Universal Variations of Hodge Structures and Finite Orbits of the Mapping Class Group.

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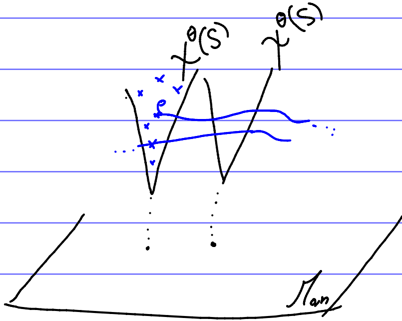
0. Motivation
1. Complex Variations of Hodge structures ($\mathbb{C}$VHS)
2. Finite orbits $\rightsquigarrow$ (universal) $\mathbb{C}$VHS.
3. $SL_2\mathbb{C}$-universal $\mathbb{C}$VHS

0. Motivation

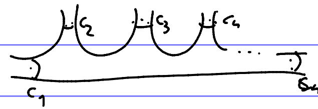
$S = S_{0,n}$, $\rho : \pi_1 S \longrightarrow SL_2\mathbb{C}$.

$$\left\{\begin{array}{c}\text{algebraic}\\ \text{leaves of}\\ \text{isomonodromic}\\ \text{foliation}\end{array}\right\} \xleftrightarrow[\substack{\text{Iwasaki '08}\\ \text{Cousin '17}\\ \text{Cousin-Heu '21}}]{} \left\{\begin{array}{c}\text{Finite orbits}\\ \text{of Mapping}\\ \text{Class}\\ \text{Group}\end{array}\right\} \xleftrightarrow[\substack{\text{Fuchs}\\ \text{Schlesinger}}]{(n=4)} \left\{\begin{array}{c}\text{Algebraic}\\ \text{Solutions}\\ \text{to Painlevé VI}\end{array}\right\}$$



$$PMod^+(S_{0,n}) \cong \pi_1 \mathcal{M}_{0,n}$$
$$= \mathrm{Diff}^+(S_{0,n}) / \mathrm{Diff}^0$$

$S = S_{0,n}$

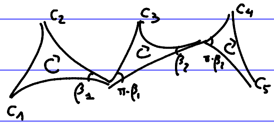


Thm [B'-Maret '24]

Finite Orbits with elliptic peripheral monodromies, $\rho(c_i) \neq 1$ are exactly

- finite image.
- abelian image.
- ($n=6$) one one-parameter family
- ($n=5$) two one-par. family and 1 special case. (Conjecture Tykhyy '15)
- ($n=4$) ⇝ list of Lisovyy-Tykhyy '14.

Key: study the case



Today: reduce the classification to this case.

1/ Complex Variations of Hodge Structures

$\mathcal{E}, D$ flat bundle over X is (polarized) $\mathbb{C}$VHS $\rightsquigarrow$ $\mathcal{E} = V_1 \oplus \ldots \oplus V_\ell$

$$D = \begin{bmatrix} D_1 & \nu_1 & & (0) \\ \eta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \nu_{\ell-1} \\ (0) & & \eta_{\ell-1} & D_\ell \end{bmatrix}$$

(V_i, D_i) h_i-unitary.

$\nu_1, \ldots, \nu_{\ell-1}$ $(0,1)$-forms.

$\eta_1, \ldots, \eta_{\ell-1}$ $(1,0)$-forms.

$\nu_i^{*_{h_i h_{i+1}}} = \eta_i$

$\rightsquigarrow$ to understand via Non-abelian Hodge Correspondence. (Simpson '92)

<u>rmk</u>: The monodromy is $\hookrightarrow SU(p,q) \subset SL_n\mathbb{C}$.

Non-Abelian Hodge Correspondence

Corlette
Donaldson
Uhlenbeck Hitchin
Simpson Yau Yokohama
+ ...

$$\left\{\begin{matrix}\text{Flat Bundles}\\ \text{over } S\\ \text{rk } n\\ (E,D)\end{matrix}\right\}/\sim \quad \xleftrightarrow{\ NAH\ } \quad \left\{\begin{matrix}\text{Polystable Higgs}\\ \text{Bundles } (\mathcal{E},\phi)\\ \mathcal{E} \text{ holomorphic}, \Lambda^n\mathcal{E}=\mathcal{O}\\ \phi\in H^0(K\,\mathrm{End}_0(\mathcal{E}))\end{matrix}\right\}/\sim$$

$$\left\{\begin{matrix}\text{Harmonic}\\ \text{Bundles}\\ (\mathcal{E},\bar\partial,\phi,H) \text{ s.t.}\\ D=\bar\partial+\partial^{*_H}+\phi+\phi^{*_H} \text{ flat}\end{matrix}\right\}/\sim$$

$\mathbb{C}^\times$-action $z\cdot(\mathcal{E},\phi) = (\mathcal{E},z\phi)$

tractible definition

<u>Simpson</u>:

$$\mathbb{C}^\times\text{-fixed pts} \longleftrightarrow \text{CVHS} \longleftrightarrow \left\{\begin{matrix}(\mathcal{E},\bar\partial)= V_1\oplus\dots\oplus V_\ell\\ \phi(V_i)\subset K\cdot V_{i+1}\end{matrix}\right.$$

Examples

1. Unitary $\Longrightarrow$ CVHS $(\Phi = 0)$

2. $(\mathcal{E}, \phi) = K^{1/2} \oplus K^{-1/2}$, $\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\rightsquigarrow$ Poincaré Uniformization.

3. $(\mathcal{E}, \phi) = L \oplus L^{-1}$, $\phi = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$ $\rightsquigarrow$ Branched hyperbolic structures.

Fact: In rk 2, these are essentially all the CVHS.

(branched $\mathbb{H}$-structures)

2/ Finite Orbits $\dashrightarrow$ CVHS

$\rho : \pi_1 S_{g,n} \longrightarrow G$ finite MCG-orbit, irreducible

Birman Sequence $\quad \begin{array}{c} S_{g,n} \to \mathcal{M}_{g,n+1} \\ \downarrow \\ \mathcal{M}_{g,n} \end{array} \quad 1 \to \pi_1 S_{g,n} \to MCG_{g,n+1} \xrightarrow{p} MCG_{g,n} \to 1$

ρ finite orbit $\rightsquigarrow$ $\Gamma = \mathrm{Stab}([\rho]) \underset{f.i.}{<} MCG_{g,n}$

$\tilde{\Gamma} = p^{-1}(\Gamma) \underset{f.i.}{<} MCG_{g,n+1} \simeq \pi_1 \mathcal{M}_{g,n+1}$

$\gamma \in \tilde{\Gamma} \rightsquigarrow p(\gamma) \cdot \rho = g_\gamma \rho g_\gamma^{-1}$ $\quad g_\gamma = \hat{\rho}(\gamma)$ $\rightsquigarrow$ $\hat{\rho} : \tilde{\Gamma} \to G$ extends ρ.

Corlette–Simpson Alternative

V quasiprojective, $\rho : \pi_1 V \to SL_2\mathbb{C}$ irreducible, w. quasi-unipotent monodromy at infinity

① If ρ nonrigid; $\exists Y$ (virtually) 1-orbifold, $f: V \to Y$, $\eta : \pi_1 Y \to SL_2\mathbb{C}$ s.t. $\rho = \eta \circ f_*$

② ρ rigid $\Rightarrow$ ρ integral $\rightsquigarrow$ in $SU(n, \mathcal{O}_L)$
CVHS

Apply to $\hat{\rho}$ $\rightsquigarrow$ ① ρ "Pullback" Doran '01; Watanabe '08, Diana '17

$\rightsquigarrow$ ② ρ integral + CVHS for <u>any</u> Riemann Surface Structure

universal

Universal $\mathbb{C}$VHS in $SL_2\mathbb{C}$ $(g=0)$

① $SU(2)$ ✓

② A compact component $\mathcal{C}$ in $\mathcal{X}^{\Theta}(SL_2\mathbb{R})$ Benedetto-Goldman '01
Deroin-Tholozan '16 ; Mondello '16

<u>Prop</u>: $\rho \in \mathcal{C}$, S_{ρ} Riemann Surf. Structure $\rightsquigarrow \exists!\, f : \tilde{S} \to \mathbb{H}$ holomorphic ρ-equivariant.

<u>Schwarz</u> $\rightsquigarrow f$ 1-Lipschitz $\rightsquigarrow \rho$ is totally elliptic

$\rho(\text{scc}) \subset$ Elliptics

Universal CVHS in $PSL(2,\mathbb{R})$

Higgs Bundle $\quad L_\bullet \oplus L_\bullet^{-1} \qquad \phi = \begin{pmatrix} 0 & \delta \\ \eta & 0 \end{pmatrix}$

weights $\alpha_1, \dots, \alpha_n > \frac{1}{2}$

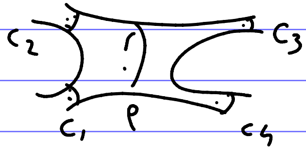
$\leftrightarrow$ angles $\theta_1, \_\_\_, \theta_n > \pi$ $\qquad \underbrace{K_S L_\bullet^{2}}_{n-2-2d^\bullet L} < 0 \Rightarrow \eta = 0$

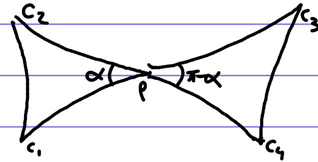
$\rightsquigarrow d^\bullet L = \frac{n-1}{2}$

Parabolic Degree.

Stability? ok provided $\sum \theta_i > 2(n-1)\pi$ $\quad \left(d^\bullet(L_\bullet) < 0 \right)$

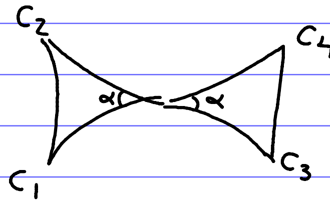
Case $n=4$:





Euler Class -1

OR



Euler Class 0.

Cantat-Loray '07 Totally Elliptic $\Longleftrightarrow$ Euler Class $= -1$ $\Longleftrightarrow$ Universal CVHS.

Recap: ρ finite orbit, irreducible.

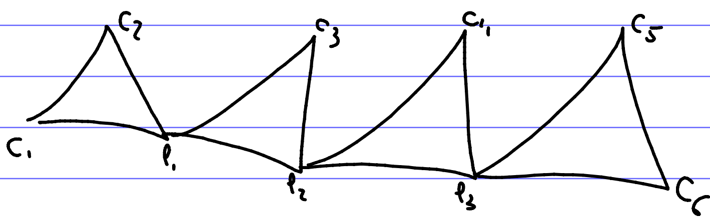
1: non unipotent monodromy at infinity? $\rightsquigarrow$ Loray-Pereira-Touzet '14

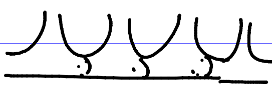
2: parabolics (non-elliptic)? $\rightsquigarrow$ Lam-Landesman-Litt '23.

3: Birman + Corlette-Simpson $\rightsquigarrow$ ρ Pullback? $\rightsquigarrow$ Watanabe '08, Diarra '17

4: ρ integral, universal CVHS $\rightsquigarrow$ All Galois conjugates are $SU(2)$ on Triangle Chains

$\rightsquigarrow$ ρ finite image on some Galois conjugate is



for 

Thank You!