On the exact WKB analysis of a difference equation satisfied by the Gauss hypergeometric function

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Introduction

The Gauss hypergeometric function

$$\phi(x,a,b,c) = F(a,b,c;x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m \tag{1}$$

satisfies

$$\{x(\vartheta + a)(\vartheta + b) - \vartheta(\vartheta + c - 1)\} F(a,b,c;x) = 0, \tag{2}$$

$$\frac{\vartheta + a}{a} F(a,b,c;x) = F(a+1,b,c;x). \tag{3}$$

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$$\frac{\vartheta + a}{a} F(a, b, c; x) = F(a + 1, b, c; x). \tag{3}$$

$$\Rightarrow \ \left\{(1 - x)(a + 1)\sigma^2 + \{(x - 2)(a + 1) + c - bx\}\, \sigma + a - c + 1\right\} \phi = 0, \tag{G}$$

where $\vartheta = x\dfrac{d}{dx}$, $\sigma = \dfrac{\vartheta + a}{a}$ and $\sigma^2 = \dfrac{(\vartheta + a + 1)(\vartheta + a)}{(a + 1)a}$.

This equation (G) has the following formal solutions

$$f^1(a) = a^{-b}(1 + O(a^{-1})), \tag{4}$$

$$f^2(a) = \left(\frac{1}{1-x}\right)^a a^{b-c}(1 + O(a^{-1})). \tag{5}$$

The Borel transform of these power series are

$$f_B^1(\xi) = \xi^{b-1}\left(\frac{1}{\Gamma(b)} + O(\xi)\right), \tag{6}$$

$$f_B^2(\xi) = (\xi - \log(1-x))^{c-b-1}\left(\frac{1}{\Gamma(c-b)} + O(\xi - \log(1-x))\right). \tag{7}$$

It is clear that the singular points of this function are located at

$$\xi = 2\pi i n, \quad \log(1-x) + 2\pi i n \quad (n \in \mathbb{Z}).$$

Theorem (Katsushima)

Let $s_1 = 0, s_2 = \log(1-x)$, $C_1 = \frac{1}{\Gamma(b)}(\frac{x}{x-1})^{b-c+1}$, $C_2 = \frac{e^{-\pi i(c-1)}}{\Gamma(c-b)}\frac{x^{1-b}}{x-1}$ and

$$F^j(a) = \int_{s_j}^{\infty} e^{-a\xi} f_B^j(\xi)\, d\xi \quad (j = 1, 2).$$

Then, the following connection formulas hold.

1. $\arg a = 0 \;\to\; \frac{\pi}{2} - 0$

$$F^1(a) \to F^1(a) + F^2(a)\frac{C_1}{C_2}e^{2\pi i(a-b)}\frac{1-e^{2\pi i(c-b-1)}}{1-e^{2\pi i(a-b)}}, \quad F^2(a) \to F^2(a). \tag{8}$$

2. $\arg a = \frac{\pi}{2} - 0 \;\to\; \frac{\pi}{2} + 0$

$$F^1(a) \to F^1(a)\frac{1-e^{2\pi i(a-1)}}{1-e^{2\pi i(a-b)}}, \quad F^2(a) \to F^2(a)\frac{1-e^{2\pi i(a-b)}}{1-e^{2\pi i(a-c+1)}}. \tag{9}$$

3. $\arg a = \frac{\pi}{2} + 0 \;\to\; \pi$

$$F^1(a) \to F^1(a), \quad F^2(a) \to F^2(a) + F^1(a)\frac{C_2}{C_1}\frac{1-e^{2\pi i(b-1)}}{1-e^{2\pi i(a-b)}}. \tag{10}$$

Purpose of this talk

To discuss Katsushima's result from the viewpoint of the exact WKB analysis by introducing a large parameter η to the difference equation (G).

$$\left\{(1-x)(a+1)\sigma^2 + \left\{(x-2)(a+1) + c - bx\right\}\sigma + a - c + 1\right\}\phi = 0. \tag{G}$$

An approach through the exact WKB analysis

By introducing a large parameter η by $a = \eta\alpha,\ b = \eta\beta,\ c = \eta\gamma$, we obtain

$$\left[(1-x)(\alpha+\eta^{-1})\sigma_\alpha^2 + \left\{(x-2)(\alpha+\eta^{-1}) + \gamma - \beta x\right\}\sigma_\alpha + \alpha - \gamma + \eta^{-1}\right]\psi = 0, \quad (G_\eta)$$

where σ_α is a shift operator with $\alpha \mapsto \alpha + \eta^{-1}$. By regarding $\sigma_\alpha = \sum \frac{1}{m!}\eta^{-m}\partial_\alpha^m$, we can construct WKB solutions

$$\psi_\pm = \exp\left(\eta\int^\alpha T^{(\pm)}(\alpha,\eta)\,d\alpha\right), \quad T^{(\pm)}(\alpha,\eta) = \sum_{n=0}^\infty \eta^{-n}T_n^{(\pm)}(\alpha). \qquad (11)$$

Here $\lambda = e^{T_0}$ is a root of $(1-x)\alpha\lambda^2 + \{(x-2)\alpha + \gamma - \beta x\}\lambda + \alpha - \gamma = 0$.

$$\Longrightarrow \quad T_0^{(\pm)}(\alpha) = \log\frac{(2-x)\alpha + \beta x - \gamma \pm \sqrt{D}}{2(1-x)\alpha}$$

$$\text{with} \quad D = x^2\alpha^2 - 2(\beta x + \gamma - 2\beta)x\alpha + (\gamma - \beta x)^2.$$

Definition

turning point $\iff$ $D = 0$, i.e. $\alpha = \dfrac{(\sqrt{\beta(x-1)} \pm \sqrt{\gamma - \beta})^2}{x}$

A-type Stokes curve $\iff$ $\Im \displaystyle\int_{\alpha_0}^{\alpha} (T_0^{(+)}(\alpha) - T_0^{(-)}(\alpha))\, d\alpha = 0$,

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logarithmic turning point $\iff$ logarithmic singularity of $T_0(\alpha)$,

i.e. $\alpha = 0, \ \gamma$

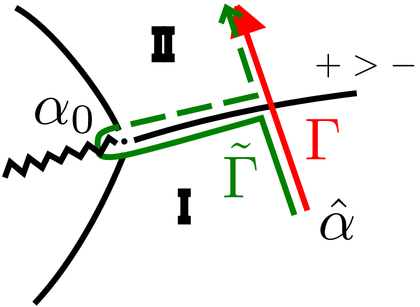
L-type Stokes curve $\iff$ $\Im \displaystyle\int_{l_0}^{\alpha} \{T_0(\alpha) - (T_0(\alpha) + 2\pi i)\}\, d\alpha = 0,$

where l_0 is a logarithmic turning point.

$\iff$ $\Re\alpha = 0, \ \Re(\alpha - \gamma) = 0.$

Analysis near the turning point

As was shown in [AKKoT], three A-type Stokes curves emanate from a simple turning point and Stokes phenomena described by the Voros connection formula are observed on them.



$$\begin{cases} \psi_+^{(\mathrm{I})} \to \psi_+^{(\mathrm{II})} + \exp\left(\int_{\tilde{\Gamma}} \eta T \, d\alpha\right), \\ \psi_-^{(\mathrm{I})} \to \psi_-^{(\mathrm{II})}, \end{cases}$$

where $\psi_\pm^{(j)}$ is a Borel sum of $\psi_\pm$ on the Stokes region j ($j = \mathrm{I}, \mathrm{II}$).

We assign a label $+>-$ to a Stokes curve if the following equation holds on the Stokes curve.

$$\Re \int_{\alpha_0}^{\alpha} (T_0^{(+)}(\alpha) - T_0^{(-)}(\alpha)) \, d\alpha > 0. \tag{12}$$

Analysis near the logarithmic turning point

(I) We write the equation (G_η) as $L\psi = 0$. Then, the second-order difference operator L can be factorized near $\alpha = 0$ as

$$L = (1-x)(\sigma_\alpha - g)(\alpha\sigma_\alpha - f), \tag{12}$$

where $f = f_0 + \eta^{-1} f_1 + \cdots$, $g = g_0 + \eta^{-1} g_1 + \cdots$ with $f_0(0)g_0(0) \neq 0$ and $f_j,\ g_j$ $(j \in \mathbb{N})$ are holomorphic at $\alpha = 0$.

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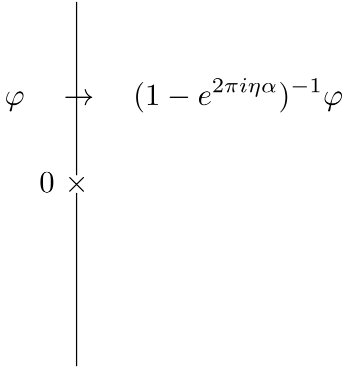
(II) $(\alpha\sigma_\alpha - f)\psi = 0$ can be transformed to the following canonical equation.

$$(\alpha\sigma_\alpha - f)\psi = 0 \quad \Longleftrightarrow \quad (\alpha\sigma_\alpha - 1)\varphi = 0. \tag{13}$$

$$\psi = \exp\left(\eta \int \exists p \, d\alpha\right)\varphi$$

<u>Remark</u> $\varphi = \dfrac{\eta^{\eta\alpha}}{\Gamma(\eta\alpha)}$ is a solution of the canonical equation.

(III) Let $\varphi = \exp \int \eta \hat{T}(\alpha, \eta)\, d\alpha$ be WKB solutions of the canonical equation. Then, on a L-type Stokes curve for the canonical equation emanating from $\alpha = 0$, the following connection formula holds.



$$\varphi \quad \to \quad (1 - e^{2\pi i \eta \alpha})^{-1} \varphi$$

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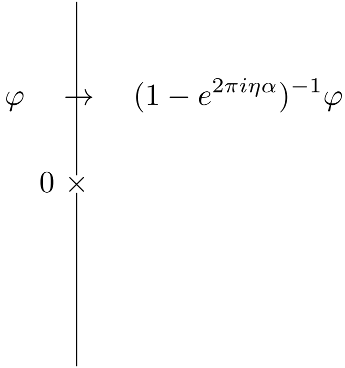

(IV) (II) implies that $T - \hat{T}$ is holomorphic at a logarithmic turning point.

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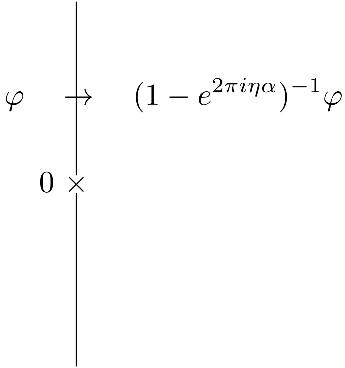
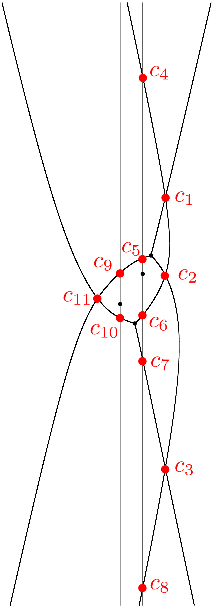


(IV) (II) implies that $T - \hat{T}$ is holomorphic at a logarithmic turning point.

$\Rightarrow$ A connection formula for (G_η) is the same as the above one.

Stokes geometry

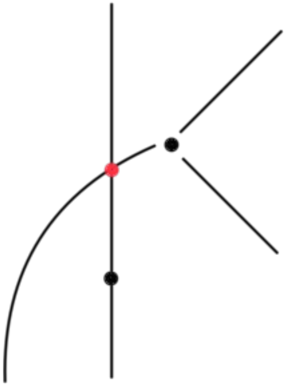


Put $x = 3+0.1i$, $\beta = 1.2+0.3i$, $\gamma = 0.6+0.8i$.

There are 11 crossing points of Stokes curves.

They are divided into several different types.

Type I (Hermite-Weber type)



$$\left\{ \left(\alpha + 2\eta^{-1}\right) \sigma_\alpha^2 - x\sigma_\alpha + 1 \right\} \psi = 0$$

A crossing point of an A-type Stokes curve and a L-type Stokes curve.

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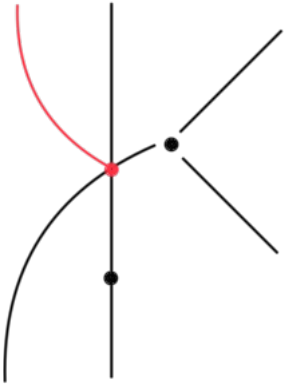
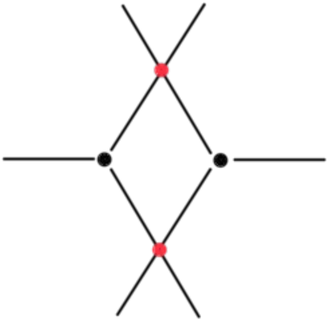


$$\left\{ \left(\alpha + 2\eta^{-1} \right) \sigma_\alpha^2 - x\sigma_\alpha + 1 \right\} \psi = 0$$

A crossing point of an A-type Stokes curve and a L-type Stokes curve.

$\Longrightarrow$ One A-type new Stokes curve emanates from the crossing point.

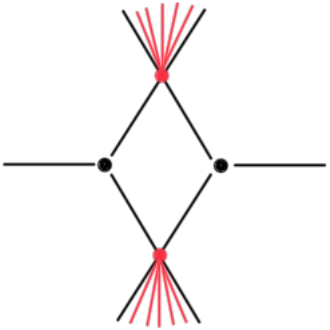
Type II (Bessel type)



$$\left(\sigma_\alpha^2 - \frac{2\alpha}{x}\sigma_\alpha + 1\right)\psi = 0$$

A crossing point of two A-type Stokes curves.

Type II (Bessel type)



$$\left(\sigma_\alpha^2 - \frac{2\alpha}{x}\sigma_\alpha + 1\right)\psi = 0$$

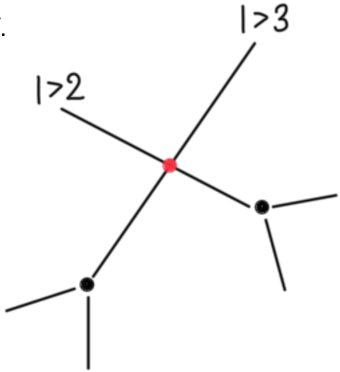
A crossing point of two A-type Stokes curves.

$\Longrightarrow$ Infinitely many new Stokes curves emanate from the crossing point.

(Infinitely many A-type & one L-type)

Type III (non-ordered)

cf.



$$\left(\eta^{-3}\partial_x^3 + p(x)\eta^{-2}\partial_x^2 + q(x)\eta^{-1}\partial_x + r(x)\right)\psi = 0$$

$$\psi = \psi_j \quad (j = 1, 2, 3) : \text{WKB solutions}$$

A crossing point of two A-type Stokes curves with the dominance relation indicated in the figure.

Type III (non-ordered)

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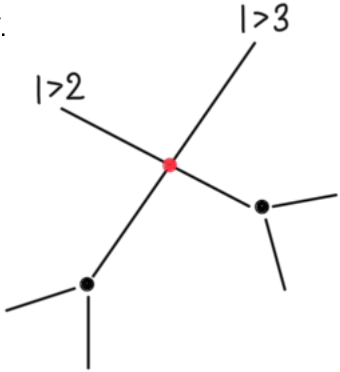

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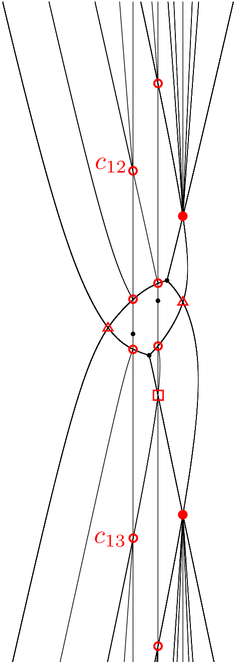
$$\psi = \psi_j \quad (j = 1, 2, 3) : \text{WKB solutions}$$

A crossing point of two A-type Stokes curves with the dominance relation indicated in the figure.

$\Longrightarrow$ No new Stokes curve appears.



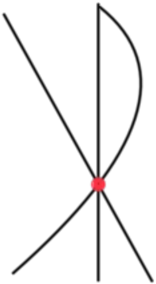
○ : **Hermite-Weber type**

● : **Bessel type**

△ : **non-ordered**

□ : **Different type crossing point where Stokes coefficients change**

Type IV ("exceptional")



A crossing point of two A-type Stokes curves and one L-type Stokes curve.

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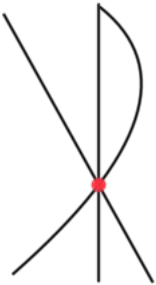
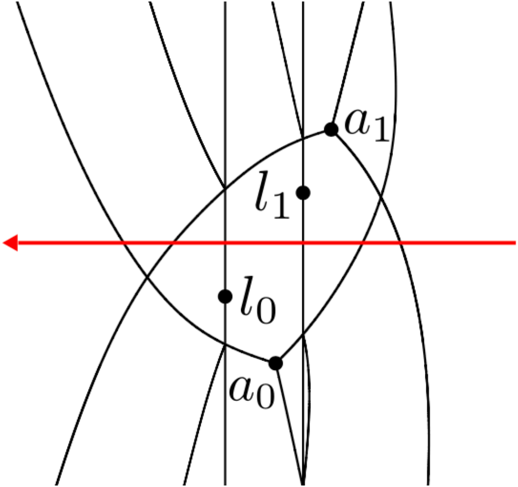


A crossing point of two A-type Stokes curves and one L-type Stokes curve.

$\Longrightarrow$ Stokes coefficients for the L-type Stokes curve change.

Path of analytic continuation



path of analytic continuation
near the real axis

Here a_0, a_1 are simple turning points and l_0, l_1 are logarithmic turning points.

Analytic continuation of $\psi_\pm$ along the path

As a basis of solutions we take WKB solutions normalized as follows.

$$\psi_\pm = \exp\left(\eta \int_{a_0}^{\alpha} T_0^{(\pm)}\, d\alpha\right) \exp\left(\int^{\alpha} T_1^{(\pm)}\, d\alpha\right) \exp\left\{\int_{\infty}^{\alpha} \left(\eta^{-1} T_2^{(\pm)} + \cdots\right)\, d\alpha\right\}. \quad (14)$$

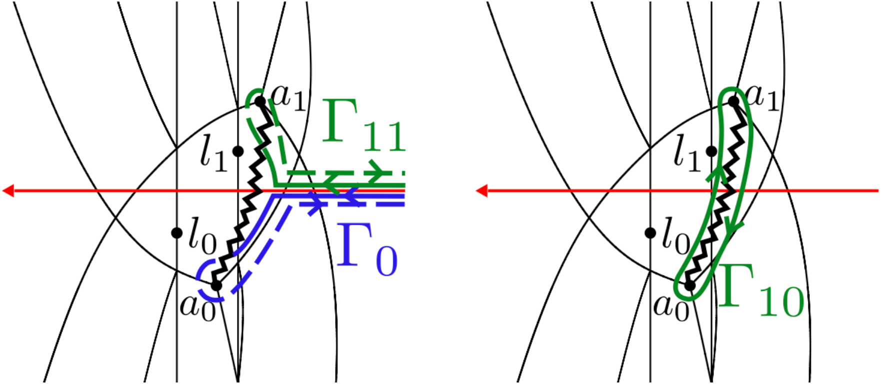
Then, the analytic continuation of $\psi_\pm$ is given by the following formuila.

$$\begin{aligned}\psi_+ \to &\left(1 - e^{-2\pi i\eta(\alpha-\gamma)}\right)\left(e^{V_0} + e^{V_1}\right)\psi_+ + \Big\{\left(1 - e^{2\pi i\eta\alpha}\right) + \left(1 - e^{-2\pi i\eta(\alpha-\gamma)}\right) \\ &\times \left(e^{2\pi i\eta\alpha} + e^{V_0+V_1+2\pi i\eta(\alpha-\gamma)} + e^{V_1-V_0+2\pi i\eta\alpha} + e^{2\pi i\eta(\alpha-\gamma)}\right)\Big\}\psi_-, \quad (15)\\ \psi_- \to &\left(1 - e^{-2\pi i\eta(\alpha-\gamma)}\right)\psi_+ \\ &+ \left(1 - e^{-2\pi i\eta(\alpha-\gamma)}\right)\left(e^{-V_0+2\pi i\eta\alpha} - e^{-V_1+2\pi i\eta(\alpha-\gamma)}\right)\psi_-. \quad (16)\end{aligned}$$

Here V_j $(j = 0, 1)$ denotes the following Voros coefficients:

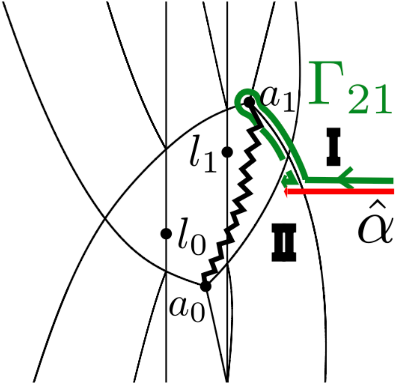
$$V_0 = \int_{\Gamma_0} \eta \left(T - T_0 - \eta^{-1} T_1 \right) \, d\alpha, \tag{17}$$

$$V_1 = \int_{\Gamma_{10}} \eta \left(T_0 + \eta^{-1} T_1 \right) \, d\alpha + \int_{\Gamma_{11}} \eta \left(T - T_0 - \eta^{-1} T_1 \right) \, d\alpha. \tag{18}$$



Sketch of Proof

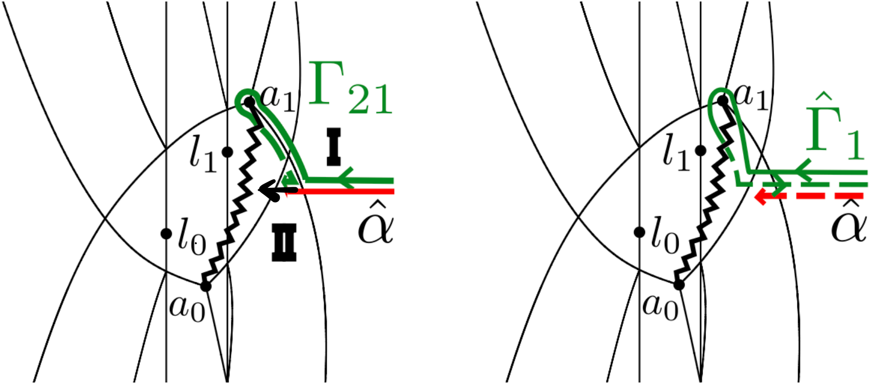
$$\psi_\pm^{(j)} = \exp \int_{\hat{\alpha}}^{\alpha} \eta T^{(\pm)}\, d\alpha \text{ : Borel sum of } \psi_\pm \text{ on the Stokes region } j$$



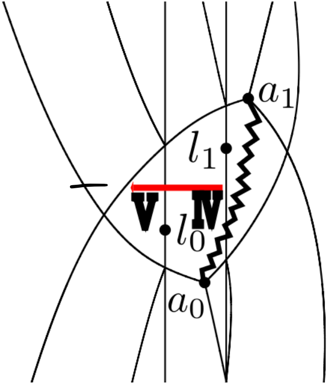
$$\begin{cases} \psi_+^{(\mathrm{I})} \to \psi_+^{(\mathrm{II})} + \exp\left(\int_{\Gamma_{21}} \eta T\, d\alpha\right) \\ \psi_-^{(\mathrm{I})} \to \psi_-^{(\mathrm{II})}. \end{cases}$$

Sketch of Proof

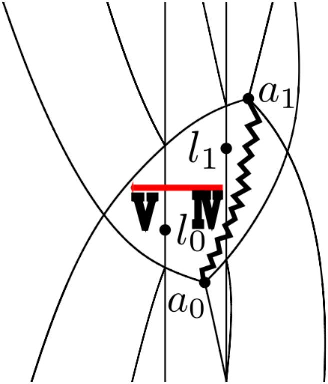
$$\psi_{\pm}^{(j)} = \exp \int_{\hat{\alpha}}^{\alpha} \eta T^{(\pm)}\, d\alpha : \text{Borel sum of } \psi_{\pm} \text{ on the Stokes region } j$$



$$\begin{cases} \psi_+^{(\mathrm{I})} \to \psi_+^{(\mathrm{II})} + \exp\left(\int_{\Gamma_{21}} \eta T\, d\alpha\right) = \psi_+^{(\mathrm{II})} + \exp\left(\int_{\hat{\Gamma}_1} \eta T\, d\alpha\right)\psi_-^{(\mathrm{II})}, \\ \psi_-^{(\mathrm{I})} \to \psi_-^{(\mathrm{II})}. \end{cases}$$



$$\begin{cases} \psi_+^{(\mathsf{IV})} \to \psi_+^{(\mathsf{V})}, \\ \psi_-^{(\mathsf{IV})} \to \left(1 - e^{2\pi i \eta \alpha}\right) \psi_-^{(\mathsf{V})}. \end{cases}$$



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Future problems

1. To compute the Voros coefficients V_j explicitly so that we can compare our formula (15) and (16) with Katsushima's result.
2. To extend the computation to more general (hypergeometric-type) difference equations.
3. To establish the general theory for the exact WKB analysis of difference equations.

References

[AKKoT] T. Aoki, T. Kawai, T. Koike and Y. Takei : On the exact WKB analysis of operators admitting infinitely many phases, Adv. Math., Vol.181, 2004.

[K] Y. Katsushima : The Stokes phenomena of additive linear difference equations, doi: 10.15083/00008462, 2015.

References

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Thank you for your attention !