Rank 3 representations of Painlevé systems

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Basic notations

Fix the following data:

- X a smooth projective curve (later $X = \mathbb{C}P^1$)
- $\mathbb{C}P^1 = \operatorname{Spec} \mathbb{C}[z] \cup \operatorname{Spec} \mathbb{C}[w]$ with $w = z^{-1}$
- $r \geq 2$ (rank of the vector bundle over X), $G = \mathrm{GL}(r, \mathbb{C})$
- $D = \sum m_i p_i$ effective divisor on X (later with $|D| = 3$)
- $K_X(D)$ twisted canonical bundle

Moduli space of meromorphic Higgs bundles

A rank r meromorphic Higgs bundle on X consists of the data

- a holomorphic rank r vector bundle $\mathcal{E}$ over X
- a meromorphic Higgs field $\theta \in H^0(X, End(\mathcal{E}) \otimes K_X(D))$

$\mathcal{M}_{r,d}(X, D)$: moduli space of isomorphism classes of stable meromorphic Higgs bundles of given rank r and degree d over X

Proposition (E. Markman '94)

- *$\mathcal{M}_{r,d}(X, D)$ is a holomorphic Poisson manifold*
- *symplectic leaves of maximal rank of $\mathcal{M}$ are given by fixing the gauge equivalence class of its irregular part at D and the corresponding eigenvalues of the residue of θ (c.f. next page).*

The cases when the maximal symplectic leaves are of complex dimension 2, have been listed by P. Boalch.

Holomorphic symplectic leaves

In the coordinate chart $z - p_i$ and in a local trivialization of $\mathcal{E}$ at p_i:

$$\theta = \sum_{k=-m_i}^{\infty} A_k (z - p_i)^k \mathrm{d}z,$$

- The irregular part of θ is $\int \sum_{k=-m_i}^{-2} A_k (z - p_i)^k \mathrm{d}z$
- the residue of θ at $p_i \in D$ is $A_{-1}(z - p_i)^{-1} \mathrm{d}z$.

To get symplectic leaves, we need to fix the eigenvalues of the irregular part and the adjoint orbit of the residue in each eigenspace. Moreover, we may choose Dolbeault parabolic weights $\{\alpha_{p_i}^j\}$ at all $p_i \in D$.

Definition

$\mathcal{M}_{Dol}$: Dolbeault moduli space of rank r meromorphic, parabolic, α-stable Higgs bundles over $\mathbb{C}P^1$ with the above fixed data.

Hitchin–Kobayashi correspondence

h is a Hermitian-Einstein metric in $\mathcal{E}$ if the connection associated to $(\mathcal{E}, \theta, h)$:

$$\nabla = D_h^+ + \theta + \theta^{\dagger_h}$$

is integrable on $X \setminus D$. Here, $D_h^+ = \bar{\partial}_{\mathcal{E}} + \partial_h$ stands for the Chern connection associated to $\bar{\partial}_{\mathcal{E}}, h$.

At each p_i, with respect to a frame compatible with the parabolic filtration, we require the asymptotic behavior

$$h \approx \mathrm{diag}(|z|^{2\alpha^j_{p_i}} |\ln |z|^2|^k),$$

$k =$ the weight induced by the nilpotent part of the residue.

Theorem (Biquard–Boalch'04, Mochizuki'11)

A Hermitian-Einstein metric for $(\mathcal{E}, \theta)$ with the given behavior exists if and only if $(\mathcal{E}, \theta)$ is α-polystable.

Meromorphic connections

Definition

A meromorphic connection on a holomorphic vector bundle E over X is a $\mathbb{C}$-linear map

$$\nabla^{1,0} \colon E \to E \otimes K_X(D)$$

satisfying Leibniz' rule.

Let $\{\beta^j_{p_i}\}$ denote the de Rham parabolic weights at p_i, and μ_i the eigenvalues of the residues of $\nabla^{1,0}$.

Let $\mathcal{D}_X$ be the sheaf of analytic differential operators on X. A left module $\mathbb{M}$ over $\mathcal{D}_X$ is holonomic if it is finitely generated and torsion. Then a meromorphic connection induces a holonomic $\mathcal{D}_X$-module structure on the sheaf E:

$$\partial_t(e) = \nabla^{1,0}_{\partial_t} e.$$

Formal decomposition of holonomic $\mathcal{D}$-modules

For $c \in C = \mathrm{Sing}(\mathbb{M})$, we set $\mathbb{M}_c = \mathbb{C}[\![t]\!]\langle\partial_t\rangle \otimes_{\mathbb{C}[t]\langle\partial_t\rangle} \mathbb{M}$, with t a local chart centered at c.

Theorem (Hukuhara–Levelt–Turrittin)

Let $\mathbb{M}$ be a holonomic $\mathcal{D}_X$-module induced by a meromorphic connection with pole at $\mathrm{Spec}\,\mathbb{C}[t]/(t)$. Then $\exists\ d \in \mathbb{N}$ and a finite Galois extension $K = \mathbb{C}(\!(u,t)\!)/(u^d - t)$ of $\mathbb{C}(\!(t)\!)$ s.t.

$$K \otimes_{\mathbb{C}[t]} \mathbb{M} \cong \bigoplus_{q \in K/\mathbb{C}[\![u]\!]} (\mathbb{C}[\![u]\!], \mathrm{d} + \mathrm{d}q) \otimes_{\mathbb{C}[\![u]\!]} \mathcal{F}_q$$

for some locally free $\mathbb{C}[\![u]\!]$-modules $\mathcal{F}_q$ endowed with a regular singular connection.

Irregular type, slope, etc.

Definition

- The irregular type of $\mathbb{M}_c$ is the formal sum $\sum_q \mathrm{rk}(\mathcal{F}_q)q$. It can be thought of as a marked finite cover of S^1.
- The slope of q is $\lambda(q) = \frac{\deg_u q}{d}$.
- The Swan conductor (or irregularity) of $\mathbb{M}_c$ is $\mathrm{Sw}(\mathbb{M}_c) = \sum_q \mathrm{rk}(\mathcal{F}_q)\lambda(q)$.
- The exponential torus of an irregular type $\sum m_q q$ is $\left(\prod_q T^{m_q}\right)/\mathbb{C}^\times$, where $T^{m_q} \subset \mathrm{GL}(m_q, \mathbb{C})$ is the standard maximal torus.
- The formal monodromy of $\mathbb{M}_c$ is the product $H \in \prod_q \mathrm{GL}(m_q, \mathbb{C})$ of the monodromies of $\mathcal{F}_q$.

Irregular de Rham moduli space

Definition

$\mathcal{M}_{dR}$: de Rham moduli space of β-stable integrable connections with irregular singularities of fixed irregular types and residues at D.

The transformation of the irregular types from Dolbeault to de Rham local forms is $\frac{1}{2}q_i \mapsto q_i$.
The transformation of the ν_i eigenvalues of the residues and the α_i parabolic weights from the Dolbeault side, obeys Simpson's formulas:

$$\alpha_i = \operatorname{Re}\mu_i, \quad \nu_i = \frac{\mu_i - \beta_i}{2}.$$

Non-abelian Hodge moduli space

Theorem (Biquard–Boalch'04)

For fixed irregular types $\sum m_q q$ and adjoint orbits $\mathcal{O}_i$, there exist moduli spaces $\mathcal{M}_{Dol}$ of gauge equivalence classes of α-stable meromorphic Higgs bundles having given irregular type, and with residue at p_i belonging to $\mathcal{O}_i$. The spaces $\mathcal{M}_{Dol}$ are real 4-dimensional complete hyperKähler manifolds. Moreover, endowed with another complex structure J of the hyperKähler family, the spaces $\mathcal{M}_{Dol}$ are $\mathbb{R}$-analytically isomorphic to moduli spaces $\mathcal{M}_{dR}$ of β-stable integrable connections with irregular singularities of fixed irregular types and residues in given adjoint orbits. In particular, there exist diffeomorphisms

$$NAH\colon \mathcal{M}_{Dol} \longrightarrow \mathcal{M}_{dR}.$$

Cases related to root system $\widetilde{E}_6$

The case of interest to us is the one related to the root system $\widetilde{E}_6$, that has numerical data $r = 3$ and length$(D) = 3$.
Three different partitions realize the possible divisors D: $(1,1,1), (2,1)$ and (3).
The first one will be referred to as the logarithmic case, and the other two as the irregular ones.
In both irregular cases, the Jordan type of the leading-order coefficient of θ may belong to three combinatorially different adjoint orbits

- semisimple orbits (untwisted),
- orbits with a Jordan block of size 2 and a Jordan block of size 1 (minimally twisted),
- and finally orbits with a Jordan block of size 3 (maximally twisted).

All together 7 combinatorical cases.

Joshi–Kitaev–Treharne systems

The 6 irregular cases (related to the root system $\widetilde{E}_6$) of rank 3 connections, were studied by Joshi, Kitaev, and Treharne'07,'09, in order to give the 3×3 Lax representations of the Painlevé equations.
We will refer to these as $JKT*$-cases, where

$$* \in \{VI, V, IVa, IVb, II, I\}.$$

- Cases $JKTVI, JKTV, JKTIVa$ correspond to $D = \{0\} + 2 \cdot \{\infty\}$.
- Cases $JKTIVb, JKTII, JKTI$ correspond to $D = 3 \cdot \{\infty\}$.

The corresponding de Rham and Dolbeault moduli spaces are denoted by $\mathcal{M}_{dR}^{JKT*}$ and $\mathcal{M}_{Dol}^{JKT*}$

Joshi–Kitaev–Treharne systems

Summary of the investigated cases		
JKT system	polar divisor of ∇	type of irregular singularity at ∞
JKTVI	$D = \{0\} + 2\{\infty\}$	untwisted
JKTV	$D = \{0\} + 2\{\infty\}$	minimally twisted
JKTIVa	$D = \{0\} + 2\{\infty\}$	maximally twisted
JKTIVb	$D = 3\{\infty\}$	untwisted
JKTII	$D = 3\{\infty\}$	minimally twisted
JKTI	$D = 3\{\infty\}$	maximally twisted

Example - *JKTII* case

$D = 3 \cdot \{\infty\}$, such that the irregular singularity $p_0 = \infty$ is minimally twisted. Local normal form of the meromorphic Higgs field at ∞:

$$\theta = \left[\begin{pmatrix} a_0 & 1 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_1 \end{pmatrix} w^{-3} + \begin{pmatrix} 0 & 0 & 0 \\ b_0 & b_1 & 0 \\ 0 & 0 & b_2 \end{pmatrix} w^{-2} + \begin{pmatrix} 0 & 0 & 0 \\ c_0 & c_1 & 0 \\ 0 & 0 & c_2 \end{pmatrix} w^{-1} + O(1) \right] \otimes \mathrm{d}w,$$

where $a_0, a_1, b_i, c_i \in \mathbb{C}$ $(i = 0, 1, 2)$ are fixed, and $a_0 \neq a_1$. The irregular type of θ is given by the Laurent polynomials:

$$\frac{1}{2} q_{0,1} = q_{\pm} = \lambda_1 w^{-1/2} + \lambda_2 w^{-1} + \lambda_3 w^{-3/2} - \frac{1}{2} a_0 w^{-2}$$

$$\frac{1}{2} q_2 = -b_2 w^{-1} - \frac{1}{2} a_1 w^{-2}.$$

with corresponding residue eigenvalues $0, c_1, c_2$. The coefficients λ_i $(1 \leq i \leq 3)$ depend algebraically on b_i, c_i.

Example - *JKTII* (cont'd)

The de Rham local form of ∇ at ∞ is defined as

$$\nabla = \mathrm{d} + \mathrm{d}Q_i + \Lambda_i \frac{\mathrm{d}z}{z} + O(1)\,\mathrm{d}z,$$

The de Rham irregular type:

$$q_{0,1} = 2(\lambda_1 w^{-1/2} + \lambda_2 w^{-1} + \lambda_3 w^{-3/2} - \frac{1}{2} a_0 w^{-2})$$
$$q_2 = 2(-b_2 w^{-1} - \frac{1}{2} a_1 w^{-2}).$$

The corresponding de Rham local form:

$$\nabla = \mathrm{d} + \left[\begin{pmatrix} 2a_0 & 1 & 0 \\ 0 & 2a_0 & 0 \\ 0 & 0 & 2a_1 \end{pmatrix} w^{-3} + \begin{pmatrix} 0 & 0 & 0 \\ b_0' & b_1' & 0 \\ 0 & 0 & b_2' \end{pmatrix} w^{-2} + \begin{pmatrix} 0 & 0 & 0 \\ c_0' & c_1' & 0 \\ 0 & 0 & c_2' \end{pmatrix} w^{-1} + O(1) \right] \otimes \mathrm{d}w$$

where b_i', c_i' correspond to b_i, c_i via Simpson's transformation.

Stokes local systems – irregular types

$G = \mathrm{GL}(3, \mathbb{C})$
$T \subset G$ maximal torus with Lie algebra $\mathfrak{t}$

$$Q_i = \sum_{k=-m_i}^{-1} A'_{k-1}(z - p_i)^k,$$

Q_i irregular type singles out a subgroup

$$H_i = \{g \in G | \ \mathrm{Ad}_g(A'_j) = A'_j \text{ for all } j \geq 1\} \subset G$$

called its centralizer.
$\mathcal{R} \subset \mathfrak{t}^\vee$: set of roots of the Lie algebra $\mathfrak{g}$ of G w.r.t. $\mathfrak{t}$
$\widetilde{X}$: real oriented blow-up of X at p_i
∂_i: fiber of $\widetilde{X}$ over p_i (boundary circle)
Elements of ∂_i are called directions at p_i, and are denoted by d.

Stokes local systems – Stokes directions and subgroups

Definition

A direction d is called a singular direction supported by $\alpha \in \mathcal{R}$ if it is the tangent direction to one of the components of the curve

$$\mathrm{Im}(\alpha \circ Q(z_i)) = 0, \quad \mathrm{Re}(\alpha \circ Q(z_i)) < 0.$$

$\mathcal{R}(d) = \{\alpha | \alpha \text{ supports singular direction } d\}$
$\mathbb{S}\mathrm{to}_d(Q_i) \subset G$: closed subgroup corresponding to the Lie subalgebra spanned by $\mathfrak{g}_\alpha$ for all $\alpha \in \mathcal{R}(d)$.

$$\mathbb{S}\mathrm{to}(Q_i) = \prod_{d \in \partial_i} \mathbb{S}\mathrm{to}_d(Q_i),$$

Stokes representations

For all i, fix a base point b_i near $p_i \rightsquigarrow$ fundamental groupoid Π of the irregular curve. For all singular directions d at p_i, make a puncture on $\widetilde{X}$ "close to" $d \in \partial_i$.

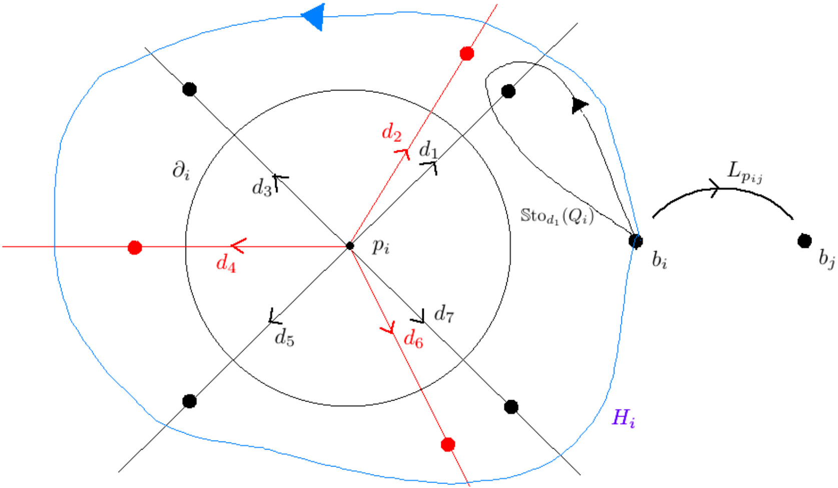
Stokes representation: representation ρ of Π in G, satisfying:

- on a loop based at b_i winding around a puncture corresponding to a singular direction d, ρ takes values in $\mathbb{S}\mathrm{to}_d(Q_i)$,
- on a loop based at base point b_i around p_i, ρ takes values in H_i.

Theorem (P. Boalch'14)

The space $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ *of Stokes representations of* Π *in* G *is a smooth complex affine variety and is canonically a quasi-Hamiltonian* H*-space, where* $H = H_1 \times \cdots \times H_m \subset G^m$.

Stokes representations for JKTII



RHB correspondence

Theorem (Riemann–Hilbert–Birkhoff correspondence, P. Boalch'14)

The isomorphism classes of meromorphic connections on algebraic principal G-bundles on X with irregular type Q_i at p_i correspond bijectively to the $H_1 \times \cdots \times H_m$-orbits in $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$.

The subset of connections whose associated Stokes representation is irreducible carries a natural structure of regular holomorphic Poisson variety, called the wild character variety.

Betti symplectic leaves

Any symplectic leaf of the wild character variety is given as the quasi-Hamiltonian reduction of $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ at $\prod_i \mathcal{C}_i$ for some set of conjugacy classes $\mathcal{C}_i \subset H_i$:

$$\mu_H^{-1}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_m)/H_1 \times \cdots \times H_m,$$

where

$$\mu_H \colon \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \to H_1 \times \cdots \times H_m$$
$$\{(C_i, h_i, S_1^i, S_2^i, \ldots)\}_{i=1}^m \mapsto (h_1, \ldots, h_m).$$

The wild character variety determined by G, the irregular curve $(\mathbb{C}P^1, \mathbf{p}, \{Q_i\}_{i=1}^m)$ and the conjugacy classes $\{\mathcal{C}_i\}_{i=1}^m$ is the holomorphic symplectic variety above.

RHB for symplectic leaves

For fixed $\Lambda_i \in \mathfrak{gl}(r, \mathbb{C})$, we let $\mathcal{C}_i$ be the conjugacy class of $\exp(2\pi\sqrt{-1}\Lambda_i)$.

Theorem (RHB, Symplectic version)

The wild character variety determined by G, the irregular curve $(\mathbb{C}P^1, \mathbf{p}, \{Q_i\}_{i=1}^m)$ and the conjugacy classes $\{\mathcal{C}_i\}_{i=1}^m$ is $\mathbb{C}$-analytically isomorphic to the moduli space $\mathcal{M}_{dR}$ parameterizing gauge-equivalence classes of (irreducible) irregular connections having the same irregular types Q_i and residues belonging to the adjoint orbit of Λ_i at p_i.

The analytical monodromy about singular point p is

$$M_p = L_p H \left(\prod_{\varphi} S_\varphi \right) L_p^{-1},$$

where φ ranges through the set of Stokes directions in positive

Stokes directions

Consider the Stokes local systems associated to the *JKT*-systems. Let the eigenvalues of the irregular part of ∇ at ∞ be (with $d = 2$ or 3):

$$q_i = \lambda_1 w^{-1/d} + \cdots + \lambda_k w^{-k/d}, \quad q_j = \lambda'_1 w^{-1/d} + \cdots + \lambda'_k w^{-k/d}$$

Consider their difference:

$$q_i - q_j = (\lambda_1 - \lambda'_1)w^{-1/d} + \ldots + (\lambda_l - \lambda'_l)w^{-l/d}$$

for some $k, l, d \in \mathbb{Z}_+$, so that $(\lambda_l - \lambda'_l) \neq 0$, and at least one of λ_k, λ'_k is nonzero.

Stokes directions for the pair $\{q_i - q_j\}$ form the set of $\varphi \in S^1$ that are tangent at 0 to the smooth components of the curve

$$\mathrm{Re}\left((\lambda_l - \lambda'_l)w^{-l/d}\right) = 0, \qquad w = |w|e^{\sqrt{-1}\varphi}.$$

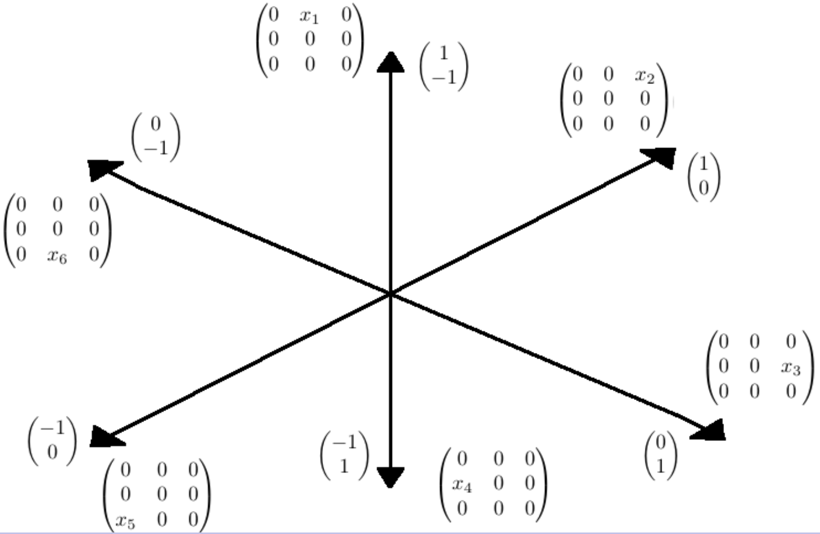
Invariant monomials

Consider the conjugation action of the maximal torus of G on matrices:

$$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\alpha & x_1 & x_2\\ x_4 & \beta & x_3\\ x_5 & x_6 & \gamma\end{pmatrix}\cdot\begin{pmatrix}\lambda^{-1} & 0 & 0\\ 0 & \mu^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\alpha & \lambda\mu^{-1}x_1 & \lambda x_2\\ \lambda^{-1}\mu x_4 & \beta & \mu x_3\\ \lambda^{-1}x_5 & \mu^{-1}x_6 & \gamma\end{pmatrix}$$

The ring of invariant elements of the coordinate ring of Stokes data is generated by the monomials where the sum of the weight vectors add up to 0. These are: x_1x_4, x_2x_5, x_3x_6, $x_1x_3x_5$, $x_2x_4x_6$.

A_2 root system with the weight vectors



Example - *JKTII*

In the minimally twisted case, the exponential torus is 1-dimensional, i.e. the possible base changes are given by $\{e_+, e_-, e_2\} \leftrightarrow \{\lambda e_+, \lambda e_-, \lambda^{-2} e_2\}$ for $\lambda \in \mathbb{C}^\times$:

$$H_2 = \begin{pmatrix} 0 & -\alpha^{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

for $\alpha \in \mathbb{C}^\times$. The ring of invariant monomials is

$$\mathbb{C}[x_1, x_2, x_3, x_5, x_6]^{H_2} = \mathbb{C}[U, V, W, R, T]/\mathcal{I}$$

where $U := x_2 x_5$, $V := x_3 x_6$, $W := x_1$, $R := x_2 x_6$, $T := x_1 x_3 x_5$, and $\mathcal{I} = (UVW - RT)$.

Example - *JKTII*

$q_2 = -\frac{1}{2} a_1 w^{-2} + O(w^{-1})$ and
$q_{0,1} = -\frac{1}{2} a_0 w^{-2} + \lambda_3 w^{-3/2} + O(w^{-1})$
The Stokes directions are:

- $\varphi = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$, solving $\mathrm{Re}(w^{-3/2}) = 0$, corresponding to (q_0, q_1) pairs,
- $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, solving $\mathrm{Re}(w^{-2}) = 0$, corresponding to $(q_{0,1}, q_2)$ pairs.

Thus, the Stokes matrices are in the appropriate order:

$$S_1 = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_5 & x_6 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 & 0 \\ x_4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S_5 = \begin{pmatrix} 1 & 0 & x_8 \\ 0 & 1 & x_9 \\ 0 & 0 & 1 \end{pmatrix}, S_6 = \begin{pmatrix} 1 & x_7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_{11} & x_{12} & 1 \end{pmatrix}.$$

Example - *JKTII*

For the topological monodromy $M_\infty = H\prod_{i=7}^{1} S_i = I$, or equivalently $S_3S_2S_1 = (HS_7S_6S_5S_4)^{-1}$, leading to the equations

$$\alpha^{-1} = x_2x_5 + x_3x_6 + x_1x_3x_5 + 1 = U + V + T + 1$$
$$W = \alpha(UW + R + TW + VW) + 1$$
$$UVW = RT$$

After linear transformations $U = X - 1$, $V = \alpha^{-1}Y - 1$, $W = Z$, this yields the equation of the wild character variety as an affine algebraic surface:

$$\mathcal{M}_B^{JKTII} = \{(X, Y, Z) \in \mathbb{C}^3 | XYZ - X - \alpha^{-1}Y - Z + 1 + \alpha^{-1} = 0\}.$$

Agrees with the result of M. van der Put, M. Saito'09 describing the rank 2 JMU representations of the Painlevé II system.

The main result I.

Theorem

Under suitable choices of parameters, the affine cubic surfaces describing the JKT wild character varieties are isomorphic to those of the corresponding Painlevé case.
Moreover there are concrete correspondences with the parameters of the Painlevé wild character varieties.

Summary		
JKT system	equation of $\mathcal{M}_B$ moduli space	$\dim\mathcal{P}$
JKTVI	$\gamma XYZ + \alpha X^2 + \beta Y^2 + \gamma Z^2 + c_1X + c_2Y + c_3Z + c_4 = 0$	4
JKTV	$XYZ + X^2 + Y^2 + c_1X + c_2Y + c_3Z + c_4 = 0$	3
JKTIVa	$XYZ + X^2 + c_1X + c_2Y + c_3Z + c_4 = 0$	2
JKTIVb	$XYZ + Y^2 + c_1X + c_2Y + c_3Z + c_4 = 0$	2
JKTII	$XYZ - X - \alpha^{-1}Y - Z + 1 + \alpha^{-1} = 0$	1
JKTI	$XYZ + X + Y + 1 = 0$	0

The main result II.

Theorem

Fourier–Laplace transformation $\mathcal{F}$ of $\mathcal{D}$-modules induces a hyperKähler isometry between the moduli spaces of rank 2 irregular connections associated to the Painlevé equations and the corresponding rank 3 JKT moduli spaces of irregular connections.

- Strategy: show that $\mathcal{F}$ is algebraic in both the Dolbeault and de Rham complex structures, and preserves the Dolbeault holomorphic symplectic form.
- J. Douçot'24 analyzed the de Rham parts of the same isomorphisms from the perspective of supernova graphs.
- P. Boalch'12 described more generally certain isomorphisms between moduli spaces of irregular connections defined over the trivial bundle given by different "readings" of so-called supernova graphs.

(Geometric) Fourier–Laplace transformation

- Let $\widehat{X} = \operatorname{Spec} \mathbb{C}[\hat{z}] \cup \operatorname{Spec} \mathbb{C}[\hat{w}]$ be another copy of $\mathbb{C}P^1$ ($\hat{w} = \hat{z}^{-1}$).
- Denote the projection morphisms by $p \colon X \times \widehat{X} \to X$, $\hat{p} \colon X \times \widehat{X} \to \widehat{X}$.
- Denote by ψ the sheaf of $\mathcal{D}_{X \times \widehat{X}}$-modules that is the free $\mathcal{O}_{X \times \widehat{X}}$-module of rank 1, endowed with the connection

$$\mathrm{d} + \mathrm{d}(z\hat{z}) = \mathrm{d} + \hat{z}\,\mathrm{d}z + z\,\mathrm{d}\hat{z}.$$

- A minimal extension $\mathbb{M}_{\min}$ of $\mathbb{M}$ over C is defined by
  1. $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} \mathbb{M}_{\min} \cong \mathbb{C}(z) \otimes_{\mathbb{C}[z]} \mathbb{M}$ (it is an extension), and
  2. $\mathbb{M}_{\min}$ admits no quotient or submodules supported in dimension 0 (it is minimal).

Definition

The Fourier–Laplace transform of $\mathbb{M}$ is the $\mathcal{D}_{\widehat{X}}$-module $\widehat{\mathbb{M}} = \mathcal{F}(\mathbb{M}) = R\hat{p}_*(p^*\mathbb{M}_{\min} \otimes \psi)[1]$

Local Fourier–Laplace transformation

Aim: understand the contribution of the singularities of $\mathbb{M}$ at its singular points C to the singularities of its Fourier transform $\widehat{\mathbb{M}}$. We use the definition of G. Laumon'87 of local versions $\mathcal{F}^{0,\widehat{\infty}}, \mathcal{F}^{\infty,\hat{0}}, \mathcal{F}^{\infty,\widehat{\infty}}$ of the functor $\mathcal{F}$ for ℓ-adic sheaves. B. Malgrange'91: gave $\mathcal{D}$-module versions.

Theorem (G. Laumon, B. Malgrange)

The functors $\mathcal{F}^{0,\widehat{\infty}}, \mathcal{F}^{\infty,\hat{0}}, \mathcal{F}^{\infty,\widehat{\infty}}$ are exact. Furthermore, assume that $\mathbb{M}$ is a holonomic $\mathcal{D}_X$-module with singular set $C \subseteq \{0, \infty\}$, and if $0 \in C$ then it is a regular singularity, and ∞ is an irregular singularity. By the corresponding data of $\mathbb{M}$, the followings can be completely described:

- *Swan-conductor at a singular point and the rank of $\widehat{\mathbb{M}}$*
- *list of singularities of $\widehat{\mathbb{M}}$ and the formal types of $\widehat{\mathbb{M}}$ at its singular points*

Example - *JKTII*

The only (irregular) singularity of $\mathbb{M}$ being placed at ∞, we choose

$$A'_{-3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad A'_{-2} = \begin{pmatrix} q_1 & 0 & 0 \\ q_3 & q_1 & 0 \\ 0 & 0 & q_2 \end{pmatrix}$$

for $q, q_3 \neq 0$ and arbitrary q_1, q_2. Then, $\widehat{\mathbb{M}}$ has a regular singularity at $\hat{z} = q_2$. Moreover, it has an irregular singularity at $\hat{z} = \widehat{\infty}$ with

$$\mathrm{Sw}(\widehat{\mathbb{M}}_{\widehat{\infty}}) = \mathrm{Sw}(\mathbb{M}_\infty) = 2 \cdot \frac{3}{2} + 2 = 5, \quad \mathrm{rk}(\widehat{\mathbb{M}}_{\widehat{\infty}}) = 5 - 3 = 2.$$

Example - *JKTII*

Formally, $\widehat{\mathbb{M}}_{\widehat{\infty}}$ decomposes into a sum of two rank 1 modules, of slopes 3 and 2 respectively, so that

$$\hat{Q} = \hat{A}_{-4}\hat{w}^{-3} + \hat{A}_{-3}\hat{w}^{-2} + \hat{A}_{-2}\hat{w}^{-1}$$

with

$$\hat{A}_{-4} = \begin{pmatrix} q_3^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}_{-3} = \begin{pmatrix} * & 0 \\ 0 & -q^{-1} \end{pmatrix}.$$

This gives a special case of the JMU representation of Painlevé II, namely one eigenvalue of $\hat{A}_{-4}$ vanishes.

Dolbeault spaces

- We construct a surface Z^* for each $* \in \{VI, V, IVa, IVb, II, I\}$ such that the Beauville–Narasimhan–Ramanan (or BNR or spectral) correspondence establishes an algebraic isomorphism between $\mathcal{M}_{\mathrm{Dol}}^{JKT*}$ and the moduli space $\mathcal{M}_{\mathrm{Muk}}^{*}$ of sheaves on Z^* with suitably fixed Chern classes.
- The surface Z^* can be defined as a suitable multiple blow-up of the Hirzebruch surface of index one $\mathbb{F}_1$, followed by the blow-down of its section at ∞:

$$\omega \colon Z^* \dashrightarrow \mathbb{F}_1.$$

Spectral description

Consider $K_{\mathbb{C}P^1}(D) \cong \mathcal{O}_{\mathbb{C}P^1}(1)$, and $p\colon \mathbb{F}_1 \to \mathbb{C}P^1$ (the ruling).
$\mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)$ the relative ample bundle of p, with canonical sections:

$$\zeta \in H^0(\mathbb{F}_1, p^* K_{\mathbb{C}P^1}(D) \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)), \quad \xi \in H^0(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1)).$$

The section $\xi = 0$ is denoted by σ_∞: section at infinity of p.

Definition

The *spectral curve* $\Sigma_{(\mathcal{E},\theta)}$ of $(\mathcal{E},\theta)$ is the algebraic curve determined by:

$$\chi_\theta(\zeta,\xi) = \det(\zeta I_{\mathcal{E}} - \xi p^*\theta) = \zeta^3 + F_\theta \zeta^2 \xi + G_\theta \zeta \xi^2 + H_\theta \xi^3 = 0.$$

The *spectral sheaf* is the coherent sheaf $\mathcal{S}_{(\mathcal{E},\theta)}$ on $\mathbb{F}_1$ defined by:

$$0 \to p^*(\mathcal{E} \otimes K_{\mathbb{C}P^1}(D)^{-1}) \xrightarrow{\zeta I_{\mathcal{E}} - \xi p^*\theta} p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_1|\mathbb{C}P^1}(1) \to \mathcal{S}_{(\mathcal{E},\theta)} \to 0.$$

Hitchin fibration

The characteristic coefficients $F_\theta, G_\theta, H_\theta$ satisfy:

$$(F_\theta, G_\theta, H_\theta) \in H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)) \oplus H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)^{\otimes 2}) \oplus H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}(D)^{\otimes 3}) = \mathcal{B}$$

$\mathcal{B}$ is called the Hitchin base.

Proposition

For fixed values of the parameters in the local forms of θ, the set of possible $F_\theta, G_\theta, H_\theta$ forms an affine subspace of $\mathcal{B}$ of dimension 1.

Definition

The map h, called the *Hitchin fibration* is

$$h : \mathcal{M}_{Dol}^{JKT*} \to \mathbb{C}$$
$$(\mathcal{E}, \theta) \mapsto (F_\theta, G_\theta, H_\theta)$$

BNR-correspondence

Theorem (Beauville–Narasimhan–Ramanan correspondence, Sz. '17)

There is an equivalence of categories between the groupoid of irregular Higgs bundles with prescribed polar part and the relative Picard scheme parameterizing torsion-free sheaves of rank 1 *and of given degree on holomorphic curves in a given homology class* $[F_{\infty}^{JKT*}] \in H_2(Z^*, \mathbb{Z})$ *on a certain multiple blow-up* Z^* *of the Hirzebruch surface* $\mathbb{F}_1$*.*

Here, F_{∞}^{JKT*}: fiber at infinity. As a consequence:

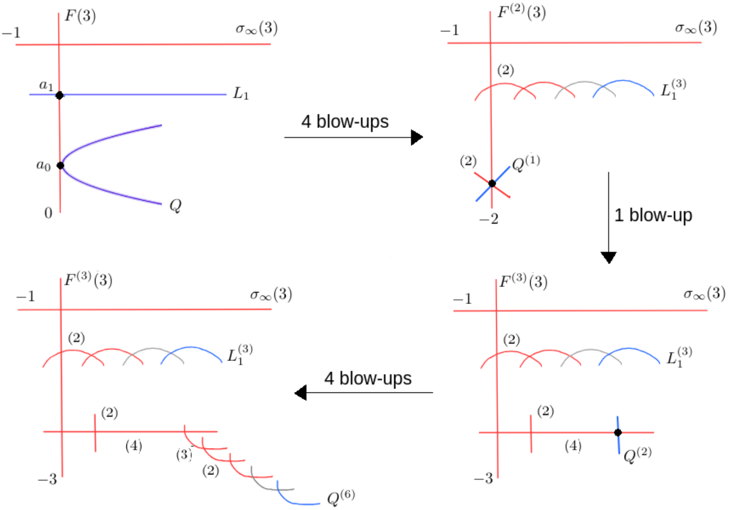
Proposition

For generic $b_0 \in \mathcal{B}$*, the fiber* $h^{-1}(b_0)$ *of the Hitchin fibration is a torsor over the Jacobian* $\mathrm{Jac}(\Sigma_{(\mathcal{E},\theta)})$ *for any* $(\mathcal{E},\theta) \in h^{-1}(b_0)$*. In particular,* $h^{-1}(b_0)$ *is a smooth elliptic curve.*

Example - Construction of Z^{II}

- Consider the curves $C_1 = Q + L_1$, and $C_2 = 3F + 3\sigma_\infty$, with corresponding divisors $D_1 = (f_1), D_2 = (f_2)$ (for some f_i functions) on $\mathbb{F}_1$.
- Consider
$$\{(\lambda_1 f_1 + \lambda_2 f_2) | [\lambda_1 : \lambda_2] \in \mathbb{C}P^1\}$$
complete family of linearly equivalent divisors (i.e. a pencil).
- Blow up the pencil at its base points $C_1 \cap C_2$, and blow-down σ_∞
- It is easy to see that the fiber F_∞^{JKTII} at infinity comes from the red curve, and it is of type $\tilde{E}_7$.
- We see in particular that $H^0(Z^*, \mathcal{O}(F_\infty^{JKTII})) \cong \mathbb{C}^2$, and $[F_\infty^{JKTII}] = -[K_{Z^*}]$.

Example - Construction of Z^{II}



Summary of the six cases

Summary			
JKT system	polar divisor of θ	type of irregular singularity at ∞	Dynkin diagram of F_∞^{JKT*}
JKTVI	$D = \{0\} + 2\{\infty\}$	untwisted	I_0^* (or $\widetilde{D}_4$)
JKTV	$D = \{0\} + 2\{\infty\}$	minimally twisted	I_1^* (or $\widetilde{D}_5$)
JKTIVa	$D = \{0\} + 2\{\infty\}$	maximally twisted	$\widetilde{E}_6$
JKTIVb	$D = 3\{\infty\}$	untwisted	$\widetilde{E}_6$
JKTII	$D = 3\{\infty\}$	minimally twisted	$\widetilde{E}_7$
JKTI	$D = 3\{\infty\}$	maximally twisted	$\widetilde{E}_8$

Algebraic Nahm transform

Our aim is to show:

Theorem

Fourier–Laplace transformation establishes an algebraic symplectic isomorphism with respect to the Dolbeault complex structures between each $\mathcal{M}_{Dol}^{JKT*}$ and the corresponding $\mathcal{M}_{Dol}^{P*}$.

For each Z^*, there exists a birational morphism $Z^* \dashrightarrow X \times \widehat{X}$, defined on the complement of the blow-down of σ_∞,
In particular, the projection morphisms $p\colon Z^* \dashrightarrow X$, $\hat{p}\colon Z^* \dashrightarrow \widehat{X}$ are well-defined away from the blow-down of σ_∞.

Definition

The (algebraic) Nahm transform of $(\mathcal{E}, \theta) \in \mathcal{M}_{\mathrm{Dol}}^{JKT*}$ is

$$\mathcal{N}(\mathcal{E}, \theta) = (\hat{p}_* \mathcal{S}_{(\mathcal{E},\theta)}, \hat{p}_*(p^* z \cdot)\, \mathrm{d}\hat{z}).$$

Properties of Nahm transformation

$\mathbf{\Omega}_{\mathrm{Dol}}^{JKT*} \in H^{2,0}(\mathcal{M}_{\mathrm{Dol}}^{JKT*})$, and $\mathbf{\Omega}_{\mathrm{Dol}}^{P*} \in H^{2,0}(\mathcal{M}_{\mathrm{Dol}}^{P*})$: holomorphic symplectic 2-forms.

Proposition

Nahm transformation is an algebraic symplectic isomorphism

$$\mathcal{N} \colon (\mathcal{M}_{Dol}^{JKT*}, \mathbf{\Omega}_{Dol}^{JKT*}) \to (\mathcal{M}_{Dol}^{P*}, \mathbf{\Omega}_{Dol}^{P*}).$$

Moreover, there exists a commutative diagram of diffeomorphisms between smooth 4-manifolds:

$$\begin{array}{ccc} \mathcal{M}_{Dol}^{JKT*} & \xrightarrow{\mathcal{N}} & \mathcal{M}_{Dol}^{P*} \\ {\scriptstyle NAH}\downarrow & & \downarrow{\scriptstyle NAH} \\ \mathcal{M}_{dR}^{JKT*} & \xrightarrow{\mathcal{F}} & \mathcal{M}_{dR}^{P*} \end{array}$$

Mukai moduli spaces and conclusion

- Consider the log-Calabi–Yau pairs (Z^*, F_∞^{JKT*}).
- We let $\mathcal{M}^*_{\mathrm{Muk}}$ denote the moduli space of simple coherent sheaves $\mathcal{S}$ on Z^* such that
  - $c_0(\mathcal{S}) = 0 \in H^0(Z^*, \mathbb{Z})$,
  - $c_1(\mathcal{S}) = PD[F_\infty^{JKTII}] \in H^2(Z^*, \mathbb{Z})$,
  - $c_2(\mathcal{S}) = PD[pt] \in H^4(Z^*, \mathbb{Z})$,

  (PD=Poincaré duality, and $[pt]$=homology class of a point)
- It is a smooth symplectic quasi-projective surface with holomorphic symplectic form $\mathbf{\Omega}^*_{\mathrm{Muk}} \in H^{2,0}(\mathcal{M}^*_{\mathrm{Muk}})$.

Proposition

The isomorphism between $\mathcal{M}_{Dol}^{JKT*}$ and $\mathcal{M}^*_{Muk}$ preserves their holomorphic symplectic 2-forms $\mathbf{\Omega}_{Dol}^{JKT*}$ and $\mathbf{\Omega}^*_{Muk}$, and the same holds for the isomorphism between $\mathcal{M}_{Dol}^{P*}$ and $\mathcal{M}^*_{Muk}$ too.

Thank you for your attention!