Moduli space of irregular rank two parabolic bundles over the Riemann sphere and its compactification

Web-seminar on Painlevé Equations and related topics

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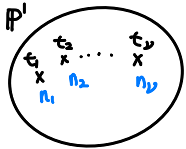
Setting

- $\{t_1, t_2, \ldots, t_\nu\}$ $(\nu > 0)$: distinct points on $\mathbb{P}^1$
- $n_1, n_2, \ldots, n_\nu$: positive integers
- $D := \sum_{i=1}^{\nu} n_i[t_i]$: effective divisor on $\mathbb{P}^1$
  - $n := \deg(D) = \sum_{i=1}^{\nu} n_i$
- E : rank 2 vector bundle on $\mathbb{P}^1$



Irregular parabolic bundles

- In this talk, we are concerned with *rank 2* (*irregular*) *parabolic bundles over* $(\mathbb{P}^1, D)$:
  - **Bhosle '92 '96, Yokogawa '93 (generally).**

Definition

- We say $(E, \boldsymbol{l})$ (where $\boldsymbol{l} = \{l_i\}_{1 \leq i \leq \nu}$) is a (*irregular quasi*) *parabolic bundle of rank 2 and of degree d over* $(\mathbb{P}^1, D)$ if
  - E is a rank 2 vector bundle on $\mathbb{P}^1$ with $\deg(E) = d$ and
  - l_i is a free $\mathcal{O}_{n_i[t_i]}$-submodule of $E|_{n_i[t_i]}$ for each i.
- We call l_i a (*quasi*) *parabolic structure at t_i*.

- When $n_i = 1$ $(1 \leq i \leq \nu)$, that is, D is reduced, the notion of parabolic bundles above coincides with the notion of rank 2 quasi-parabolic bundles over $(\mathbb{P}^1, D)$.

When D is reduced

- Fix weights $\boldsymbol{w} = (w_1, \dots, w_n) \in [0,1]^n$, $n = \deg(D)$.
- For a line bundle $L \subset E$, define:

$$\mathrm{stab}_L(E, \boldsymbol{l}) = \underbrace{\deg(E) - 2\deg(L)}_{usual} + \sum_{L|_{t_i} \neq l_i} w_i - \sum_{L|_{t_i} = l_i} w_i \in \mathbb{R}.$$

- $\mathrm{stab}(E, \boldsymbol{l}) := \inf \{\mathrm{stab}_L(E, \boldsymbol{l}) \ ; \ L \subset E$ a line bundle$\}$.

Definition

$(E, \boldsymbol{l})$ is $\boldsymbol{w}$-semistable (resp. $\boldsymbol{w}$-stable) $\iff$ $\mathrm{stab}(E, \boldsymbol{l}) \geq 0$ (resp. > 0)

- For generic $\boldsymbol{w}$, $\boldsymbol{w}$-semistable $=$ $\boldsymbol{w}$-stable.
- $\mathfrak{Bun}_{\boldsymbol{w}}(D, d)$: the moduli space of rank 2 $\boldsymbol{w}$-stable parabolic bundles of degree d over $(\mathbb{P}^1, D)$.
- For generic $\boldsymbol{w}$, $\mathfrak{Bun}_{\boldsymbol{w}}(D, d)$ is smooth projective and irreducible of dimension $n-3$ (or empty).
  - **Mehta–Seshadri '80, Bhosle '89, Bhosle–Biswas '12** (general cases).
  - **Bauer '91** ($r = 2, g = 0$).

Motivation

- Parabolic structures appear from a generic connection $\nabla\colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$.
  - They are the filtrations w.r.t. the diagonalization of the principal part of a connection matrix at each t_i.

Parabolic structures are important to study of geometry of the isomonodromic deformations of ∇ (or Painlevé type equations)

- Parabolic connections are triples $(E, \nabla\colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D), \boldsymbol{l})$
- The isomonodromic deformations are vector fields on the moduli space of parabolic connections.
  - **Arinkin-Lysenko '97, Inaba–Iwasaki–Saito '06, Inaba '13** (D is reduced)
  - **Arinkin-Fedorov '12, Inaba–Saito '13** (D is not necessary reduced)
- The moduli space of parabolic connections has an algebraic symplectic form.
  - The isomonodromic deformations have Hamiltonian description.

Motivation

The correspondence

$$(E, \nabla, \boldsymbol{l}) \longrightarrow (E, \boldsymbol{l})$$

is useful for studying the moduli space of parabolic connections.

- This correspondence gives a map from the moduli space of parabolic connections to the moduli space of parabolic bundles.
  - Remark that $(E, \boldsymbol{l})$ $\boldsymbol{w}$-stable $\Longrightarrow (E, \nabla, \boldsymbol{l})$ $\boldsymbol{w}$-stable.
    The converse is false.
- But we may expect this map has nice properties:
  - **affine bundle, lagrangian fibration**.
  - **Loray–Saito '15** have given Darboux coordinates on the moduli space of parabolic connections ($r = 2$, $g = 0$, D is reduced).

We would like to understand the moduli space of parabolic connections via this map. In order to do that, first of all, we try to understand the moduli space of parabolic bundles well.

Motivation

- We consider only the specific case $r = 2$ and $g = 0$.
- But isomonodromic deformations in this situation are interesting. Indeed, these isomonodromic deformations correspond to
  - **Painlevé equation** P_{VI} $(D = t_1 + t_2 + t_3 + t_4)$.
  - **Painlevé equations** P_V, P_{IV}, $P_{III(D_6^{(1)})}$, P_{II} $(D = 2t_1 + t_2 + t_3,\ D = 3t_1 + t_2,\ D = 2t_1 + 2t_2,\ D = 4t_1\ )$
  - **Garnier system (some irregular Garnier system)** when $n \geq 5$.
- Moreover, we cannot apply some general arguments to this situation $(g = 0)$.

Known results when D is reduced

- Take $\boldsymbol{\lambda} = (\lambda_1^\pm, \ldots, \lambda_n^\pm) \in \mathbb{C}^{2n}$ where $\sum_{i=1}^n (\lambda_i^+ + \lambda_i^-) = -d$.
- We say $(E, \boldsymbol{l})$ is ***λ-flat*** if $\exists \nabla$ s.t. $(E, \nabla, \boldsymbol{l})$ is $\boldsymbol{\lambda}$-parabolic connection. (The eigenvalues of the residue matrix of ∇ at each i are $\lambda_i^\pm$.)

Proposition

For generic $\boldsymbol{\lambda}$, and $g = 0$, we have for $(E, \boldsymbol{l})$:

simple $\Longleftrightarrow$ $\boldsymbol{\lambda}$-flat $\Longleftrightarrow$ undecomposable (**Arinkin–Lysenko**)

undecomposable $\Longleftrightarrow$ $\boldsymbol{w}$-stable for some $\boldsymbol{w}$ (**Loray–Saito**)

- The moduli space $\mathfrak{Bun}(D, d)$ of $\boldsymbol{\lambda}$-flat parabolic bundles is a non separated scheme, obtained by patching together smooth projective charts $\mathfrak{Bun}_{\boldsymbol{w}}(D, d)$ for a finite set of values for $\boldsymbol{w}$ (one for each chamber).

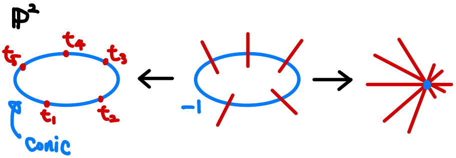
Known results when D is reduced

- We assume that $w_1 = w_2 = \cdots = w_n = w$.
- Moreover we set $n = 5$. ($\mathfrak{Bun}_{\boldsymbol{w}}(D, d)$ is a surface).

Proposition (Bauer '91, Loray–Saito '15, Donagi–Pantev '19)

- $\mathfrak{Bun}_{\boldsymbol{w}}(D, -1)$ is empty when $0 < w < \frac{1}{5}$;
- $\mathfrak{Bun}_{\boldsymbol{w}}(D, -1) = \mathbb{P}^2$ when $\frac{1}{5} < w < \frac{1}{3}$;
- $\mathfrak{Bun}_{\boldsymbol{w}}(D, -1)$ is a del Pezzo surface of degree 4 when $\frac{1}{3} < w < \frac{3}{5}$;
- $\mathfrak{Bun}_{\boldsymbol{w}}(D, -1)$ is a del Pezzo surface of degree 5 when $\frac{3}{5} < w < 1$.

- **We would like to extend these results to the cases where D is not necessary reduced.**



Irregular rank 2 connections

- $\nabla \colon E \longrightarrow E \otimes \Omega^1_{\mathbb{P}^1}(D)$: connection
- Locally: $\nabla = d + \left(\frac{A^{(i)}_{n_i}}{z_i^{n_i}} + \cdots + \frac{A^{(i)}_1}{z_i} + \text{holomorphic} \right) dz_i$
- We assume unramified: $A^{(i)}_{n_i}$ has distinct eigenvalues.
  - **Hukuhara, Levelt, Turritin**: there exists a unique formal splitting $E = \hat{L}_i^+ \oplus \hat{L}_i^-$ such that $\hat{L}_i^+, \hat{L}_i^-$ are ∇-invariant.
    In a good formal trivialization of E: $\hat{L}_i^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\hat{L}_i^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and

$$\nabla = d + \begin{pmatrix} \frac{\lambda^+_{i,n_i}}{z_i^{n_i}} + \cdots + \frac{\lambda^+_{i,1}}{z_i} & 0 \\ 0 & \frac{\lambda^-_{i,n_i}}{z^n} + \cdots + \frac{\lambda^-_{i,1}}{z} \end{pmatrix} dz_i$$

  - **Formal invariants**: $(\lambda_i^+, \lambda_i^-)$, where $\lambda_i^\pm = \left(\frac{\lambda^\pm_{i,n_i}}{z_i^n} + \cdots + \frac{\lambda^\pm_{i,1}}{z_i} \right) dz_i$.
  - **Parabolic structure**: $\hat{L}_i^+$ modulo $z_i^{n_i}$ (eigendirection of principal part)

Flatness criterium

- $\boldsymbol{\lambda} = (\lambda_i^+, \lambda_i^-)_{1\le i\le \nu}$, where $\lambda_i^{\pm} = \left(\frac{\lambda_{i,n_i}^{\pm}}{z_i^{n_i}} + \cdots + \frac{\lambda_{i,1}^{\pm}}{z_i}\right) dz_i$
  - $\sum_{i=1}^{\nu} \mathrm{Res}(\lambda_i^+ + \lambda_i^-) = -d$
  - $\sum \mathrm{Res}_{t_i} \lambda_i^{\epsilon_i} \notin \mathbb{Z}$ whatever are $\epsilon_i \in \{+,-\}$
- We say $(E, \boldsymbol{l})$ is ***$\boldsymbol{\lambda}$-flat*** if $\exists \nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ s.t. the formal invariants of ∇ at t_i are $(\lambda_i^+, \lambda_i^-)$.
- Then: $(E, \boldsymbol{l})$ simple $\Rightarrow$ $(E, \boldsymbol{l})$ is $\boldsymbol{\nu}$-flat $\Rightarrow$ $(E, \boldsymbol{l})$ is undecomposable.

Proposition (Arinkin-Fedorov '12, K.–Loray–Saito)

Let $(E, \boldsymbol{l})$ be undecomposable. Then $(E, \boldsymbol{l})$ is $\boldsymbol{\lambda}$-flat iff, for any nilpotent endomorphism N of $(E, \boldsymbol{l})$, we have

$$\sum_{i=1}^{\nu} \mathrm{Res}(N|l_i \cdot (\lambda_i^+ - \lambda_i^-)) = 0,$$

where $N|l_i$ denotes the action induced by N on l_i.

Examples

- $D = 2[0] + 2[1] + [\infty]$, $E = \mathcal{O} \oplus \mathcal{O}(1)$
- the parabolic structure is given by

$$\begin{pmatrix} c_0 x \\ 1 \end{pmatrix} \mod x^2, \quad \begin{pmatrix} c_1(x-1) \\ 1 \end{pmatrix} \mod (x-1)^2, \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ at } \infty.$$

  - When $(c_0, c_1) \neq (0,0)$, $(E, \boldsymbol{l})$ is undecomposable.
  - There is (up to homothecy) one non trivial nilpotent endomorphism

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{that preserves the parabolic structure.}$$

- Criterium writes

$$\mathrm{Res}(c_0 x(\lambda_0^+ - \lambda_0^-)) + \mathrm{Res}(c_1(x-1)(\lambda_1^+ - \lambda_1^-)) = 0,$$

which define a point on $(c_1 : c_2) \in \mathbb{P}^1$. That is, if c_0, c_1 satisfies

$$c_0(\lambda_{0,2}^+ - \lambda_{0,2}^-) + c_1(\lambda_{1,2}^+ - \lambda_{1,2}^-) = 0,$$

then the parabolic bundle with these c_0, c_1 is $\boldsymbol{\lambda}$-flat.

Introduction

(1) When D is reduced,

$$\text{simple} \iff \boldsymbol{\lambda}\text{-flat} \iff \text{undecomposable}$$

$$\text{undecomposable} \iff \boldsymbol{w}\text{-stable for some } \boldsymbol{w} \qquad (*)$$

- When D is not reduced,
  - simple $\Longrightarrow$ $\boldsymbol{\lambda}$-flat $\Longrightarrow$ undecomposable
  - $\exists$ a undecomposable parabolic bundle which is not $\boldsymbol{\lambda}$-flat.
  - $\exists$ a $\boldsymbol{\lambda}$-flat parabolic bundle which is not simple.
  - $\exists$ a undecomposable parabolic bundle which is not $\boldsymbol{w}$-stable for any $\boldsymbol{w}$.
- To give a counterpart of $(*)$, we will define *tameness* later.

(2) The moduli space of $\boldsymbol{w}$-stable parabolic bundles is **not** projective in general.

  - We impose that each l_i is free in the definition of parabolic bundles.
  - To construct a good compactification of the moduli space, we will define a *refined parabolic bundle*.

Refined parabolic bundles

Definition

- We say $(E, \{l_{i,\bullet}\}_{i\in I})$ is a *refined parabolic bundle of rank 2 and of degree d* if
  - E is a rank 2 vector bundle on $\mathbb{P}^1$ with $\deg(E) = d$ and
  - $l_{i,\bullet} = \{l_{i,k}\}_{1\leq k\leq n_i}$ is a filtration

$$E|_{n_i[t_i]} \supset l_{i,n_i} \supset l_{i,n_i-1} \supset \cdots \supset l_{i,1} \supset 0$$

  of $\mathcal{O}_{n_i[t_i]}$-modules where the length of $l_{i,k}$ is k.
- We call this filtration a *refined parabolic structure* at t_i.

- For a parabolic str. l_i, we have a refined parabolic str. $l_i \supset z_i l_i \supset \cdots \supset z_i^{n_i-1} l_i$.
- The notion of refined parabolic bundles is contained in the notion of parabolic bundles in **Yokogawa '93**.
- To construct good compactifications of some moduli spaces of unramified irregular parabolic **connections**, **Miyazaki '13** used the refined parabolic structures.
  - The compactifications are isomorphic to Okamoto's initial condition spaces

Type of refined parabolic structures

- Set $V_j^{(i)} := E(-(n_i - j + 1)[t_i])/E(-n_i[t_i])$ for $j = 1, 2, \dots, n+1$.
- Then we have a filtration of $\mathcal{O}_{n_i[t_i]}$-modules:

$$E|_{n_i[t_i]} = V_{n_i+1}^{(i)} \supset V_{n_i}^{(i)} \supset V_{n_i-1}^{(i)} \supset \cdots \supset V_2^{(i)} \supset V_1^{(i)} = 0$$

Definition

- Let $l_{i,k}$ be an $\mathcal{O}_{n_i[t_i]}$-submodule of $E|_{n_i[t_i]}$ with $\mathrm{length}(l_{i,k}) = k$.

  We define a partition of the integer k: $\mu^{(i)} := (\mu_{n_i}^{(i)}, \mu_{n_i-1}^{(i)}, \dots, \mu_1^{(i)})$ by

$$\mu_j^{(i)} := \mathrm{length}\left((V_{j+1}^{(i)} \cap l_{i,k})/(V_j^{(i)} \cap l_{i,k})\right) \in \{0, 1, 2\}$$

- For a refined parabolic structure at t_i, we have a sequence of partitions of the integers $n_i, n_i - 1, \dots, 1$ (standard tableau).

  We call the sequence the *type of the refined parabolic structure*.

Examples $n_i = 3$. That is, $z_i^3 = 0$

(i) $(1,1,1) \supset (0,1,1) \supset (0,0,1)$ (which are ordinary parabolic structures)

$$(\mathbb{C} + \mathbb{C}z_i + \mathbb{C}z_i^2)\begin{pmatrix} f \\ 1 \end{pmatrix} \supset (\mathbb{C} + \mathbb{C}z_i)\begin{pmatrix} z_i f \\ z_i \end{pmatrix} \supset \mathbb{C}\begin{pmatrix} z_i^2 f \\ z_i^2 \end{pmatrix}$$

(ii) $(0,1,2) \supset (0,0,2) \supset (0,0,1)$

$$\underbrace{\mathbb{C}\begin{pmatrix} z_i^2 \\ 0 \end{pmatrix} \oplus (\mathbb{C} + \mathbb{C}z_i)\begin{pmatrix} 0 \\ z_i \end{pmatrix}}_{\text{fix}} \supset \mathbb{C}\begin{pmatrix} z_i^2 \\ 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ z_i^2 \end{pmatrix} \supset \mathbb{C}\begin{pmatrix} k_1 z_i^2 \\ k_2 z_i^2 \end{pmatrix}, \quad (k_1 : k_2) \in \mathbb{P}^1$$

(iii) $(0,1,2) \supset (0,1,1) \supset (0,0,1)$

$$\underbrace{\mathbb{C}\begin{pmatrix} z_i^2 \\ 0 \end{pmatrix} \oplus (\mathbb{C} + \mathbb{C}z_i)\begin{pmatrix} 0 \\ z_i \end{pmatrix}}_{\text{fix}} \supset (\mathbb{C} + \mathbb{C}z_i)\begin{pmatrix} k z_i^2 \\ z_i \end{pmatrix} \supset \mathbb{C}\begin{pmatrix} 0 \\ z_i^2 \end{pmatrix}, \quad (k : 1) \in \mathbb{P}^1$$

- If we fix l_{i,n_i} (for the last 2 items), the refined parabolic bundles in the last 2 items are parametrized by a chain of 2-projective lines.

Stability of refined parabolic bundles

- We take weights: $\boldsymbol{w} = (\boldsymbol{w}_1, \dots, \boldsymbol{w}_n), \boldsymbol{w}_i = (w_{i,n_i}, \dots, w_{i,1}) \in [0,1]^n$ such that
$$0 \leq w_{i,n_i} \leq \cdots \leq w_{i,1} \leq 1 \quad \text{for any } i \in \{1, 2, \dots, \nu\}.$$
- Set
$$\epsilon_{i,k}(L) := \begin{cases} -1 & \text{when } \mathrm{length}((l_{i,k} \cap L|_{n_i[t_i]})/(l_{i,k-1} \cap L|_{n_i[t_i]})) \neq 0 \\ 1 & \text{when } \mathrm{length}((l_{i,k} \cap L|_{n_i[t_i]})/(l_{i,k-1} \cap L|_{n_i[t_i]})) = 0 \end{cases},$$
$$\mathrm{stab}_L^{\boldsymbol{w}}(E, \boldsymbol{l}) := \deg(E) - 2\deg(L) + \sum_{i=1}^{\nu} \sum_{k=1}^{n_i} \epsilon_{i,k}(L) \cdot w_{i,k}.$$
- $\mathrm{stab}^{\boldsymbol{w}}(E, \boldsymbol{l}) := \inf \{\mathrm{stab}_L^{\boldsymbol{w}}(E, \boldsymbol{l}) \ ; \ L \subset E \text{ a line bundle}\}$.

Definition

A refined parabolic bundle $(E, \boldsymbol{l})$ is *$\boldsymbol{w}$-stable* (resp. *$\boldsymbol{w}$-semistable*) if
$$\mathrm{stab}^{\boldsymbol{w}}(E, \boldsymbol{l}) > 0 \qquad (\text{resp. } \geq 0).$$

- Construction of the moduli space of $\boldsymbol{w}$-stable refined parabolic bundles is due to **Yokogawa '93**.

Main result (1)

- For a line subbundle $L \subset E$ and $i \in I := \{1, 2, \ldots, \nu\}$,

$$N_i(L) := \max_{k' \in \{1,2,\ldots,n_i\}} \left\{ \sum_{k=1}^{k'} \epsilon_{i,k}(L) \right\}.$$

  We set $I_L^+ = \{i \in I \mid N_i(L) > 0\}$.

- We say $(E, \{l_{i,\bullet}\}_{i \in I})$ is *tame* if this refined parabolic bundle satisfies the following condition: For any line subbundles L such that $\deg(E) \leq 2\deg(L)$,
  - I_L^+ is not empty, and
  - $-\deg(E) + 2\deg(L) + 1 \leq \sum_{i \in I_L^+} N_i(L)$

Theorem (K.–Loray–Saito)

- If $(E, \{l_{i,\bullet}\}_{i \in I})$ is a refined parabolic bundle, then

  undecomposable and tame $\Longleftrightarrow$ w-stable for some w

- If l_{i,n_i} are free for any i, then

  simple $\Longleftrightarrow$ w-stable for some w

Example

- $D = 2[0] + 2[1] + [\infty]$, $E = \mathcal{O} \oplus \mathcal{O}(1)$, $L = \mathcal{O}(1)$
- the parabolic structure is given by

$$\begin{pmatrix} c_0 x \\ 1 \end{pmatrix} \quad \text{mod} \quad x^2, \quad \begin{pmatrix} c_1(x-1) \\ 1 \end{pmatrix} \quad \text{mod} \quad (x-1)^2, \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ at } \infty.$$

  - Assume that $c_0 c_1 \neq 0$. So it is undecomposable.
  - $\epsilon_{1,1}(L) = -1, \epsilon_{1,2}(L) = 1$, $\epsilon_{2,1}(L) = -1, \epsilon_{2,2}(L) = 1$, $\epsilon_{3,1}(L) = 1$.
  - $N_0(L) = \max\{-1, -1+1\} = 0$, $N_1(L) = \max\{-1, -1+1\} = 0$, $N_3 = 1$. So $\sum_{i \in I_L^+} N_i(L) = 1$.
  - On the other hand, $-\deg(E) + 2\deg(L) + 1 = 2$

- So this (refined) parabolic bundle is **not** tame

  (remark that this parabolic bundle is **not** simple. We can exclude this example.)

Elementary transformations of refined parabolic bundles

- A filtration of sheaves associated to a refined parabolic structure $l_{i_0,\bullet}$:

$$E \supset E_{i_0}^{(n_{i_0})} \supset E_{i_0}^{(n_{i_0}-1)} \supset \cdots \supset E_{i_0}^{(1)} \supset E_{i_0}^{(0)} = E(-n_{i_0}[t_{i_0}])$$

- By this filtration, we may define a new filtration as follows:

$$E'_{i_0} = E_{i_0}^{(n_{i_0})} \supset E_{i_0}^{(0)} \supset E_{i_0}^{(1)}(-[t_{i_0}]) \supset E_{i_0}^{(2)}(-2[t_{i_0}]) \supset \cdots$$
$$\cdots \supset E_{i_0}^{(n_{i_0}-1)}(-(n_{i_0}-1)[t_{i_0}]) \supset E_{i_0}^{(n_{i_0})}(-n_{i_0}[t_{i_0}]).$$

- By this filtration, we obtain a new refined parabolic structure $\mathrm{elm}(l_{i_0,\bullet})$ at t_{i_0}.

Definition

We fix i_0 where $i_0 \in I := \{1, 2, \ldots, \nu\}$. We define the *elementary transformation of $(E, \{l_{i,\bullet}\}_{i\in I})$ at t_{i_0}* by

$$\mathrm{elm}_{i_0}(E, \{l_{i,\bullet}\}_{i\in I}) = (E'_{i_0}, \{l_{i,\bullet}\}_{i\in I\setminus\{i_0\}} \cup \{\mathrm{elm}(l_{i_0,\bullet})\}).$$

The degree of E'_{i_0} is $d - n_{i_0}$.

Properties of the elementary transformations

- T_0: a sequence of integers the $n_{i_0}, n_{i_0}-1, \ldots, 2, 1$:
$$(1^{n_{i_0}}) \supset (1^{n_{i_0}-1}) \supset \cdots \supset (1^2) \supset (1)$$
- T_k: a sequence of integers the $n_{i_0}, n_{i_0}-1, \ldots, 2, 1$:
$$(1^{n_{i_0}-2}, 2^1) \supset \cdots \supset (1^{n_{i_0}-k-1}, 2^1) \supset (1^{n_{i_0}-k}, 2^0) \supset \cdots \supset (1^1, 2^0)$$
for $k = 1, 2, \ldots, n_{i_0}-1$.

Proposition (K.–Loray–Saito)

- If the type of $l_{i_0,\bullet}$ is T_0, then the type of $\mathrm{elm}(l_{i_0,\bullet})$ is T_0.
- If $l_{i_0,\bullet}$ is a generic refined parabolic str. with type T_k, then the type of $\mathrm{elm}(l_{i_0,\bullet})$ is T_{n_0-k}.

- $0 \le w_{i,n_i} \le \cdots \le w_{i,1} \le 1$ $(i = 1, 2, \ldots, \nu)$
$$w'_{i_0,n_{i_0}-k} := 1 - w_{i_0,k+1} \quad \text{and} \quad w'_{i,k} := w_{i,k} \ (i \neq i_0)$$

Proposition (K.–Loray–Saito)

- $\mathrm{stab}_L^{\boldsymbol{w}}(E, \boldsymbol{l}) = \mathrm{stab}_{L'}^{\boldsymbol{w}'}(\mathrm{elm}_{i_0}(E, \{l_{i,\bullet}\}_{i\in I}))$

$\frac{1}{n} < w < \frac{1}{n-2}$

- We assume that $w_{i,n_i} = \cdots = w_{i,2} = w_{i,1} = w$ for any i.
- $\overline{\mathfrak{Bun}_{\boldsymbol{w}}}(D, d)$: the moduli space of $\boldsymbol{w}$-stable rank 2 refined parabolic bundles with degree d over $(\mathbb{P}^1, D)$.

Proposition

Set $\frac{1}{n} < w < \frac{1}{n-2}$ and $d = 1$. **$\boldsymbol{w}$-stable iff satisfying the following conditions**:

- $E \cong \mathcal{O} \oplus \mathcal{O}(1)$;
- $(E, \boldsymbol{l})$ is an undecomposable parabolic bundle (l_i is free $\forall i$);
- $l_i^{\mathrm{red}} \notin \mathcal{O}(1) \subset \mathbb{P}(E)$ $\forall i$

- For $(E, \boldsymbol{l}) \in \overline{\mathfrak{Bun}_{\boldsymbol{w}}}(D, 1)$,

$$0 \longrightarrow (\mathcal{O}(1), \emptyset) \longrightarrow (E, \boldsymbol{l}) \longrightarrow (\mathcal{O}, \boldsymbol{l}) \longrightarrow 0.$$

- So we have the following isomorphisms:

$$\overline{\mathfrak{Bun}_{\boldsymbol{w}}}(D, 1) \cong \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}(-1) \otimes \Omega^1_{\mathbb{P}^1}(D))^* \cong \mathbb{P}^{n-3}$$

When $n = 5$

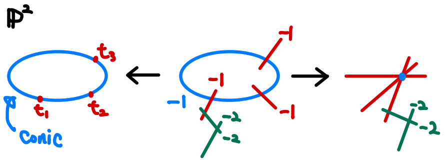
- We set $n = \deg(D) = 5$. ($\overline{\mathfrak{Bun}}_{\boldsymbol{w}}(D, d)$ is a surface).

Theorem (K.–Loray–Saito)

- $\overline{\mathfrak{Bun}}_{\boldsymbol{w}}(D, 1)$ is empty when $0 < w < \frac{1}{5}$;
- $\overline{\mathfrak{Bun}}_{\boldsymbol{w}}(D, 1) = \mathbb{P}^2$ when $\frac{1}{5} < w < \frac{1}{3}$;
- $\overline{\mathfrak{Bun}}_{\boldsymbol{w}}(D, 1)$ is a weak del Pezzo surface of degree 4 when $\frac{1}{3} < w < \frac{3}{5}$;
- $\overline{\mathfrak{Bun}}_{\boldsymbol{w}}(D, 1)$ is a weak del Pezzo surface of degree 5 when $\frac{3}{5} < w < 1$.

- $D = 3[t_1] + [t_2] + [t_3]$
- The chain of (-2)-curves corresponding to the refined parabolic structures with type

$$(0,1,2) \supset (0,1,1) \supset (0,0,1) \quad \text{and} \quad (0,1,2) \supset (0,0,2) \supset (0,0,1)$$



Geometry of a weak del Pezzo surface

- We assume that $w_{i,n_i} = \cdots = w_{i,2} = w_{i,1} = \frac{1}{2}$ for any i.
- We put $D = 2[t_1] + [t_2] + [t_3] + [t_4]$.
- Then $\overline{\mathfrak{Bun}}_{\frac{1}{2}}(D, 1) \cong \mathrm{wdP}^{(4)}_{A_1}$
  - Here, $\mathrm{wdP}^{(4)}_{A_1}$ is a weak del Pezzo surface of degree 4 whose configuration of (-2)-curves is A_1.

- **By this identification, we would like to recover the geometry of $\mathrm{wdP}^{(4)}_{A_1}$ from the modular point of view.**
- In particular, we consider the automorphism group of the surface $\mathrm{wdP}^{(4)}_{A_1}$.

- We have $\mathrm{elm}_i \colon \overline{\mathfrak{Bun}}_{\frac{1}{2}}(D, 1) \xrightarrow{\cong} \overline{\mathfrak{Bun}}_{\frac{1}{2}}(D, 1 - n_i)$.
- So, if $n_i + n_j$ is even, then

$$\mathrm{elm}_{i,j} := \mathcal{O}(\frac{n_i + n_j}{2}) \otimes (\mathrm{elm}_i \circ \mathrm{elm}_j)$$

gives an automorphism of $\overline{\mathfrak{Bun}}_{\frac{1}{2}}(D, 1)$.

Automorphism group of $\mathrm{wdP}_{A_1}^{(4)}$

- **Now we reconstruct all automorphisms of $\mathrm{wdP}_{A_1}^{(4)}$ via elementary transformations.**
- $D = 2[t_1] + [t_2] + [t_3] + [t_4]$
- Remark that, if t_1, t_2, t_3, t_4 have a generic cross-ratio, $\mathrm{Aut}(\mathbb{P}^1, D) = \langle \mathrm{id} \rangle$.

Proposition

If $\mathrm{Aut}(\mathbb{P}^1, D) = \langle \mathrm{id} \rangle$, then

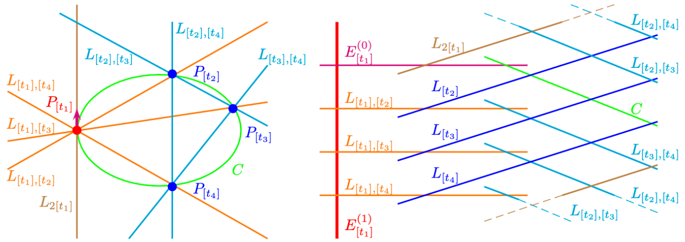
$$\mathrm{Aut}(\mathrm{wdP}_{A_1}^{(4)}) \cong \langle \mathrm{elm}_{2,3}, \mathrm{elm}_{3,4}, \mathrm{elm}_{1,1} \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

- First, we may check

$$\mathrm{Aut}(\mathrm{wdP}_{A_1}^{(4)}) \supset \langle \mathrm{elm}_{2,3}, \mathrm{elm}_{3,4}, \mathrm{elm}_{1,1} \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

Automorphism group of $\mathrm{wdP}^{(4)}_{A_1}$

- For showing $\mathrm{Aut}(\mathrm{wdP}^{(4)}_{A_1}) \subset \langle \mathrm{elm}_{2,3}, \mathrm{elm}_{3,4}, \mathrm{elm}_{1,1} \rangle$, first, we investigate how $\mathrm{elm}_{2,3}, \mathrm{elm}_{3,4}, \mathrm{elm}_{1,1}$ act on the negative curve on $\mathrm{wdP}^{(4)}_{A_1}$.



- Let $\phi \in \mathrm{Aut}(\mathrm{wdP}^{(4)}_{A_1})$.
- Second, we are able to show that there exists $\phi_0 \in \langle \mathrm{elm}_{2,3}, \mathrm{elm}_{3,4}, \mathrm{elm}_{1,1} \rangle$ such that $\phi_0 \circ \phi$ stabilizes all negative curves.
- $\phi_0 \circ \phi$ induces a linear automorphism on $\mathbb{P}^1$ which preserves the conic C and fixes the 4-points on this conic. Then $\phi_0 \circ \phi = \mathrm{id}$.