Quantum Interaction and Number Theory, Representation Theory - modular forms a bit beyond, infinite symmetric group, Fuchsian ODE

Masato Wakayama

NTT Institute for Fundamental Mathematics,
(Kyushu University and Tokyo Inst. Tech.)

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I: Quantum Rabi model (QRM) and asymmetric quantum Rabi model (AQRM) – Degeneration and Hidden symmetry.

II: Heat kernel of the QRM.

III: Heun pictures for the AQRM and Non-commutative harmonic oscillator (NCHO) – Covering models.

IV: Number theory behind the NCHO.

QRM: quantum Rabi model
AQRM: asymmetric quantum Rabi model
NCHO: non-commutative harmonic oscillator

A large part of the talk is based on the joint works with Cid Reyes-Bustos (**CRB**, NTT) and Kazufumi Kimoto (**KK**, Univ. Ryukyus). The study of NCHO was by jointly begun with Alberto Parmeggiani (Bologna, 1998) and later with Takashi Ichinose (Kanazawa, 2003).

Part I: Quantum Rabi model and asymmetric quantum Rabi model

- 1936: I.I. Rabi introduced the semi-classical Rabi model to study the effect of a weak magnetic field on an oriented atom possessing nuclear spin.
- 1963: E. Jaynes and F. Cummings considered the fully quantized version of the Rabi model (= QRM). Jaynes-Cummings' model, the rotating wave approximation of QRM, was also introduced.
- 2011: D. Braak proved "integrability" of the QRM using the $\mathbb{Z}_2$-symmetry. *Breakthrough!*
- 2018: F. Yoshihara, *et al.* confirmed that the experimental measurements matched the theoretical values predicted by the QRM & AQRM (using superconducting artificial atoms.

QRM

The Hamiltonian of **quantum Rabi model (QRM)**:

$$H_{\text{Rabi}} = \omega a^\dagger a + \Delta\sigma_z + g\sigma_x(a^\dagger + a)$$

acting on $L^2(\mathbb{R}; \mathbb{C}^2) = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where

- $2\Delta > 0$: the energy difference between the two levels system,
- $g > 0$: the coupling strength between the two-level system and the bosonic mode with frequency ω (may assume $\omega = 1$),
- $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: the Pauli matrices,
- $a^\dagger$ and a: the creation and annihilation operators; $[a, a^\dagger] = 1$.

AQRM

The Hamiltonian $H^{\epsilon}_{\text{Rabi}}$ of the **asymmetric QRM** [i] :

$$H^{\epsilon}_{\text{Rabi}} = \omega a^{\dagger} a + \Delta \sigma_z + g \sigma_x (a^{\dagger} + a) + \epsilon \sigma_x,$$

where $\epsilon \in \mathbb{R}$ (may assume $\omega = 1$)

- is a self-adjoint unbounded operator,
- has only a discrete spectrum with uniformly bounded (with multiplicity ≤ 2), and
- does not have the $\mathbb{Z}_2$-symmetry (parity) <u>except when $\epsilon = 0$</u> (i.e. $H_{\text{Rabi}} = H_+ \oplus H_-$).

[i] Jaynes-Cummings' model (the RWA of QRM) has a $U(2)$-symmetry.

Classification of spectrum

Classification of eigenvalues λ of the AQRM:

1. **Exceptional**: if $\lambda = N \pm \epsilon - g^2$ for $N \in \mathbb{Z}_{\geq 0}$.
2. **Regular:** if λ is not exceptional. Regular eigenvalues are known to be non-degenerate.

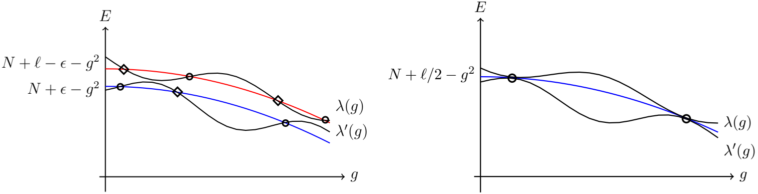
Exceptional eigenvalues λ are

1. **Juddian**: Exceptional eigenvalues with polynomial solutions (eigenfunctions), i.e. terminating power series.
2. **Non-Juddian exceptional**: Exceptional eigenvalues that are not Juddian.

Degeneracy on the spectrum of AQRM

The picture illustrates how exceptional solutions appear in the spectral curves of the AQRM.

- $2\epsilon \notin \mathbb{Z} \Rightarrow$ No apparent degeneracy (Left)
- $2\epsilon \in \mathbb{Z} \Rightarrow$ Exist an apparent degeneracy (Right).



$\circ$: Judddian

$\diamond$: non-Juddian

Constraint polynomials

<u>*Constraint polynomials*</u> $P_N^{(N,\epsilon)}(x,y)$[ii] associated to the Juddian solutions defined by the following recurrence relation ($k=N$) gives the clear description of the degeneracy of the spectrum of AQRM.

The recurrence relation $\{P_k^{(N,\epsilon)}(x,y)\}_{k\geq 0}$:

$$
\begin{aligned}
P_0^{(N,\epsilon)}(x,y) &= 1,\\
P_1^{(N,\epsilon)}(x,y) &= x+y-1-2\epsilon,\\
P_k^{(N,\epsilon)}(x,y) &= (kx+y-k^2-2k\epsilon)P_{k-1}^{(N,\epsilon)}(x,y)\\
&\qquad\qquad - k(k-1)(N-k+1)xP_{k-2}^{(N,\epsilon)}(x,y).
\end{aligned}
$$

[ii]Derived by the confluent Heun picture *or* $U(\mathfrak{sl}_2)$ pictures of AQRM.

Constraint polynomials and relations

How to get Juddian solutions?

Juddian solutions via constraint relation[iii]

If g and Δ satisfy

$$P_N^{(N,\epsilon)}((2g)^2, \Delta^2) = 0$$

then $\lambda = N + \epsilon - g^2$ is a Juddian eigenvalue.

[iii] Z.M. Li, M. Batchelor: J. Phys. A: Math. Theor. **48** (2015). *Add.* **49** (2016).

Divisibility of constraint polynomials

What was remarkable is there *exists a crossing* (i.e. degeneracy)! It was *not expected* before (Li-Batchelor's numerical observation for $\epsilon = 1$ in 2015-16), but when $\epsilon = \frac{\ell}{2} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

Theorem[iv]

For $\ell, N \in \mathbb{Z}_{\geq 0}$, $\exists A_N^{\ell}(x, y) \in \mathbb{Z}[x, y]$ s.t.

$$P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y) = A_N^{\ell}(x, y) P_N^{(N,\ell/2)}(x, y).$$

The polynomial $A_N^{\ell}(x, y)$[v] is positive for any $x, y > 0$. □

[iv] KK, CRB, MW: IMRN **2021-12**. Conjectured in MW: J. Phys. A: Math. Theor. **50** (2017).

[v] For a fixed degree $\ell \in \mathbb{Z}_{\geq 0}$, the polynomial equation $A_N^{\ell}(x, y) = 0$ defines certain algebraic curve depending on the parameter N: the case $\ell = 2$ is parabolic, $\ell = 3$ gives an elliptic curve and $\ell = 4$ super elliptic curve, etc.

Degeneracy on the spectrum of AQRM

Theorem[vi]

The degeneracy of the spectrum of $H^{\epsilon}_{\text{Rabi}}$ occurs only when $\epsilon = \ell/2$ for $\ell \in \mathbb{Z}_{\geq 0}$. Furthermore, in this case

- all degenerate eigenvalues are Juddian and any Juddian eigenvalue is degenerate,
- any non-Juddian exceptional solution is non-degenerate.

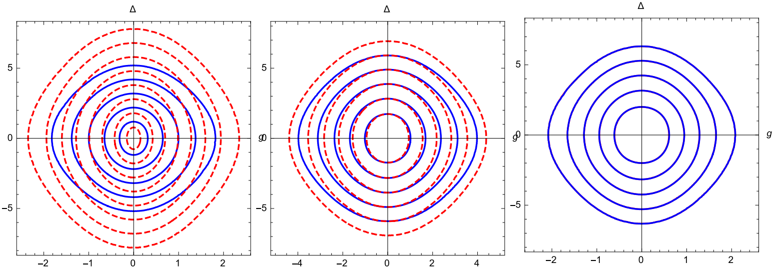
[vi] KK, CRB, MW: IMRN **2021-12**.

Illustration of degeneracy

The constraint relation

$$P_N^{(N,\epsilon)}((2g)^2, \Delta^2) = 0$$

describes a curve in the g, Δ-plane:



$\epsilon = 0.3$ $\epsilon = 1$ $\epsilon = 3/2$

Curves: $P_5^{(5,\epsilon)}((2g)^2, \Delta^2) = 0$ and $P_8^{(8,-\epsilon)}((2g)^2, \Delta^2) = 0$.
Eigenvalues: $\lambda = 5 + \epsilon - g^2$ and $\lambda = 8 - \epsilon - g^2$.

When $\epsilon \to 3/2$ the curves are going to be identical.

"Hidden symmetry" in the AQRM

"It has been observed that the AQRM, which does not possess any obvious symmetry, exhibits energy level crossings, which are often associated with symmetries. ... even though simple inspection of the model and its Hamiltonian does not reveal the nature of this symmetry..." S. Ashhab: *Attempt to find the hidden symmetry in the asymmetric quantum Rabi model*, Phys. Rev. A **101** (2020).

$\mathcal{P} := \exp(i\pi a^\dagger a)$: the photon number parity operator ($\mathcal{P}^2 =$id).

Theorem[vii] [viii]

$\exists Q_0^{(\ell)} (\notin \mathbb{C}[H_{\mathrm{Rabi}}^{\ell/2}]) \in \mathrm{Mat}_2(\mathbb{Q}[a, a^\dagger])$(unique up to const.) satisfying

1. $J_\ell := \mathcal{P} Q_0^{(\ell)}$ (self-adjoint) $\Rightarrow$ $[H_{\mathrm{Rabi}}^{\ell/2}, J_\ell] = 0$,
2. $J_\ell^2 = p_\ell(H_{\mathrm{Rabi}}^{\ell/2}; g, \Delta)$ for $\exists p_\ell \in \mathbb{R}[x, g, \Delta]$ of degree ℓ w.r.t. x,
3. $[H_{\mathrm{Rabi}}^{\ell/2}, \mathcal{P} Q] = 0$ ($Q \in \mathrm{Mat}_2(\mathbb{C}[a, a^\dagger])$) $\Rightarrow$ $Q = Q_0^{(\ell)} p(H_{\mathrm{Rabi}}^{\ell/2})$ for $\exists p \in \mathbb{C}[x]$.

[vii] V. Mangazeev, M. Batchelor, V. Bazhanov: J. Phys. A: Math. Theor. **54** (2021) shows the statements 1 and 2 ($\ell \leq 2$).

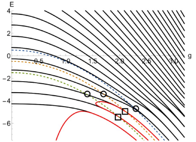
[viii] CRB, D. Braak, MW: J. Phys. A: Math. Theor. **54** (2021).

The degenerate eigenvalues may know the entire spectrum of the system.

Main conjecture

$$\ell \in \mathbb{Z}_{\geq 0}, N \geq \ell \Rightarrow p_\ell(N + \tfrac{\ell}{2} - g^2; g, \Delta) = A^\ell_N((2g)^2, \Delta^2)$$

The main conjecture is true ⇒



$\Delta = 4, \ell = 3$

Grey: spectral curve on (g,E)-plane

Red: curve $p_\ell(x, g, \Delta) = 0$

Dotted line: the baseline curve $E = i - \frac{\ell}{2} - g^2 (i = 0,1,2)$

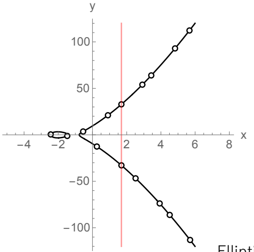
○, □: exceptional eigenvalue

Note: 1. The main conjecture[ix] is described also by an (explicit) *determinant expression* of $p_\ell(x, g, \Delta)$.
2. (Generalized) adiabatic approximation of the spectrum is known to be given by the constraint polynomials (or degeneration).

[ix] CRB, MW: Comm. Numb. Theor. Phys. **16** (2022).

Points in the elliptic curve given by $y^2 = p_3(x; g, \Delta)$ for $\Delta = 3/7$ and $g \approx 0.899$.

x=E (eigen), y= the eigen. of J_E ($J_E{}^2 = p_\ell(E; g, \Delta)$)



Elliptic curve

Conjecture: $p_\ell(E)>0$,
if E is not a non-Juddian exceptional

Part II: Heat kernel of the quantum Rabi model

The heat kernel $K_{\mathrm{Rabi}}(x, y, t)$ of H_{Rabi} is the (matrix valued) function satisfying[x]

- $\frac{\partial}{\partial t} K_{\mathrm{Rabi}}(x, y, t) = -H_{\mathrm{Rabi}} K_{\mathrm{Rabi}}(x, y, t)$ for all $t > 0$,
- $\lim_{t \to 0} K_{\mathrm{Rabi}}(x, y, t) = \delta_x(y) \mathbf{I_2}$ for $x, y \in \mathbb{R}$.

[x]Note:

$$e^{-tH_{\mathrm{Rabi}}} \phi(x) = \int_{-\infty}^{\infty} K_{\mathrm{Rabi}}(x, y, t) \phi(y) dy \quad (\phi \in C_c^{\infty}(\mathbb{R}; \mathbb{C}^2))$$

Analytical formula of the heat kernel

Theorem[xi] (Uniformly convergent series of iterated integrals)

The heat kernel of the QRM is given by

$$K_{\text{Rabi}}(t,x,y) = \widetilde{K}_0(x,y,g,t) \sum_{\lambda=0}^{\infty} (t\Delta)^{\lambda} \Phi_\lambda(x,y,g,t).$$

For $\lambda \geq 0$ the 2×2 matrix-valued function $\Phi_\lambda(g,t)$ is given by

$$\Phi_\lambda(x,y,g,t) = \int \cdots \int_{0 \leq \mu_1 \leq \cdots \leq \mu_\lambda \leq 1} e^{\phi(\mu_\lambda,t)+\xi_\lambda(\boldsymbol{\mu_\lambda},t)} \begin{bmatrix} (-1)^\lambda \cosh & (-1)^{\lambda+1} \sinh \\ -\sinh & \cosh \end{bmatrix} \times (\theta_\lambda(x,y,\boldsymbol{\mu_\lambda},t))\, d\boldsymbol{\mu_\lambda}.$$

Here $\boldsymbol{\mu_\lambda} = (\mu_1, \mu_2, \cdots, \mu_\lambda)$ and $d\boldsymbol{\mu_\lambda} = d\mu_1 d\mu_2 \cdots d\mu_\lambda$, $(\boldsymbol{\mu_0} = 0,\ d\boldsymbol{\mu_0} = 1)$.
$\widetilde{K}_0(x,y,g,t) = e^{tg^2} K(x,y,t)$, $K(x,y,t)$ being Mehler's kernel of the harmonic oscillator.

[xi] CRB, MW: ATMP (To appear), J. Phys. A: Math. Theor. **54** (2021).

Small history

- for the <u>*Kondo model*</u> (a model for a quantum impurity coupled to a large reservoir of non-interacting electrons), a special matrix coefficient of the heat kernel was obtained by Anderson, Yuval and Hamann (Phys. Rev. B **1**, (1970)), Very similar to the above!
- the expression for the average of the Heisenberg operator at time t for the Spin-Boson model was computed Legget et al. (Rev. Mod. Phys. **59** (1987)), sm
- the expression of the heat kernel by a Feynman-Kac formula was given by Hirokawa and Hiroshima (Comm. Stoch. Anal. **8** (2014)).

Note: The methods above are based on path-integral or probabilistic techniques, but <u>*no analytical formula*</u> has been known (at least now).

Heat Kernel Formula – Geometrical Interpretation

$\mathcal{O}_\lambda := \{\sigma \in \mathbb{Z}_2^\infty : \ |\sigma| = \lambda\} = \mathfrak{S}_\infty \underbrace{[1,1,\ldots,1]}_{\lambda}$ ($\mathfrak{S}_\infty$- orbit)

(length $|\cdot| = \mathfrak{S}_\infty$-invariant)

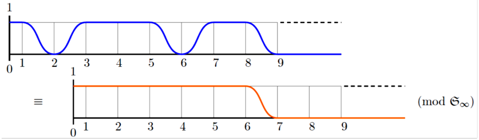
$(\underbrace{[1,1,\ldots,1]}_{\lambda} = \underbrace{(1,1,\ldots,1,}_{\lambda}0,0,\ldots\ldots) \in \mathbb{Z}_2^\infty)$

$\mathbb{Z}_2^\infty = \coprod_{\lambda=0}^\infty \mathcal{O}_\lambda$ (the orbital decomposition of $\mathbb{Z}_2^\infty$)

$$K_{\mathrm{QRM}}(t) = \sum_{\lambda=0}^{\infty} (t\Delta)^{2\lambda} \int_{\mathcal{O}_\lambda} \exp[\textit{Explicit expressions for exponential functions of } g,t,x,y,\mu_\lambda]\, d\mu_\lambda$$

(Orbital integral can be regarded as an integral over λ – simplex)

[equivalence class of paths]



Heat Kernel Formula – Algebraic (Group Theoretic Interpretation)

Irreducible decomposition of $\mathbb{Z}_2^\infty$ *under the action of* $\mathfrak{S}_\infty$:

$$\mathfrak{S}_\infty \curvearrowright \mathbb{Z}_2^\infty = \bigoplus_{\lambda=0}^{\infty} \mathrm{Ind}_{\mathfrak{S}_{\Pi_\lambda}}^{\mathfrak{S}_\infty} 1$$

$\mathfrak{S}_{\Pi_\lambda}$: Young subgroup $\mathfrak{S}_\infty$ corresponding to the partition

(Young diagram) $\Pi_\lambda = \Big(\underbrace{\square\square\cdots\square\square}_{\lambda}, \underbrace{\square\square\square\square\square\cdots}_{\infty}\Big).$

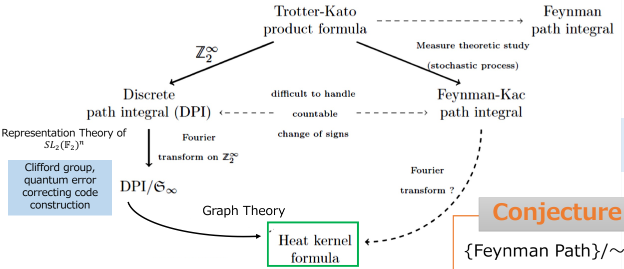
There is an idea (due to Ludvig Faddeev) that the construction of any irreducible representation can be obtained by Feynman path integrals for a Lie group via co-adjoint orbits. This is **reminiscent** of our analytical formula of the heat kernel.

Derivation process of the analytical formula

Product Formulas of Trotter-Kato ➡Heat Kernel, Propagator

A, B: Bounded operators from below ➡ $e^{-t(A+B)} = \lim_{N\to\infty} (e^{-tA/N} e^{-tB/N})^N.$

(No suitable additive measure)



$\mathbb{Z}_2^\infty := \lim_{\to n} \mathbb{Z}_2^n$
$\mathfrak{S}_\infty$: infinite symmetric group

Conjecture

{Feynman Path}/∼ ≈ { Discrete Path }

(equivalence classes by some infinite dimensional algebraic system e.g. group)

Part III: Heun pictures for the QRM and NCHO

By the correspondence (to the Bargmann space $\mathcal{B}$ picture)

$$\underline{a^\dagger \to z \text{ and } a \to \partial_z},$$

the eigenvalue problem $H^{\epsilon}_{\mathrm{Rabi}}\varphi = \lambda\varphi$ is equivalent[xii] to finding $\psi_1, \psi_2 \in \mathcal{B}$ satisfying

$$\begin{aligned}(z\partial_z + \Delta)\psi_1 + (g(z + \partial_z) + \epsilon)\psi_2 &= \lambda\psi_1,\\ (g(z + \partial_z) + \epsilon)\psi_1 + (z\partial_z - \Delta)\psi_2 &= \lambda\psi_2.\end{aligned}$$

[xii] The convergence condition of the (entire function) solution is automatically satisfied for this type of differential equations.

Confluent Heun picture for QRM

Appropriate change of the variable, the system can be transformed into the 2nd order (<u>*confluent Heun*</u>) ODE $\mathcal{H}_i^{\epsilon}(\lambda)f=0$, $\mathcal{H}_i^{\epsilon}$ $(i=1,2)$ [xiii], e.g., $i=1$:

$$\mathcal{H}_1^{\epsilon}(\lambda)=\frac{d^2}{dy^2}+\left(-4g^2+\frac{a+1}{y}+\frac{a-2\epsilon}{y-1}\right)\frac{d}{dy}+\frac{-4g^2ay+\mu+4\epsilon g^2-\epsilon^2}{y(y-1)}.$$

Here $a:=-(\lambda+g^2-\epsilon)$ and

$$\mu:=(\lambda+g^2)^2-4g^2(\lambda+g^2)-\Delta^2 \quad \text{(in the accessory parameter)}.$$

[xiii] D. Braak: PRL **107** (2011).

NCHO

Hamiltonian of the **non-commutative harmonic oscillator**[xiv]:

$$Q = Q_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left\{ -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \right\} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left\{ x\frac{d}{dx} + \frac{1}{2} \right\},$$

Assume $\alpha, \beta > 0$ and $\alpha\beta > 1$.

- a positive self-adjoint unbounded operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$
- Q has only a discrete spectrum with uniformly bounded multiplicity (≤ 2)[xv]:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots (\uparrow \infty).$$

[xiv] A. Parmeggiani, MW: PNAS **98** (2001).
[xv] MW: Proc. Japan Acad. **89** (2013).

Eigenvalue problem

Oscillator (or Weil) representation $(\pi', \mathbb{C}[y])$ of $\mathfrak{sl}_2 = \mathbb{C}[H, E, F]$:

$$\pi'(H) = y\partial_y + \frac{1}{2}, \quad \pi'(E) = y^2/2, \quad \pi'(F) = -\partial_y^2/2.$$

By an isometry $L^2(\mathbb{R}) \ni \varphi_n \mapsto y^n \in \overline{\mathbb{C}[y]}$, φ_n being the nth Hermite function, we have

Theorem[xvi]

There exists a quadratic element $\mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2)$ (explicitly given by α, β and λ) such that

$$Q\varphi = \lambda\varphi \;\Leftrightarrow\; \pi'(\mathcal{R})u = 0 \, (u \in \overline{\mathbb{C}[y]}).$$

[xvi] H. Ochiai: CMP **217** (2001).

Heun picture via Laplace intertwiner

Define the representation ϖ_a $(a \in \mathbb{C})$ of $\mathfrak{sl}_2$:

$$\varpi_2(H) = z\partial_z + \frac{1}{2},\ \varpi_2(E) = z^2(\frac{1}{2}z\partial_z + a),\ \varpi_2(F) = -\frac{1}{2z}\partial_z + \frac{a-1}{2z^2},$$

Define the modified Laplace transform $\mathcal{L}_a$ $(\Re a \geq 1)$:

$$(\mathcal{L}_a u)(z) = \int_0^\infty u(yz)e^{-\frac{y^2}{2}}y^{a-1}dy.$$

Theorem[xvii][xviii]

The restriction of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) to even (resp. odd) functions is an intertwiner between the representations π' and ϖ_1 (resp. ϖ_2).

[xvii] H. Ochiai: CMP **217** (2001): odd case.

[xviii] MW: IMRN **2016:3**: even case and a general principal series ϖ_a.

Heun picture of the NCHO

Theorem [xix]

There exist linear bijections:

$$\{\varphi \in L^2(\mathbb{R},\mathbb{C}^2) \mid Q\varphi = \lambda\varphi,\, \varphi(-x) = \pm\varphi(x)\} \xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^{\pm} f = 0\}.$$

- Ω: a simply-connected domain in $\mathbb{C}$ $(0, 1 \in \Omega,\ \alpha\beta \notin \Omega)$
- $H_\lambda^{\pm} = H_\lambda^{+}(w, \partial_w)$:

$$H_\lambda^{+}(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-p}{w} + \frac{-\frac{1}{2}-p}{w-1} + \frac{p+1}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{1}{2}\left(p+\frac{1}{2}\right)w - q^{+}}{w(w-1)(w-\alpha\beta)},$$

$$H_\lambda^{-}(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{1-p}{w} + \frac{-p}{w-1} + \frac{p+\frac{3}{2}}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{3}{2}pw - q^{-}}{w(w-1)(w-\alpha\beta)}.$$

- p and $q^{\pm}$ are explicitly given by α, β and λ.

[xix] H. Ochiai (2001), MW (2016).

NCHO（η-NCHO） & QRM (Asymmetric QRM).

The Weil Representation

NCHO ($\eta - NCHO$) $\xleftarrow[(\simeq \pi'_a (a=1,2))]{\text{Oscillator representation } \pi'}$ $\mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2)$ (degree 2) $\xrightarrow[\pi'_a \simeq \varpi_a \text{ (non-unitary principal series)}]{\text{Generalized Laplace transform } \mathcal{L}_a \text{ (Intertwiner)}}$ Heun ODE

4 regular singular points
0,1, t(=αβ), ∞

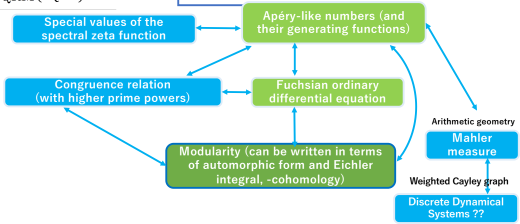
Heun ODE $\Downarrow$ confluence process

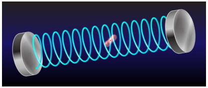
$\mathcal{K} \in \mathcal{U}(\mathfrak{sl}_2)$ (degree 2) Confluent Heun ODE

$\varpi_{\tilde{a}}$; Segal-Bargmann space representation ; QRM (AQRM)

NCHO is a covering model of QRM.

Number theoretic structure behind NCHO





Courtesy of APS/Alan Stonebraker (2011)
Light-matter interaction

○ Hamiltonian Q^η of η-NCHO :

$$Q^\eta := Q + 2i\eta\sqrt{\alpha\beta - 1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\eta \in \mathbb{R}).$$

Comparison between the η-NCHO and AQRM

	η-NCHO		AQRM	
	$\eta \notin \frac{1}{2}\mathbb{Z}$	$\eta \in \frac{1}{2}\mathbb{Z}$	$\eta \notin \frac{1}{2}\mathbb{Z}$	$\eta \in \frac{1}{2}\mathbb{Z}$
Type	$\lambda \in \Sigma_0$		Exceptional eigenvalues	
Form	$\lambda = \frac{2\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}\left(L + \frac{1}{2} + 2\eta\right)$		$E = N + \eta - g^2$	
Multiplicity	1	2	1	1 or 2
Eigenfunction	one polynomial or one Heun polynomial[a]	one polynomial and one Heun polynomial	one Juddian[b] or one non-Juddian exceptional[c]	one non-Juddian exceptional or two Juddian
Degeneracy		same parity		different parity[d]

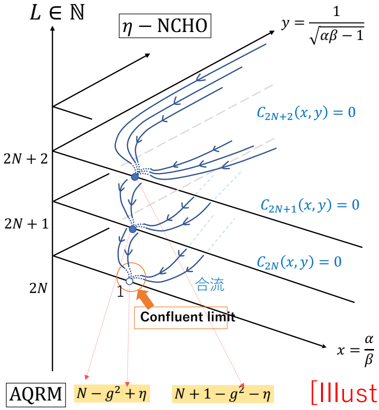
[a] Heun polynomial does not include usual polynomials.

[b] Juddian (or quasi-exact) solutions are given by the product of a polynomial and an exponential factor.

[c] Non-Juddian exceptional solutions are eigenfunctions that are not Juddian.

[d] There remains a delicate point for the definition of parity except when $\eta = 0$.

Simple and rough illustration



[Illustration]

Observation of the Covering

η-NCHO and Asymmetric quantum Rabi model

○ Confluence process : $\alpha\beta \to \infty,\ \alpha\beta\left(\frac{\alpha-\beta}{\alpha+\beta}\right) \to 0$, (particularly, $\frac{\alpha}{\beta} \to 1$).

○ On the algebraic curves $C_L(x,y) = 0$, the eigrnvalue $L + \frac{1}{2} + 2\eta$ of the η − NCHO is identical.

○ Along the curves $C_{2N}(x,y) = 0$ *and* $C_{2N+1}(x,y) = 0$, the eigenvalue $L + \frac{1}{2} + 2\eta$ $(L = 2N, 2N+1)$ is pushed down the same Juddian eigenvalue $N - g^2 + \eta$ of the AQRM by the confluence procedure.

xx

[xx]Joint work with CRB is in progress.

Part IV: Number theory behined the NCHO

Spectral zeta function[xxi] for the NCHO:

$$\zeta_Q(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} \quad (\Re(s) > 1)$$

- is meromorphically continued to the whole $\mathbb{C}$ with a unique simple pole at 1,
- has trivial zeros at the non-positive even integers, but no (known) functional equation. [xxii]

The main interest is special values $\zeta_Q(k)\,(k \geq 2)$.[xxiii]

[xxi] T. Ichinose, MW: CMP **258** (2005).
[xxii] For QRM: S. Sugiyama: Nagoya Math. J. (2018) & CRB, MW: J. Phys. A. (2021) (different way).
[xxiii] $\zeta_Q(s) = 2(\alpha^2 - 1)^{-\frac{s}{2}}(2^s - 1)\zeta(s)$ when $\alpha = \beta$.

Special values for $k \geq 2$ $(\alpha \neq \beta)$[xxiv] [xxv] [xxvi]

$$\zeta_Q(k) = 2\left(\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\right)^k \left(\zeta(k,1/2) + \sum_{0<2j\leq k}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{2j} R_{k,j}(\kappa)\right)$$

$$\text{with } R_{k,j}(\kappa) := \sum_{1\leq i_1<i_2<\cdots<i_{2j}\leq k} \int_{[0,1]^k} \frac{2^k du_1 \ldots du_k}{\sqrt{\mathcal{W}_k(\boldsymbol{u};\kappa;i_1,\ldots,i_{2j})}} \quad \left(\kappa := \frac{1}{\sqrt{\alpha\beta-1}}\right),$$

$$\mathcal{W}_k(\boldsymbol{u};\kappa;i_1,\ldots,i_{2j}) := \det\left(\Delta_k(\boldsymbol{u}) + \kappa\Xi_k(i_1,\ldots,i_{2j})\right)\prod_{r=1}^{k}(1-u_r^4).$$

($\Delta_k(\boldsymbol{u}) + \kappa\Xi_k(\boldsymbol{i}) \in \mathrm{Sym}_k^{\times}$ for $\boldsymbol{u} \in (0,1)^k$: be explicitly given.)

Ex.
$$R_{2,1}(\kappa) = \int_{[0,1]^2} \frac{4\,du_1\,du_2}{\sqrt{(1-u_1^2u_2^2)^2 + \kappa^2(1-u_1^4)(1-u_2^4)}},$$

$$R_{3,1}(\kappa) = 3\int_{[0,1]^3} \frac{8\,du_1\,du_2\,du_3}{\sqrt{(1-u_1^2u_2^2u_3^2)^2 + \kappa^2(1-u_1^4)(1-u_2^4u_3^4)}}.$$

[xxiv] T. Ichinose, MW: Kyushu J. Math. **59** (2005).
[xxv] *Elliptic integral* expression for $\zeta_Q(2)$: H. Ochiai: Ramanujan J. **15** (2008), Pfaff's formula for ${}_2F_1$ and Clausen' identity between ${}_2F_1$ and ${}_3F_2$.
[xxvi] KK, MW: Ann. Inst. Henri Poincaré - D (To appear).

Apéry-like numbers for $\zeta_Q(k)$ $(k \geq 2)$

Define the *Apéry-like numbers* $J_k(n)$[xxvii] associated to $R_{k,1}(\kappa)$ of $\zeta_Q(k)$.

$$R_{k,1}(\kappa) = \frac{k}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_k(n) \kappa^{2n},$$

with

$$J_k(n) = \sum_{r=1}^{k-1} 2^k \int_{[0,1]^k} \frac{(1-u_1^4 \cdots u_r^4)^n (1-u_{r+1}^4 \cdots u_k^4)^n}{(1-u_1^2 \cdots u_k^2)^{2n+1}} \, du_1 \cdots du_k.$$

[xxvii] Various *congruence relations* among $J_k(n)$ hold similar to the original Apéry numbers possesses. KK, MW: Kyushu J. Math. **60** (2006) (+ Elliptic integral expression for $\zeta_Q(2)$.) L. Long, R. Osburn, H. Swisher: Proc. Amer. Math. Soc. **144** (2016), J.-C. Liu: J. Math. Anal. Appl. **467** (2018).

Recurrence relations

The Apéry-like numbers $J_k(n)$ satisfy

$$4n^2 J_k(n) - (8n^2 - 8n + 3) J_k(n-1) + 4(n-1)^2 J_k(n-2) = 4J_{k-2}(n-1)$$

for $k \geq 2$ and $n \geq 2$.

Equivalently, $w_k(z) := \sum_{n=0}^{\infty} J_k(n) z^n$ $(w_0(z) = 0)$ satisfies

$$\left\{ z(1-z)^2 \frac{d^2}{dz^2} + (1-3z)(1-z)\frac{d}{dz} + z - \frac{3}{4} \right\} w_k(z) = w_{k-2}(z).$$

Remark: The recurrence relation of $J_k(n)$ (and the differential equation of $w_k(z)$) has the same homogeneous part for all k.

Family of Elliptic curves

Note that

$$\left\{ z(1-z)^2 \frac{d^2}{dz^2} + (1-3z)(1-z)\frac{d}{dz} + z - \frac{3}{4} \right\} w_2(z) = 0$$

is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion.

In fact, each elliptic curve in the family is birationally equivalent to one of the curves

$$(1-u^2v^2)^2 + x^2(1-u^4)(1-v^4) = 0$$

appearing in the denominator of the integrand of $R_{2,1}(x)$.

Modular interpretation for $w_2(t)$

Recall the elliptic theta functions

$$\theta_2(\tau)=\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2},\quad \theta_3(\tau)=\sum_{n=-\infty}^{\infty} q^{n^2/2},\quad \theta_4(\tau)=\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2}.$$

Theorem[xxviii]

Set $t=t(\tau)=-\frac{\theta_2(\tau)^4}{\theta_4(\tau)^4}$, then

$$\begin{aligned} w_2(t) &= \frac{J_2(0)}{1-t}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{t}{t-1}\right) \\ &= J_2(0)\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2} = J_2(0)\frac{\eta(2\tau)^{22}}{\eta(\tau)^{12}\eta(4\tau)^8}, \end{aligned}$$

is a $\Gamma(2)$-modular form of weight 1.

[xxviii] KK, MW: Proc. Conf. *L*-functions, World Scientific (2007).

Modular interpretations for $w_4(t)$

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Proposition

$$w_4(t) = \pi^2 w_2(t) + W_1(t), \qquad W_1(t) = \frac{2\pi i}{16\pi^2} w_2(t) \boldsymbol{G}_1'(\tau).$$

Here $\boldsymbol{G}_1'(\tau)$ is the derivative of an automorphic integral (Eichler form) for $\mathfrak{G}(2) := \langle T^2, S \rangle \subset \Gamma(2)$ of weight -2.

The automorphic integral $\boldsymbol{G}_k(\tau)$ is given by

$$\boldsymbol{G}_k(\tau) = \underbrace{\int_0^q \cdots \int_0^q}_{4k-1} f(\tau)^k \frac{dq}{q} \cdots \frac{dq}{q}, \quad \text{where } f(\tau) = \theta_2(\tau)^4 \theta_4(\tau)^4.$$

Differential Eisenstein series

We fix the branch $-\pi \le \arg z < \pi$ for $z \in \mathbb{C}$. Recall the *generalized Eisenstein series* for $s \in \mathbb{C}$ s.t. $\Re(s) > 2$ and $a, b \in \{0, 1, \ldots, N-1\}$:

$$G(s, x, \tau) := \sideset{}{'}\sum_{m,n\in\mathbb{Z}} (m\tau + n + x)^{-s}, \quad G(s, \tau) := G(s, 0, \tau),$$

$$G^{(N;a,b)}(s, \tau) := \sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv a \pmod{N} \\ n\equiv b \pmod{N}}} (m\tau + n)^{-s}.$$

The *differential Eisenstein series* are defined by

$$\mathrm{dG}_m(\tau) := \frac{\partial}{\partial s} G(s, \tau)\Big|_{s=m}, \quad \mathrm{dG}_m^{(N;a,b)}(\tau) := \frac{\partial}{\partial s} G^{(N;a,b)}(s, \tau)\Big|_{s=m}.$$

Eichler forms (slightly extended)

Differential Eisenstein series are Eichler forms (or automorphic integrals):

$$\mathrm{dG}_{-2k}(\tau) \in M_{-2k}(SL_2(\mathbb{Z}), \mathbb{C}(\tau)), \quad \mathrm{dG}^{(2;1,1)}_{-2k}(\tau) \in M_{-2k}(\mathfrak{G}(2), \mathbb{C}(\tau)).$$

Theorem[xxix]

The generating function $w_4(t)$ of Apéry-like numbers $J_4(n)$ is given by

$$w_4(t) = \frac{\pi^4}{2}\frac{\theta_4(\tau)^4}{\theta_3(\tau)^2}\left[1 + \frac{1}{\pi i}\frac{d}{d\tau}\left\{7\mathrm{dG}_{-2}(\tau) + 2\mathrm{dG}^{(2;1,1)}_{-2}(\tau)\right\}\right],$$

where $t = t(\tau) = -\theta_2(\tau)^4\theta_4(\tau)^{-4}$.

Differential of the generalized Eisenstein series are regarded as slightly extended elements of *nearly holomorphic modular forms.* [xxx]

[xxix] KK, MW: Ann. Inst. Henri Poincaré - D (To appear).

[xxx] G. Shimura: Ann. Math. **123** (1986), S. Horinaga: J. Numb. Theo. **219** (2021).

Thank you very much for your attention!

Clarification of the underlying fundamental reason is expected

Eigenvalue degeneracy ↔ **Hidden symmetry** (What physicists believe (long unknown))

Eigenvalues are essentially integers and parametrized.

Constraint polynomial

Description by hyperelliptic surfaces

Concrete description

Quotient of Constraint Polynomials (Polynomials) $\overset{\text{expectation}}{=}$ **Quadratic relation defined by the generators of the commutative operators**

Constraint Polynomials $P_N^\ell(g,\Delta)=0$

⇨

Juddian exceptional eigenvalue $N-g^2+\frac{\ell}{2}$ exists

Hidden symmetry – Degeneracy in AQRM ($\boldsymbol{\varepsilon}=\frac{\boldsymbol{\ell}}{\mathbf{2}}\in\mathbb{Z}$)

Eigenvalue (integer) contractions: integer divisors of constrained polynomials

↕ resemble

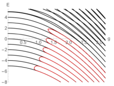
Polynomial Integral and Integer Store Reduction in Diophantine Geometry

(Deeply related to Vojta forecast)

P. Corvaja and U. Zannier, Adv. Math., 225 (2010)

Using the idea of adiabatic approximation using Constraint polynomials, an approximation of the general spectral curve can be obtained.

Zi-Min Li et al, J. Phys. A: Math. Theor., 54, 405201 (2021)



Gray: Energy and spectral curves

Red: $P_\ell=0$

Assuming the expectation, the first ℓ spectral curves of $H_{\frac{\ell}{2}}$ are very well approximated by P_ℓ

↓

Constraint polynomials can be said that the degenerate eigenvalues know the entire system.