Monodromy problem and the boundary condition for some isomonodromy deformation equations

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The Painlevé VI

- The nonlinear differential equation

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{2}\Big[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\Big]\Big(\frac{dy}{dx}\Big)^2-\Big[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\Big]\frac{dy}{dx}\\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\Big[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\Big],\ \alpha,\beta,\gamma,\delta\in\mathbb{C}.\end{aligned}$$

- A solution $y(x)$ of $\widetilde{\text{PVI}}$ has $0,1,\infty$ as critical points, is meromorphic on $\mathbb{P}^1\setminus\{0,1,\infty\}$. Following the book of Fokas-Its-Kapaev-Novokshenov, "Solving" PVI means

  a. to find the explicit asymptotics of $y(x)$ at critical points:

  $$y(x)\sim y_p(x;a_p,\sigma_p),\ \text{as } x\to p,\ p\in\{0,1,\infty\},$$

  a_p,σ_p are asymptotic parameters/integration constants;

  b. to find the explicit connection formula of $y(x)$ between two different critical points $p\neq q\in\{0,1,\infty\}$,

  $$a_p=a_p(a_q,\sigma_q),\qquad \sigma_p=\sigma_p(a_q,\sigma_q).$$

- Solved by Jimbo (generic cases) and by Shimomura, Dubrovin, Mazzocco, Guzzetti, Boalch, Kaneko,Bruno-Goryuchkina...

The isomonodromy equation ISO_n

- $\mathfrak{h}_{\text{reg}}$ the set of diagonal matrices u with distinct eigenvalues, write $u = \text{diag}(u_1, ..., u_n)$.
- Consider the nonlinear differential equation for a $n \times n$ matrix valued function $\Phi(u) : \mathfrak{h}_{\text{reg}} \to \mathfrak{gl}(n)$

$$\frac{\partial \Phi}{\partial u_k} = [\Phi, \text{ad}_u^{-1}\text{ad}_{E_{kk}}\Phi], \text{ for all } k = 1, ..., n.$$

where $\text{ad}_u^{-1}\text{ad}_{E_{kk}}\Phi$ is the unique off-diagonal matrix satisfying

$$[u, \text{ad}_u^{-1}\text{ad}_{E_{kk}}\Phi] = [E_{kk}, \Phi].$$

- In terms of components $\Phi = (\Phi_{ij}(u_1, ..., u_n))_{i,j=1,...,n}$,

$$\frac{\partial \Phi_{ij}}{\partial u_l} = \frac{\Phi_{in}\Phi_{nj}}{u_i - u_l} + \frac{\Phi_{in}\Phi_{nj}}{u_l - u_j}, \ \textit{for } i, j \neq l;$$
$$\frac{\partial \Phi_{lj}}{\partial u_l} = \frac{\Phi_{ll}\Phi_{lj}}{u_l - u_j} + \sum_{k \neq l} \frac{\Phi_{lk}\Phi_{kj}}{u_k - u_l}, \ \textit{for } j \neq l.$$

- time $(u_1, ..., u_n)$-dependent Hamiltonian equations.

The ISO_3

- By Harnad, if there exists parameters $\theta_1, \theta_2, \theta_3, \theta_\infty$

$$\begin{aligned} \mathrm{diag}(\Phi(u)) &= \mathrm{diag}(\theta_1, \theta_2, \theta_3), \\ \mathrm{eigenvalues}\Phi(u) &= 0,\ (\theta_1+\theta_2+\theta_3-\theta_\infty),\ (\theta_1+\theta_2+\theta_3+\theta_\infty), \end{aligned}$$

ISO_3 for $\Phi(u)$ is equivalent to the Painlevé VI $y(x)$ with

$$x = \frac{u_2 - u_1}{u_3 - u_1}$$

and the parameters

$$2\alpha = (\theta_\infty - 1)^2,\ 2\beta = -\theta_1^2,\ 2\gamma = -\theta_3^2,\ 2\delta = -\theta_2^2.$$

The asymptotics [a] and connection problem [b] of Painlevé VI amounts to the study of asymptotics of $\Phi(u)$ of ISO_3 as

$$\frac{u_2 - u_1}{u_3 - u_1} \to \infty,\quad \frac{u_3 - u_1}{u_2 - u_1} \to \infty,\quad \frac{u_2 - u_1}{u_2 - u_3} \to \infty,$$

respectively, and the connection problem between them.

- Following Miwa, the solutions $\Phi(u)$ have the Painlevé property: they are multi-valued meromorphic functions of $u_1, ..., u_n$ and the branching occurs when u moves along a loop around the fat diagonal

$$\Delta = \{(u_1, ..., u_n) \in \mathbb{C}^n \mid u_i = u_j, \text{for some } i \neq j\}.$$

According to Painlevé, they can be a new class of special functions.

- Analog to PVI,

(a). The parametrization of solutions by their asymptotic behaviour at critical points;

(b). The explicit connection formula from one critical point to another.

The asymptotics at a critical point

- $\frac{u_3-u_2}{u_2-u_1} \to \infty, \frac{u_4-u_3}{u_3-u_2} \to \infty, \quad \ldots \quad , \frac{u_n-u_{n-1}}{u_{n-1}-u_{n-2}} \to \infty$
- $\frac{u_3-u_2}{u_2-u_1} \to 0, \ \frac{u_4-u_3}{u_3-u_2} \to 0, \quad \ldots \quad , \frac{u_n-u_{n-1}}{u_{n-1}-u_{n-2}} \to 0$

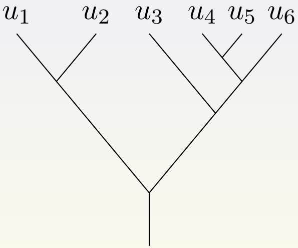


Figure: A planar binary rooted tree with 6 leaves colored by $u_1, ..., u_6$.

- $\frac{u_2-u_1}{u_3-u_2} \to 0, \ \frac{u_5-u_4}{u_6-u_5} \to 0, \ \frac{u_6-u_5}{u_4-u_3} \to 0, \ \frac{u_4-u_3}{u_3-u_2} \to 0$

Hamiltonian description

• The isomonodromy equation with respect to the derivation of u_j is generated by the time dependent quadratic Hamiltonian

$$H_j := \sum_{k\neq j} \frac{\Phi_{kj}\Phi_{jk}}{u_k - u_j},$$

where Φ_{ij}'s are the entry functions on $\mathfrak{gl}_n$.

• As $\frac{u_3-u_2}{u_2-u_1} \to \infty, \frac{u_4-u_3}{u_3-u_2} \to \infty, \quad \ldots \quad , \frac{u_n-u_{n-1}}{u_{n-1}-u_{n-2}} \to \infty$, the Hamiltonians behave like

$$H_j^0 = -\sum_{k<j} \frac{\Phi_{kj}\Phi_{jk}}{u_j - u_{j-1}}, \text{ for } j = 2, ..., n.$$

Given any matrix A, $\{H_j^0\}$ generate a Hamiltonian flow

$$\mathrm{Ad}_{\left((u_2-u_1)^{\delta_1(A)} \times \overrightarrow{\prod_{k=2,\ldots,n-1}} \left(\frac{u_{k+1}-u_k}{u_k-u_{k-1}}\right)^{\delta_k(A)}\right)} A.$$

- skew-Hermitian valued solution $\Phi(u)$ of ISO_n is real analytic on connected component $U_{\mathrm{id}} = (u_1 < u_2 < \cdots < u_n)$ of $\mathfrak{h}_{\mathrm{reg}}(\mathbb{R})$.
- $z_1 = u_2 - u_1, \quad z_2 = \frac{u_3-u_2}{u_2-u_1}, \quad \dots \quad , z_{n-1} = \frac{u_n - u_{n-1}}{u_{n-1}-u_{n-2}}$

Theorem (Xu)

For any skew-Hermitian valued solution $\Phi(u)$ of ISO_n on U_{id}, there exists a unique constant skew-Hermitan matrix Φ_0 such that as $z_k \to \infty$ for all $k = 2, ..., n-1$,

$$\Phi(u) = \mathrm{Ad}_{\left(z_1^{\delta_1(\Phi_0)} \times \overrightarrow{\prod_{k=2,\dots,n-1}} z_k^{\delta_k(\Phi_0)}\right)} \Phi_0 + O\left(z_2^{-1}, ..., z_{n-1}^{-1}\right),$$

where

$$\delta_k(\Phi_0)_{ij} = \begin{cases} (\Phi_0)_{ij}, & \textit{if} \quad 1 \le i, j \le k, \ \textit{or} \ i = j \\ 0, & \textit{otherwise.} \end{cases}$$

The converse is also true.

Parametrization of solutions in generic complex case

Theorem (Tang-X)

For any generic solution $\Phi(u)$ of ISO_n, $\exists$ $n \times n$ matrix-valued functions Φ_0, $\Phi_1(z_1)$,... $\Phi_{n-2}(z_1, ..., z_{n-2})$ such that

$$\lim_{z_k \to \infty} z_k^{\delta_{k-1}\Phi_{k-1}} \Phi_k z_k^{-\delta_{k-1}\Phi_{k-1}} = \Phi_{k-1}, \quad k = 1, ..., n-1$$

and there exists real numbers $\varepsilon_k > 0$ such that

$$\sup\{|\mathrm{Re}(\lambda_i^{(k)}(\Phi_0) - \lambda_j^{(k)}(\Phi_0))| : 1 \leqslant i, j \leqslant k\} = 1 - \varepsilon_k,$$

($\lambda_i^{(k)}$ eigenvalues of left-top $k \times k$ submatrix) and as $z_k \to \infty$,

$$\Phi(u) = \mathrm{Ad}\left(z_1^{\delta_1(\Phi_0)} \times \overrightarrow{\prod_{k=2,...,n-1}} z_k^{\delta_k(\Phi_{k-1})}\right)\Phi_0 + O\left(z_2^{-\varepsilon_2}, ..., z_{n-1}^{-\varepsilon_{n-1}}\right).$$

Converse is true. By $\Phi(u; \Phi_0)$ solution with asymtotoics Φ_0.
Relation with works of Mochizuki, Guest-Its-Lin.

Parametrization at another asymptotic zone

Theorem (Tang-X)

For any generic solution $\Phi(u)$ of ISO_n, $\exists$ $n\times n$ matrix-valued functions $\widetilde{\Phi}_0$, $\widetilde{\Phi}_1(z_1)$,... $\widetilde{\Phi}_{n-2}(z_1,...,z_{n-2})$ such that

$$\lim_{z_k\to 0} z_k^{\eta_{k-1}\widetilde{\Phi}_{k-1}}\widetilde{\Phi}_k z_k^{-\eta_{k-1}\widetilde{\Phi}_{k-1}} \quad = \quad \widetilde{\Phi}_{k-1}, \quad k=1,...,n-1$$

and there exists real numbers $\varepsilon_k>0$ such that

$$\sup\{|\mathrm{Re}(\zeta_i^{(k)}(\widetilde{\Phi}_0)-\zeta_j^{(k)}(\widetilde{\Phi}_0))| : 1\leqslant i,j\leqslant k\} = 1-\varepsilon_k,$$

($\zeta_i^{(k)}$ eigenvalues of right-bottom k submatrix) and as $z_k\to 0$,

$$\Phi(u) = \mathrm{Ad}_{\left(z_1^{\eta_1(\widetilde{\Phi}_0)}\times \overrightarrow{\prod_{k=2,...,n-1}} z_k^{\eta_k(\widetilde{\Phi}_{k-1})}\right)}\widetilde{\Phi}_0 + O\left(z_2^{\varepsilon_2},...,z_{n-1}^{\varepsilon_{n-1}}\right).$$

The converse is also true. Denote by $\Phi(u;\widetilde{\Phi}_0)$ the solution with prescribed asymtotoics $\widetilde{\Phi}_0$.

Part II: problem (b)

The connection formula between two critical points

The connection problem

- Given any fixed generic solution $\Phi(u)$ of ISO_n, we have

$$\Phi_0, \text{the boundary value } z_k \to \infty;$$
$$\widetilde{\Phi}_0, \text{the boundary value } z_k \to 0.$$

- The connection problem is to find the explicit expression of Φ_0 as a function of $\widetilde{\Phi}_0$.

- From local analysis to global analysis. As $n = 3$, it was solved by Jimbo as the connection formula for Painlevé VI.

$$\begin{tikzcd}
\Big\{\text{Solutions } \Phi(u)\Big\} \arrow[r, "\text{as } \frac{u_{k+1}-u_k}{u_k-u_{k-1}} \to \infty"] \arrow[d, "\text{as } \frac{u_{k+1}-u_k}{u_k-u_{k-1}} \to 0"'] & \Big\{\Phi_0 \in \mathfrak{gl}_n\Big\} \arrow[d] \\
\Big\{\widetilde{\Phi}_0 \in \mathfrak{gl}_n\Big\} \arrow[r] & \Big\{\text{monodromy of linear problem}\Big\}
\end{tikzcd}$$

The linear Riemann-Hilbert problem

• The isomonodromy equation ISO_n is compatibility condition of the equation for a function $F(z; u_1, ..., u_n) \in \mathrm{GL}_n$

$$\begin{aligned} \frac{\partial F}{\partial z} &= \left(u + \frac{\Phi(u)}{z}\right) \cdot F, \\ \frac{\partial F}{\partial u_k} &= \left(E_{kk} z + \mathrm{ad}_u^{-1} \mathrm{ad}_{E_{kk}} \Phi(u)\right) \cdot F, \quad k = 1, ..., n. \end{aligned}$$

• For fixed u, the ODE has a unique formal solution $\hat{F}$ at $z = \infty$. $\exists$ sectorial regions around $z = \infty$, such that on each sector there is a unique holomorphic solution with prescribed asymptotics $\hat{F}$. The transition between the solutions are measured by upper and lower triangular matrices $S_\pm(u, \Phi(u))$.

• Varying u, the Stokes matrices $S_\pm(u, \Phi(u))$ of linear system are locally constant (independent of u), and are first integrals.

• The Riemann-Hilbert problem is to express explicitly of $S_\pm(u, \Phi(u; \Phi_0))$ via the boundary value Φ_0 of $\Phi(u; \Phi_0)$.

The explicit Stokes matrices via Φ_0

- $\{\lambda_i^{(k)}\}_{i=1,\ldots,k}$ eigenvalues of left-top $k \times k$ submatrix of Φ_0.

Theorem

The sub-diagonals of the Stokes matrices $S_\pm(u, \Phi(u;\Phi_0))$ are

$$(S_+)_{k,k+1} = \sum_{i=1}^{k} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(1+\lambda_l^{(k)}-\lambda_i^{(k)})}{\prod_{l=1}^{k+1} \Gamma(1+\lambda_l^{(k+1)}-\lambda_i^{(k)})} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(\lambda_l^{(k)}-\lambda_i^{(k)})}{\prod_{l=1}^{k-1} \Gamma(1+\lambda_l^{(k-1)}-\lambda_i^{(k)})} \cdot \Delta^{1,\ldots,k-1,k}_{1,\ldots,k-1,k+1},$$

where $k = 1, \ldots, n-1$ and $\Delta^{1,\ldots,k-1,k}_{1,\ldots,k-1,k+1}(\lambda_i^{(k)} - \Phi_0)$ is the k by k minor of the matrix $\lambda_i^{(k)} - \Phi_0$ formed by the first k rows and $1, \ldots, k-1, k+1$ columns. Furthermore, the other entries are given by explicit expressions via the sub-diagonal ones.

The explicit Stokes matrices via $\widetilde{\Phi}_0$

- $\{\zeta_i^{(k)}\}_{i=1,\dots,k}$ eigenvalues of right-bottom k submatrix of $\widetilde{\Phi}_0$.

Theorem

The $(k,k+1)$ entry of the Stokes matrix $S_+(u,\Phi(u;\widetilde{\Phi}_0))$ is

$$\sum_{i=1}^{k}\frac{\prod_{l=1,l\neq i}^{k}\Gamma(1+\zeta_l^{(k)}-\zeta_i^{(k)})}{\prod_{l=1}^{k+1}\Gamma(1+\zeta_l^{(k+1)}-\zeta_i^{(k)})}\frac{\prod_{l=1,l\neq i}^{k}\Gamma(\zeta_l^{(k)}-\zeta_i^{(k)})}{\prod_{l=1}^{k-1}\Gamma(1+\zeta_l^{(k-1)}-\zeta_i^{(k)})}\cdot\Delta_{n-k,n-k+2,\dots,n}^{n-k+1,\dots,n},$$

where $k=1,\dots,n-1$ and $\Delta_{n-k,n-k+2,\dots,n}^{n-k+1,\dots,n}(\zeta_i^{(k)}-\widetilde{\Phi}_0)$ is the k by k minor of the matrix $\zeta_i^{(k)}-\widetilde{\Phi}_0$ formed by the last k rows and $n-k,n-k+2,\dots,n$ columns. Furthermore, the other entries are given by explicit expressions via the sub-diagonal ones.

- Motivated by the theory of Gelfand-Tsetlin and KZ equation, the quantization of ISO_n equation introduced by Reshetikhin.

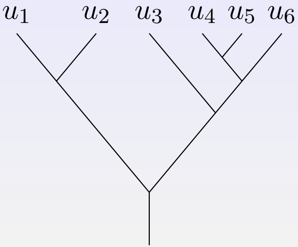
From the space of Φ_0 with the boundary condition to the space of Stokes matrices via the equivalences

$$
\begin{aligned}
& \Big\{\Phi_0 \mid \ \sup\{|\mathrm{Re}(\lambda_i^{(k)}(\Phi_0)-\lambda_j^{(k)}(\Phi_0))| : 1\leqslant i,j\leqslant k\}<1\Big\} \\
\Longleftrightarrow\ & \Big\{\text{solutions } \Phi(u;\Phi_0) \text{ of the isomonodromy equation}\Big\} \\
\Longleftrightarrow\ & \Big\{\text{meromorphic linear system of PDEs}\Big\} \\
\Longleftrightarrow\ & \Big\{\text{space of Stokes matrices } S_\pm(u,\Phi(u;\Phi_0)) = S_\pm(u,\Phi(u;\widetilde{\Phi}_0))\Big\} \\
\Longleftrightarrow\ & \Big\{\widetilde{\Phi}_0 \mid \ \sup\{|\mathrm{Re}(\zeta_i^{(k)}(\widetilde{\Phi}_0)-\zeta_j^{(k)}(\widetilde{\Phi}_0))| : 1\leqslant i,j\leqslant k\}<1\Big\}
\end{aligned}
$$

Diagram:

$$
\begin{tikzcd}
\Big\{\text{Solutions } \Phi(u)\Big\} \arrow[r, "\text{as } \frac{u_{k+1}-u_k}{u_k-u_{k-1}}\to\infty"] \arrow[d, "\text{as } \frac{u_{k+1}-u_k}{u_k-u_{k-1}}\to 0"'] & \Big\{\Phi_0\in\mathfrak{gl}_n\Big\} \arrow[d] \\
\Big\{\widetilde{\Phi}_0\in\mathfrak{gl}_n\Big\} \arrow[r] & \Big\{S_\pm(u,\Phi(u;\Phi_0)) = S_\pm(u,\Phi(u;\widetilde{\Phi}_0))\Big\}
\end{tikzcd}
$$

Other asymptotic zones



As $\frac{u_2-u_1}{u_3-u_2} \to 0$, $\frac{u_5-u_4}{u_6-u_5} \to 0$, $\frac{u_6-u_5}{u_4-u_3} \to 0$, $\frac{u_4-u_3}{u_3-u_2} \to 0$, there exists a boundary value A of a generic solution $\Phi(u)$. However, to write down explicitly $S_\pm(u, \Phi(u; A))$, one needs to compute the Stokes matrices of

$$\frac{dF}{dz} = \left(\begin{pmatrix} 0_m & 0 \\ 0 & \mathrm{Id}_{n-m} \end{pmatrix} + \frac{1}{z}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)F.$$

- Branching rules of $\mathfrak{gl}_m \times \mathfrak{gl}_{n-m} \subset \mathfrak{gl}_n$.

Remained questions

- None generic boundary value Φ_0, i.e.,

$$\sup\{|\mathrm{Re}(\lambda_i^{(k)}(\Phi_0) - \lambda_j^{(k)}(\Phi_0))| : 1 \leqslant i, j \leqslant k\} = 1,$$

should relate to the resonant cases. From explicit expression, the Stokes matrices $S_\pm(u, \Phi(u; \Phi_0))$ have poles as

$$|\lambda_i^{(k)}(\Phi_0) - \lambda_j^{(k)}(\Phi_0)| = 1.$$

- Algebraic solutions.
- The space of initial values.
- The WKB approximation (joint with A. Alekseev, A. Neitzke and Y. Zhou).
- Quantization, relations with quantum groups and crystal basis and so on (Chekhov, M. Mazzocco and V. Rubtsov).
- Difference analog, relations with elliptic quantum groups.

Thank you very much!