Joyce structures, integrable systems, and hyperkähler metrics

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Joyce structures can be formulated on holomorphic symplectic manifolds (M, Ω) (or more generally holomorphic Poisson). They involve a certain $\mathbb{C}^*$-family of **flat and symplectic** non-linear connections on $TM \to M$.

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- T. Bridgeland and I. Strachan showed that Joyce structures over a holomorphic symplectic manifold (M, Ω) encodes a **complex Hyperkähler structure** ($\mathbb{C}$-HK) on TM.

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Remark:

In the setting of Joyce structures, the simplest "class S" case has M equal to a moduli space of Riemann surfaces of fixed genus and with a quadratic differential. Fixing the complex structure of the Riemann surface gives a Lagrangian $B \subset M$.

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Problem:

Given an HK metric with compatible integrable system structure, can we describe it in a way similar to Joyce structures?

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The connection h is **flat** if

$$[H, H] \subset H\,.$$

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Non-linear connections h, together with the map ν will play an important role in what follows.

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A (pre-)Joyce structure on (M, Ω) is a holomorphic or meromorphic non-linear connection $h : \pi^*(TM) \to T(TM)$ such that

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- Then $\mathcal{A}^\epsilon$ is symplectic if the induced parallel transport preserves Ω^v.

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- ∇ induces a horizontal splitting $\mathcal{H} : \pi^*(TM) \to T(TM)$ on $\pi : TM \to M$, and hence we can write

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Hence, locally we have

$$\mathcal{A}_X^{\epsilon} = \mathcal{H}_X + \mathrm{Ham}^v(\nu_X W) + \frac{1}{\epsilon}\nu_X \, .$$

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From ϖ^ζ one can recover the original data: g, I_1, I_2, I_3.

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- I_1, I_2, I_3 are holomorphic sections of $\mathrm{End}(TM) \to M$, parallel with respect to the Levi-Civita connection of g.

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As before, from Ω^ϵ we can recover g, I_1, I_2, I_3.

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- The integrability of I_1, I_2, I_3 follows from the flatness of $\mathcal{A}_\epsilon$, while the closedness of Ω^ϵ follows from the fact that $\mathcal{A}^\epsilon$ is symplectic.

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In what follows, given a submersion $\pi : X \to M$, we will consider "complexified" non-linear connections. This means a splitting of the following short exact sequence

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A usual non-linear connection gives rise to complexified one, but the converse is not true.

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Consider a symplectic fiber bundle $\pi : X \to M$ over a complex manifold (M, I). Namely, we have a fiberwise symplectic form $\omega^v \in \Gamma(X, \Lambda^2 V^*)$, where $V = \mathrm{Ker}(\mathrm{d}\pi)$.

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- The $\mathbb{C}^*$-family of complexified non-linear connections $\mathcal{A}^\zeta : \pi^*(TM) \otimes \mathbb{C} \to TX \otimes \mathbb{C}$ defined by
$$\mathcal{A}^\zeta_X = h_X - \frac{1}{\zeta}\overline{v_X}, \quad \mathcal{A}^\zeta_{\overline{X}} = \overline{h_X} + \zeta v_X, \quad X \in \pi^*(T^{1,0}M)\,,$$
is flat and symplectic with respect to $(\pi : X \to M, \omega^v)$.

HK metrics and integrable systems

Theorem part 1 (I.T., to appear):

Given $(\pi : X \to M, \omega^v, h, v)$ as before, there is a (pseudo)-HK structure on X. The $\mathbb{C}P^1$-family of complex structures I^ζ is such that

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and they give 0 when evaluated on other combinations of h and v (and their conjugates).

Finally, if the fibers of π are compact and connected, then $(\pi : (X, I_3) \to (M, I), \omega_1 + \mathrm{i}\omega_2)$ has the structure of a complex integrable system (up to the data of polarizations of the fibers).

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There is also a converse:

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Consider a HK manifold (X, g, I_1, I_2, I_3) with complex integrable system structure $(\pi : (X, I_3) \to (M, I), \omega_1 + \mathrm{i}\omega_2)$. Then for the symplectic fiber bundle $(\pi : X \to M, \omega_3|_V)$ there are maps (h, v) such that $\mathcal{A}^\zeta$ is flat and symplectic, and such that (1) and (2) hold.

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- The antilinear map $v : \pi^*(T^{1,0}M) \to V$ is then defined by

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- We also have an induced (flat) horizontal splitting $\mathcal{H} : \pi^*(TM) \to T(TM)$ induced from ∇.

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Proposition

For a tuple $(\pi : TM \to M, \omega^\nu, h, v)$ and a local holomorphic section X of $TM \to M$, h and v admit the decomposition

$$h_X = \mathcal{H}_X + \mathrm{Ham}^v(f_X), \quad v_X = \nu_{\overline{X}} + \mathrm{Ham}^v(\overline{g_X})$$

for locally defined functions f_X and g_X on TM.

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The analogous case in Joyce structures is to start with the flat holomorphic symplectic manifold (M, Ω, ∇) and consider the Joyce structure given by

$$\mathcal{A}^\epsilon_X = \mathcal{H}_X + \frac{1}{\epsilon}\nu_X\,.$$

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We can then write a global function on TM by the formula

$$J = \mathrm{i} \sum_{\gamma} \Omega(\gamma) \sum_{n>0} \frac{e^{\mathrm{i}n\theta_\gamma}}{n^2} K_0(n|Z_\gamma|)\,,$$

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It turns out that the tuple $(\pi : TM \to M, \omega^{\nu}, h, v)$ also satisfies the requirements of the theorem, and defines an HK structure on TM. This recovers the **GMN HK metric** in the case of uncoupled BPS indices.

Example 2 continued

The resulting $\mathbb{C}^*$-family of complexified, flat, symplectic connections then has the form

$$\mathcal{A}_X^\zeta = \mathcal{H}_X + \mathsf{Ham}^v(\mathcal{H}_X J) - \frac{1}{\zeta}(\nu_X + \mathsf{Ham}^v(\nu_X \overline{J}))$$
$$\mathcal{A}_{\overline{X}}^\zeta = \overline{\mathcal{H}_X} + \mathsf{Ham}^v(\overline{\mathcal{H}_X J}) + \zeta(\overline{\nu_X} + \mathsf{Ham}^v(\overline{\nu_X} J))\,.$$

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This should be compared to the analogous case in Joyce structures, where

$$\mathcal{A}_X^\epsilon = \mathcal{H}_X + \mathsf{Ham}^v(\nu_X W) + \frac{1}{\epsilon}\nu_X\,.$$

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The function W has a simple form when the BPS indices are uncoupled and finite:

$$W = \sum_{\gamma} \frac{\Omega(\gamma)}{Z_\gamma}\mathrm{Li}_3(e^{\theta_\gamma})\,.$$

Thanks!