism φ_{Δ_M} *associated with the triplet* (M, D_M, Δ_M) *and* $E_{\Delta_M} = \varphi_{\Delta_M}(D_M) = E$.

Example 1 (Canonical triplet (M_G, D_G, Δ_G) associated with a well-generated u.g.g.r. G)**.**

$G \subset U(V)$: an irreducible well-generated (finite) unitary reflection group of rank n

(i.e., G is a finite subgroup of $U(V)$ generated by n reflections.)

Then, we can construct a triplet (M_G, D_G, Δ_G) canonically from G in the following manner.

· $\mathcal{F} = (F_1, \ldots, F_n)$: a set of basic invariants (i.e., a system of homogeneous generaters of the ring of G-invariant polynomial functions on V)

· $\boldsymbol{d} = (d_1, \ldots, d_n)$: the degrees of G (i.e., $d_i = \deg F_i$) in order of

$$0 < d_1 \leq d_2 \leq \cdots \leq d_{n-1} < d_n$$

$\boldsymbol{d}$ is uniquely determined by G (i.e., does not depend on the choise of $\mathcal{F}$).

· $M_G = V/G$: the orbit space of G
We have an isomorphism $M_G \cong \mathbb{C}^n = \{\boldsymbol{x} = (\boldsymbol{x}', x_n)\}$ via $\boldsymbol{x} = (F_1, \ldots, F_n)$.

· $D_G := \partial_{x_n}$ is uniquely determined (up to non-zero constant multiplicity) by G.

· $h_{G,\mathcal{F}}(\boldsymbol{x})$: the disciriminant of G
$h_{G,\mathcal{F}}(\boldsymbol{x})$ is a monic polynomial of degree n w.r.t. x_n:

$$h_{G,\mathcal{F}}(\boldsymbol{x}) = x_n^n - s_1(\boldsymbol{x}')x_n^{n-1} + \cdots + (-1)^n s_n(\boldsymbol{x}') \in R_G$$

and weighted homogenious w.r.t. $E = \sum_{i=1}^n w_i x_i \partial_{x_i}$, where $w_i := d_i/d_n$.

Let Δ_G be a divisor on M_G defined by the discriminant $h_{G,\mathcal{F}}(\boldsymbol{x})$.
($\Delta_G \subset M_G$ agrees with the branch locus of the quotient mapping $V \to V/G = M_G$.)

It is known that the divisor Δ_G is free and it can be confirmed that the triplet (M_G, D_G, Δ_G) satisfies the conditions (i)-(iv) in Definition 1.

Definition 3. (M_G, D_G, Δ_G) is called the *canonical triplet associated with* G.

In virtue of Proposition 1, the canonical triplet (M_G, D_G, Δ_G) associated with G admits the canonical isomorphism φ_{Δ_G}. In particular, we have $E_{\Delta_G} = \varphi_{\Delta_G}(D_G) = E$.

§2 Flat structure (without metric) and space of Okubo-Saito potentials

M : an n-dimensional complex manifold

TM: the (holomorphic) tangent bundle of M

$\Theta_M(=\mathcal{D}er_M)$: the sheaf of holomorphic sections of TM

Definition 4 (C. Sabbah)**.** A *flat structure* (or *Saito structure*) *(without metric)* on M is a 5-tuple (M, ∇, Φ, E, e) which consists of

(i) ∇ is a flat and torsion-free connection on TM,

(ii) Φ is a symmetric Higgs field on TM,

(iii) E and e are global sections on TM (called *Euler field* and *unit field* respectively)

and satisfies certain integrability condition (this integrability condition will be stated later in a concrete form).

Remark 3. The symmetric Higgs field Φ provides a commutative and associative $\mathcal{O}_M$-algebra structure with the unit e on Θ_M by

$$u \circ v := \Phi_u(v) \quad (u, v \in \Theta_M).$$

Remark 4. A metric $\langle -, - \rangle$ on M (i.e., a nondegenerate symmetric $\mathcal{O}_M$-bilinear form on Θ_M) is called *Frobenius metric* if $\exists r \in \mathbb{C}$ such that $\langle -, - \rangle$ satisfies

$$\begin{aligned} &\langle u \circ v, w \rangle = \langle u, v \circ w \rangle \\ &d\langle u, v \rangle - \langle \nabla u, v \rangle - \langle u, \nabla v \rangle = 0 \\ &E\langle u, v \rangle - \langle [E, u], v \rangle - \langle u, [E, v] \rangle = r\langle u, v \rangle \end{aligned}$$

for $\forall u, v, w \in \Theta_M$. When a flat structure (M, ∇, Φ, e, E) is equipped with a Frobenius metric $\langle -, - \rangle$, $(M, \langle -, - \rangle, \Phi, e, E)$ forms a Frobenius manifold (in the sense of B. Dubrovin).

In what follows, we assume the following conditions on (M, ∇, Φ, e, E):

Semisimplicity condition (SS): $\Phi(E) \in \mathrm{End}_{\mathcal{O}_M}(\Theta_M)$ is diagonalizable at any point on M. The eigenvalues $(z_1, \ldots, z_n)$ of $\Phi(E)$ are mutually distinct at generic points on M.

Nonresonance condition (NR): $\nabla(E) \in \mathrm{End}_{\mathcal{O}_M}(\Theta_M)$ is diagonalizable at any point on M. The eigenvalues $(w_1, \ldots, w_n)$ of $\nabla(E)$ satisfy $w_n = 1$ and $w_i - w_j \notin \mathbb{Z} \setminus \{0\}$ for $i \neq j$.

Remark 5. By the condition (SS), the $\mathcal{O}_M$-algebra structure $\circ$ decomposes into the direct product of 1-dimensional simple $\mathcal{O}_M$-algebras

$$\partial_{z_i} \circ \partial_{z_j} = \delta_{ij} \partial_{z_i}, \qquad (1 \leq i, j \leq n)$$

by taking $(z_1, \ldots, z_n)$ as a local coordinate. So, $(z_1, \ldots, z_n)$ is called a *canonical coordinate* of the flat structure (M, ∇, Φ, e, E).

Definition 5. The *discriminant locus* Δ of (M, ∇, Φ, e, E) is the divisor on M defined by $\det \Phi(E) = 0$.

Proposition 2. *Let a flat structure* (M, ∇, Φ, e, E) *be given. Then,* (M, e, Δ) *forms a triplet in Definition 1. Moreover, the triplet* (M, e, Δ) *admits the canonical isomorphism* φ_Δ *associated with* (M, e, Δ) *and we have* $\varphi_\Delta = \Phi(E)$*. In particular, we have* $E_\Delta = \varphi_\Delta(e) = \Phi_e(E) = E$*.*

Corollary 1. *Let two flat structures* $(M, \nabla^k, \Phi^k, e^k, E^k)$ $(k = 1, 2)$ *be given on* M*. Assume the equality* $(M, e^1, \Delta^1) = (M, e^2, \Delta^2)$ *holds for the triplets underlying the respective* $(M, \nabla^k, \Phi^k, e^k, E^k)$ $(k = 1, 2)$*. Then, we have* $e^1 = e^2, E^1 = E^2, \Phi^1 = \Phi^2$*.*

Proof. By the assumption and Proposition 2, we have $e^1 = e^2, E^1 = E^2, \Phi^1(E^1) = \Phi^2(E^2)$. Then, by the condition (SS), we see that $\Phi^1 = \Phi^2$ at generic points on M because the canonical coordinate $(z_1, \ldots, z_n)$ is given by the set of eigenvalues of $\Phi^1(E^1) = \Phi^2(E^2)$. Therefore, $\Phi^1 = \Phi^2$ holds at any point of M by the identity theorem. □

Remark 6. In virtue of Corollary 1, the $\mathcal{O}_M$-algebra structure $\circ$ and the Euler field E of (M, ∇, Φ, e, E) are uniquely determined by its underlying triplet (M, e, Δ).

(In other words, the "F-manifold structure with the Euler field" underlying a flat structure (M, ∇, Φ, e, E) is uniquely determined by its triplet (M, e, Δ).)

Definition 6. For a flat structure (M, ∇, Φ, e, E), a local coordinate system $\boldsymbol{t} = (t_1, \ldots, t_n)$ on an open set $U \subset M$ is said to be a *flat coordinate* if $\nabla(\partial_{t_i}) = 0$ holds for any $i \in \{1, \ldots, n\}$.

By the conditioin (NR), we may take a flat coordinate $\boldsymbol{t} = (t_1, \ldots, t_n)$ s.t.

$$e = \partial_{t_n}, \quad E = \sum_{i=1}^{n} w_i t_i \partial_{t_i}. \tag{3}$$

In the sequel, we take a flat coordinate $\boldsymbol{t}$ as satisfying (3).

We introduce (local) representation matrices of $-\Phi(E), \nabla(E) \in \mathrm{End}_{\mathcal{O}_M}(\Theta_M)$ and $\Phi \in \Omega^1(\mathrm{End}_{\mathcal{O}_M}(\Theta_M))$ by taking $(\partial_{t_1}, \ldots, \partial_{t_n})$ as a basis of $\Theta_M(U)$:

(i) $\mathcal{T} = (\mathcal{T}_{ij})_{1\le i,j\le n} \in \mathcal{O}_M(U)^{n\times n}$ is defined by

$$-\Phi_{\partial_{t_i}}(E) = \sum_{j=1}^{n} \mathcal{T}_{ij}\partial_{t_j}.$$

(ii) $\mathcal{B}_\infty = ((\mathcal{B}_\infty)_{ij})_{1\le i,j\le n} \in \mathcal{O}_M(U)^{n\times n}$ is defined by

$$\nabla_{\partial_{t_i}}(E) = \sum_{j=1}^{n} (\mathcal{B}_\infty)_{ij}\partial_{t_j}.$$

By (3), we see that $\mathcal{B}_\infty$ is a constant diagonal matrix: $\mathcal{B}_\infty = \mathrm{diag}(w_1, \ldots, w_n)$.

(iii) $\tilde{\Phi} = (\tilde{\Phi}_{ij})_{1\le i,j\le n} \in \Omega^1_M(U)^{n\times n}$ is defined by

$$\Phi(\partial_{t_i}) = \sum_{j=1}^{n} \tilde{\Phi}_{ij}\partial_{t_j}.$$

In terms of the matrices $\mathcal{T}, \mathcal{B}_\infty, \tilde{\Phi}$, the integrability condition required in Definition 4 is equivalent to that of the following linear Pfaffian system

$$\mathcal{T} dY = \tilde{\Phi} \mathcal{B}_\infty Y. \tag{4}$$

Remark 7. By fixing the variables $\boldsymbol{t}' = (t_1, \ldots, t_{n-1})$, (4) reduces to an ordinary Fuchsian differential equation (called *Okubo normal form*)

$$\mathcal{T} \frac{dY}{dt_n} = \mathcal{B}_\infty Y \tag{5}$$

w.r.t. the variable t_n. (Note that the variable t_n is specified by the unit field (=primitive vector field) $e = \partial_{t_n}$.) The completely integrable Pfaffian system (4) is equivalent to an isomonodromic deformation of (5).

Lemma 2. *Let a flat structure* (M, ∇, Φ, e, E) *be given and* $\boldsymbol{t} = (t_1, \dots, t_n)$ *be a flat coordinate on an simply-connected open set* $U \subset M$.

For $\lambda \in \mathbb{C}$ *s.t.* $\det(\mathcal{B}_\infty - 1 + \lambda) \neq 0$, *we consider the following completely integrable Pfaffian system*

$$\mathcal{T} dY = \tilde{\Phi} B_\infty Y, \tag{6}$$

where we put $B_\infty := \mathcal{B}_\infty - 1 + \lambda$. *Then, there exists an* n*-dimensional* $\mathbb{C}$*-vector space* $\mathcal{P}^{(\lambda)} \subset \mathcal{O}_M(U)$ *s.t. any solution to (6) on* U *is given as*

$$Y = -B_\infty^{-1} \begin{pmatrix} \theta_1^t(p) \\ \vdots \\ \theta_n^t(p) \end{pmatrix}, \quad p \in \mathcal{P}^{(\lambda)},$$

where $\theta_i^t = \varphi_\Delta(\partial_{t_i}) = \Phi_{\partial_{t_i}}(E) \in \mathcal{D}er_M(-\log \Delta)(U)$, $(1 \leq i \leq n)$.

Definition 7. For a flat structure (M, ∇, Φ, e, E), the n-dimensional $\mathbb{C}$-vector space $\mathcal{P}^{(\lambda)}$ in Lemma 2 is called *the space of Okubo-Saito potentials* of weight λ associated with (M, ∇, Φ, e, E).

Properties of $\mathcal{P}^{(\lambda)}$ associated with (M, ∇, Φ, e, E)

- For any $p \in \mathcal{P}^{(\lambda)}$, $Ep = \lambda p$ holds.

- $\mathcal{P}^{(\lambda)}$ is uniquely determined by (M, ∇, Φ, e, E), (i.e., does not depend on $\boldsymbol{t}$).

- Every $p \in \mathcal{P}^{(\lambda)}$ can be analytically continued over $\widetilde{M \setminus \Delta}$, where $\widetilde{M \setminus \Delta}$ denotes the universal covering space of $M \setminus \Delta$.

- Define a connection $\tilde{\nabla}^{(\lambda)}$ on $\Theta_{M\setminus\Delta}$ by

$$\tilde{\nabla}^{(\lambda)}_u v := \nabla_u v - \nabla_{E^{-1}\circ u \circ v} E + \lambda E^{-1} \circ u \circ v, \quad u, v \in \Theta_{M\setminus\Delta}.$$

  Take a basis $(p_1, \dots, p_n)$ of $\mathcal{P}^{(\lambda)}$. Then $\tilde{\nabla}^{(\lambda)}(\partial_{p_i}) = 0$ $(1 \leq i \leq n)$, i.e., $(p_1, \dots, p_n)$ forms a flat coordinate system w.r.t. the connection $\tilde{\nabla}^{(\lambda)}$.

- Let $\mathcal{P}^{(\lambda_i)}$ $(i = 1, 2)$ be two spaces of Okubo-Saito potentials of weight λ_i $(i = 1, 2)$ associated with (M, ∇, Φ, e, E). Then, they are transformed to each other by the Riemann-Liouville integrals:

$$p^{(1)}(\boldsymbol{t}) \in \mathcal{P}^{(\lambda_1)} \ \rightarrow \ p^{(2)}(\boldsymbol{t}) := \int_{\Delta} (t_n - s_n)^{\lambda_1 - \lambda_2 - 1} p^{(1)}(\boldsymbol{t}', s_n) ds_n \in \mathcal{P}^{(\lambda_2)}$$

§3 Space of Okubo-Saito potentials associated with a triplet (M, D_M, Δ_M)

In this section, we DO NOT assume the existence of a flat structure (M, ∇, Φ, e, E). Instead, we start from a triplet (M, D_M, Δ_M) with the canonical isomorphism φ_{Δ_M}.

Definition 8. Let a triplet (M, D_M, Δ_M) with the canonical isomorphism φ_{Δ_M} be given. Let $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ be an n-dimensional $\mathbb{C}$-vector space of holomorphic functions on $\widetilde{M \setminus \Delta_M}$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

$\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ is said to be a *space of Okubo-Saito potentials associated with* (M, D_M, Δ_M) if it satisfies the following conditions (i)-(iii):

(i) $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ is a linear representation space of $\pi_1(M \setminus \Delta_M)$ via the covering transformations of $\widetilde{M \setminus \Delta_M} \to M \setminus \Delta_M$.

(ii) $E_{\Delta_M} p = \lambda p$ holds for any $p \in \mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$, where $E_{\Delta_M} = \varphi_{\Delta_M}(D_M)$.

(iii) For any $m \in M$, take a simply-connected neighbourhood U_m and a local coordinate system $\boldsymbol{x} = (x_1, \dots, x_n)$ on U_m s.t. $\partial_{x_n} = D_M$.

Then, there exists $R \in GL(n, \mathcal{O}_M(U_m))$ satisfying

$$D_M R = O, \quad R_{nj} = \lambda^{-1}\delta_{nj}$$

s.t., for any $p \in \mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$,

$$Y := R \begin{pmatrix} \varphi_{\Delta_M}(\partial_{x_1})p \\ \vdots \\ \varphi_{\Delta_M}(\partial_{x_n})p \end{pmatrix}$$

gives a solution to a completely integrable Pfaffian system of rank n in the form

(7) $$T dY = \tilde{\Phi} B_\infty Y$$

where $T \in \mathcal{O}_M(U_m)^{n\times n}$ and $\tilde{\Phi} \in \Omega^1_M(U_m)^{n\times n}$ satisfy

$$D_M T = -I_n, \quad (\det T) = \Delta_M,$$
$$D_M \tilde{\Phi} = O, \quad \tilde{\Phi}_{D_M} = I_n$$

and B_∞ is a constant diagonal matrix.

Theorem 1. *Let* $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ *satisfying the conditions (i)-(iii) in Definition 8 be given. Then, there exists uniquely a flat structure* (M,∇,Φ,e,E) *whose underlying triplet* (M,e,Δ) *satisfies* $(M,e,\Delta)=(M,D_M,\Delta_M)$ *and whose space of Okubo-Saito potentials* $\mathcal{P}^{(\lambda)}$ *satisfies* $\mathcal{P}^{(\lambda)}=\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$.

Moreover, two spaces of Okubo-Saito potentials $\mathcal{P}^{(\lambda_1)}_{(M,D_M,\Delta_M)}$ *and* $\mathcal{P}^{(\lambda_2)}_{(M,D_M,\Delta_M)}$ *induce the identical flat structure* (M,∇,Φ,e,E)

$\iff$ $\mathcal{P}^{(\lambda_1)}_{(M,D_M,\Delta_M)}$ *and* $\mathcal{P}^{(\lambda_2)}_{(M,D_M,\Delta_M)}$ *are transformed to each other by use of the Riemann-Liouville integrals.*

Corollary 2. *A flat structure* (M,∇,Φ,e,E) *can be equipped with a Frobenius metric* $\langle -,-\rangle$ *if and only if the space of Okubo-Saito potentials* $\mathcal{P}^{(\lambda)}$ *admits a monodromy invariant non-degenerate symmetric* $\mathbb{C}$*-bilinear form for some value of* λ.

Properties of the monodromy of $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$

- The monodromy group is generated by n generalized reflections $\{R_1,\ldots,R_n\}$:

$$R_i \sim \mathrm{diag}(e^{2\pi\sqrt{-1}r_i},1,\ldots,1), \quad r_i \in \mathbb{C}.$$

- $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ has a "good basis" $\{a_1,\ldots,a_n\}$ (called *canonical system*) which consists of the "roots" for the generalized reflections $\{R_1,\ldots,R_n\}$.

- The local monodromy at $x_n=\infty$ is given by

$$e^{2\pi\sqrt{-1}B_\infty} = \mathrm{diag}(e^{2\pi\sqrt{-1}\lambda_1},\ldots,e^{2\pi\sqrt{-1}\lambda_n}).$$

So, $\mathcal{P}^{(\lambda)}_{(M,D_M,\Delta_M)}$ may be considered a generalization of "root system" with a prescribed "root basis" $\{a_1,\ldots,a_n\}$ (=canonical system) and "the conjugacy class of a Coxeter element" $e^{-2\pi\sqrt{-1}B_\infty}$ ($\sim c := R_1R_2\cdots R_n$).

Example 2 (Period integrals of K. Saito's primitive form)**.**

Consider a universal unfolding of a simple singularity of ADE type (for instance). The parameter space M of the unfolding is naturally equipped with the structure of a triplet (M, D_M, Δ_M). Let $\zeta^{(\lambda)}$ be a primitive form (which is defined by use of the higher residue pairings on the de Rham cohomology). Then,

$$Per^{(\lambda)}_{(M,D_M,\Delta_M)} = \left\{ \int_\gamma \zeta^{(\lambda)} \,\middle|\, \gamma \in \mathcal{H} \right\}$$

is a space of Okubo-Saito potentials associated with (M, D_M, Δ_M), where $\mathcal{H}$ denotes the local system which consists of the Milnor lattice on each fiber. (The Pfaffian system (7) is deduced from the "Gauss-Manin connection" on the de Rham cohomorogy.)

In this case, $Per^{(\lambda)}_{(M,D_M,\Delta_M)}$ admits a monodromy invariant $\mathbb{C}$-bilinear form which is induced from the intersection form of the Milnor lattice.

Hence, there exists a unique Frobenius structure $(M, \langle -, - \rangle, \Phi, e, E)$ induced by $Per^{(\lambda)}_{(M,D_M,\Delta_M)}$.

Example 3 (Canonical flat structure on M_G for a well-generated u.g.g.r. G)**.**
Let $G \subset U(V)$ be an irreducible well-generated unitary reflection group. As described in §1, there exists a triplet (M_G, D_G, Δ_G) obtained canonically from G.

The dual space V^* of V is a set of homogeneous linear functions on V, therefore V^* may be considered an n-dimensional $\mathbb{C}$-vector space of maulti-valued analytic functions on $M_G \setminus \Delta_G$ via the natural quotient mapping $\pi_G : V \to M_G = V/G$.

Actually, V^* is a space of Okubo-Saito potentials of weght $1/h$ associated with (M_G, D_G, Δ_G), where $h := d_n$ is the Coxeter number of G. (d_n is the highest degree of G.)

Hence, there exists a flat structure on M_G uniquely determined by $((M_G, D_G, \Delta_G), V^*)$, which is called the *canonical flat structure associated with* G.

(This flat structure was constructed and studied by Kato-Mano-Sekiguchi, Arsie-Lorenzoni and Konishi-Minabe-Shiraishi.)

§4 Flat structures on solutions to the sixth Painlevé equation

In the case of $n = 3$, an isomonodromic deformation of (5) is governed by a solution to the sixth Painlevé equation ($\mathrm{P_{VI}}$).

Theorem 2 (Dubrovin (1-parameter case), Arsie-Lorenzoni, Kato-Mano-Sekiguchi)**.** *There exists a correspondence between* 3*-dimensinal generically semisimple flat structures and solutions to the sixth Painlevé equation*

$$\begin{aligned} y'' =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right). \end{aligned}$$

In particular, 3*-dimensinal generically semisimple Frobenius manifolds correspond to solutions to certain* 1*-parameter family of* $\mathrm{P_{VI}}$.

Remark 8. The correspondence stated in Theorem 2 is not one-to-one but many-to-many:

> Given a 3-dimensinal semisimple flat structure, the corresponding solution to $\mathrm{P_{VI}}$ is determined up to Bäcklund transformations.
>
> Oppositely, given a solution to $\mathrm{P_{VI}}$, many flat structures may correspond to it in general. However, the underlying triplet (M, e, Δ) is unique for a solution to $\mathrm{P_{VI}}$, which means that the triplets (M, e, Δ) provide an invariant of solutions to $\mathrm{P_{VI}}$. (But this invariant is not complete.)

Definition 9. Let $y = y(t)$ be an algebraic solution to P_{VI}. Then, a complete algebraic curve Π over $\mathbb{C}$ is said to be a *(minimal) Painlevé curve* if there exist two rational functions t, y on Π:

$$
\begin{array}{ccc} & \Pi & \\ {}^{t}\swarrow & & \searrow^{y} \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array} \tag{8}
$$

satisfying the following conditions

(i) t is a Belyi function.

(ii) For every branch of t on a simply-connected open set $U \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $y = y(t)$ satisfies P_{VI} for some value of the parameter $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_t, \theta_\infty)$.

(iii) Let $\mathbb{C}(\Pi)$ denotes the field of rational functions on Π. Then $\mathbb{C}(t, y) = \mathbb{C}(\Pi)$ holds.

Example 4 (Algebraic solutions (H_3) and $(H_3)'$ by Dubrovin-Mazzocco)**.**
It is known that there are the following two algebraic solutions (H_3), $(H_3)'$ related to the Coxeter group $W(H_3)$:

(H_3): $\Pi \cong \mathbb{P}^1$, $\boldsymbol{\theta} = (0, 0, 0, -4/5)$.

A parameter representation is given by

$$y = \frac{(u-1)^2(3u+1)^2(u^2+4u-1)(119u^8-588u^6+314u^4-108u^2+7)^2}{(u+1)^3(3u-1)P(u)},$$
$$t = \frac{(u-1)^5(3u+1)^3(u^2+4u-1)}{(u+1)^5(3u-1)^3(u^2-4u-1)},$$

where $P(u)$ is a polynomial defined by

$$P(u) = 42483u^{18} - 719271u^{16} + 5963724u^{14} + 13758708u^{12} - 7616646u^{10} + 1642878u^8 - 259044u^6 + 34308u^4 - 2133u^2 + 49.$$

Frobenius potential: $\boldsymbol{w} = (w_1, w_2, w_3) = (1/5, 3/5, 1)$

$$F_{H_3} = \frac{t_1t_3^2 + t_2^2t_3}{2} + \frac{t_1^2t_2^3}{6} + \frac{t_1^5t_2^2}{20} + \frac{t_1^{11}}{3960}$$

$(H_3)'$: $\Pi \cong \mathbb{P}^1$, $\boldsymbol{\theta} = (0, 0, 0, -2/5)$.

A parameter representation is given by

$$y = \frac{(u-1)^4(3u+1)^2(u^2+4u-1)(11u^4-30u^2+3)^2}{(u+1)(3u-1)(3u^2+1)P'(u)},$$
$$t = \frac{(u-1)^5(3u+1)^3(u^2+4u-1)}{(u+1)^5(3u-1)^3(u^2-4u-1)},$$

where $P'(u)$ is a polynomial defined by

$$P'(u) = 121u^{12} - 1942u^{10} + 63015u^8 - 28852u^6 + 4855u^4 - 342u^2 + 9.$$

Frobenius potential: $\boldsymbol{w} = (w_1, w_2, w_3) = (3/5, 4/5, 1)$

$$F_{H_3'} = \frac{t_1 t_3^2 + t_2^2 t_3}{2} - \frac{t_1^4 z}{18} - \frac{7}{72} t_1^3 z^4 - \frac{17}{105} t_1^2 z^7 - \frac{2}{9} t_1 z^{10} - \frac{64}{585} z^{13}$$

where z is a solution to an algebraic equation

$$z^4 + t_1 z + t_2 = 0.$$

The algebraic solutions (H_3) and $(H_3)'$ correspond to the common triplet

$$(M_{H_3}, D_{H_3}, \Delta_{H_3}) \cong (M_{H_3'}, D_{H_3'}, \Delta_{H_3'})$$

which is isomorphic to the canonical triplet associated with $W(H_3)$. The Painlevé curve Π of (H_3) and $(H_3)'$ can be constructed from the canonical triplet $(M_{H_3}, D_{H_3}, \Delta_{H_3})$. ($\mathbb{C}(\Pi)$ is isomorphic to a splitting field of the discriminant

$$\begin{aligned} h_{H_3}(\boldsymbol{x}) =& x_3^3 - s_1(\boldsymbol{x}')x_3^2 + s_2(\boldsymbol{x}')x_3 - s_3(\boldsymbol{x}') \\ =& x_3^3 + \left(x_1^2 x_2 + \frac{x_1^5}{10}\right) x_3^2 - \left(\frac{9}{5}x_1 x_2^3 + \frac{6}{5}x_1^4 x_2^2 + \frac{x_1^{10}}{100}\right) x_3 \\ &+ \frac{27}{125}x_2^5 + \frac{23}{25}x_1^3 x_2^4 + \frac{x_1^6 x_2^3}{50} + \frac{2}{25}x_1^9 x_2^2 - \frac{x_1^{12} x_2}{100} - \frac{x_1^{15}}{1000} \end{aligned}$$

as a monic polynomial of degree 3 in x_3.)

This is the reason why the parameter representations of t for (H_3) and $(H_3)'$ mutually coincide.

The algebraic solution (H_3) corresponds to the canonical flat structure associated with $W(H_3)$, i.e., the space of Okubo-Saito potentials is V^* for a standard representation $W(H_3) \subset U(V)$. The weight of V^* is $1/10$.

Let $S_3(V^*)$ denote the degree 3 part of the symmetric tensor poduct of V^*. Then, $S_3(V^*)$ is a 10-dimensional representation space of $W(H_3)$ but not irreducible. $S_3(V^*)$ includes an irreducible representation space of dimension 3 of $W(H_3)$, which is denoted by $(V^*)'$. $(V^*)'$ comes from an outer automorphism of $W(H_3)$ and is not equivalent to V^*.

$(V^*)'$ forms a space of Okubo-Saito potentials of weight $3/10$ associated with the triplet $(M_{H_3}, D_{H_3}, \Delta_{H_3})$.

The algebraic solution $(H_3)'$ corresponds to the flat structure whose space of Okubo-Saito potentials is $(V^*)'$.

Problems.

- Classify triplets (M, D_M, Δ_M) corresponding to solutions to $\mathrm{P_{VI}}$.
- When two solutions to $\mathrm{P_{VI}}$ correspond to a common triplet (M, D_M, Δ_M), does any representation-theoritical relationship exist between their spaces of Okubo-Saito potentials?