The monodromy of meromorphic projective structures

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Web-seminar on Painlevé Equations and related topics

➥ **What are (meromorphic) projective structures?**

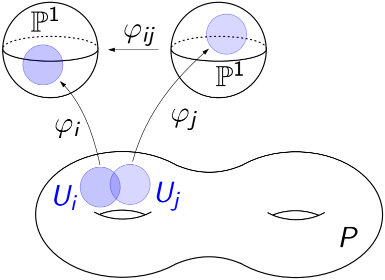
➥ **Does the monodromy data of a (meromorphic) projective structure characterizes it?**

➥ Theorem (Hejhal, 1975): Locally, yes (non-singular cases).

➥ Theorem (T.S.): Locally, yes (meromorphic cases).

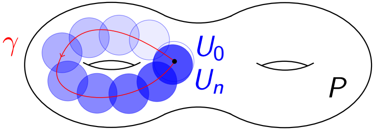
Under suitable assumptions.

Definitions



Definition

A **complex projective structure** P is a Riemann surface such that $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} =$ a restriction of a $g_{ij} \in \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}(2, \mathbb{C})$.



The **monodromy representation** of P w.r.t. φ_0:

$$\begin{aligned} \pi_1(P, \bullet) &\longrightarrow \mathrm{PGL}(2, \mathbb{C}) \\ \gamma &\longmapsto \rho_{\varphi_0}(\gamma) = g_{n,(n-1)} \circ \cdots \circ g_{2,1} \circ g_{1,0} \end{aligned}$$

Definitions

Fix S an infinitely differentiable *compact* real oriented surface.

$\mathcal{P}(S) := \{$isomorphism classes of **marked** proj. structures on $S\}$

$P_1 \sim P_2 \Leftrightarrow \exists \Phi : S \to S$ a $\mathcal{C}^\infty$-diffeomorphism $\underline{\text{isotopic to } \mathrm{id}_S}$
pulling back any chart of P_2 to a chart of P_1

$\mathcal{R}(S) := \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(2,\mathbb{C}))/\,\mathrm{PGL}(2,\mathbb{C})$

Definition (**Monodromy map**):

$$\mathrm{Mon}_S : \mathcal{P}(S) \longrightarrow \mathcal{R}(S)$$

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Two natural questions:

- Is it injective?
- Is it surjective?
- **Same questions locally?**

Relation to quadratic differentials

Fix C a smooth compact complex curve.

The set of projective structures on C is an affine space for the vector space $H^0(C, (T^*C)^{\otimes 2})$ of **quadratic differentials**.

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➥ P_1, P_2 projective structures on C
(z_1, z_2 corresponding projective coordinates, $\psi := z_2 \circ z_1^{-1}$)

$$P_1 - P_2 = \phi := \frac{\mathcal{S}_{z_1}(\psi)}{2} dz_1^{\otimes 2} \text{ (Schwarzian derivative)}$$

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- P_1 projective structure on C, and $\phi_{z_1} := \frac{q(z_1)}{2} dz_1^{\otimes 2}$

$P_1 + \phi$: charts are the solutions of $\mathcal{S}_{z_1}(\varphi) = q(z_1)$

$\Leftrightarrow$ charts are the quotients of independent solutions of $y'' + \dfrac{q(z_1)}{2} y = 0.$

The theorem of Hejhal

$\mathcal{P}(S) \longrightarrow \mathcal{T}(S)$ is an affine bundle for the vector bundle $\mathcal{Q}(S) \longrightarrow \mathcal{T}(S)$ of quadratic differentials.

Theorem (Hejhal 1975, Earle, Hubbard 1981)
If $g \geq 2$, $\mathrm{Mon}_S^{g\geq 2} : \mathcal{P}(S) \to \mathcal{R}^{\mathrm{nc}}(S)$ is a local biholomorphism.

➥ **What about projective structures with poles?**

Definition

C a complex curve.

Definition

A **meromorphic projective structure on** C is a projective structure P^* on the complement $C^* = C \setminus \Sigma$ of a finite subset $\Sigma \subset C$, such that given a holomorphic projective structure P_0 on C, the quadratic differential $\phi = P^* - P_{0|C^*}$ on C^* extends to a meromorphic quadratic differential on C.

➥ Pole orders are well-defined (do not depend on P_0).

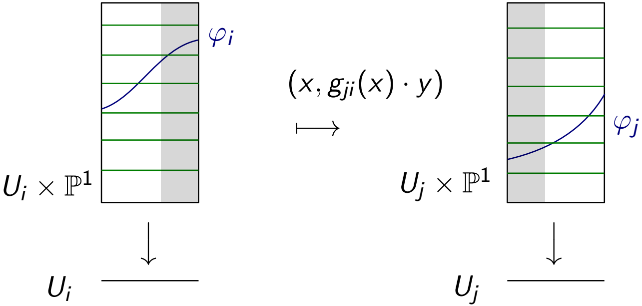
Relation to Painlevé VI equation

The space of projective structures on $\mathbb{P}^1$ with 5 singularities of orders ≤ 2, with exactly 1 apparent singularity, and fixed **residues**, is of dimension 3.

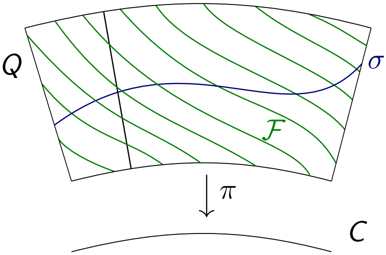
The associated isomonodromic foliation in this space is the Painlevé VI foliation.

P a (non-singular) projective structure on C, with atlas (U_i, φ_i).

$$\begin{aligned} g_{ij} : U_i \cap U_j &\longrightarrow \mathrm{PGL}(2,\mathbb{C}) \\ x &\longmapsto g_{ij}(x) \text{ whose restriction equals } \varphi_{ij} \end{aligned}$$



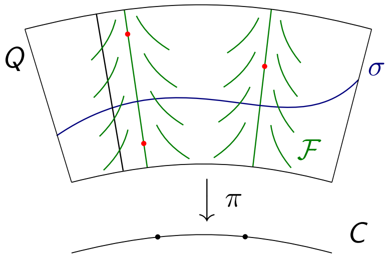
PGL$(2,\mathbb{C})$-opers on C



This defines **a** PGL$(2,\mathbb{C})$-**oper** $(\pi : Q \to C, \mathcal{F}, \sigma)$ **on** C:

- $\pi : Q \to C$ is a holomorphic $\mathbb{P}^1$-bundle,
- $\mathcal{F}$ is a Riccati foliation on Q,
- $\sigma : C \to Q$ is a holomorphic section of π, transverse to $\mathcal{F}$.

And vice versa.



meromorphic proj. structure $\longmapsto$ meromorphic oper

(without apparent sing.) (unique minimal birational model)

Moduli space of rank 2 meromorphic connections

- $(\pi : Q \to C, \mathcal{F}, \sigma)$ lifts to **a** $\mathrm{GL}(2,\mathbb{C})$-**oper** (E, ∇, L) **on** C.
- Oper condition + no apparent singularities $\Rightarrow$ opers belong to the smooth locus of the moduli space of (Inaba 2016, 2021)

$$\mathcal{M}^{\alpha,\circ}_{C,D,\Lambda}.$$

- Thanks to the wild Riemann-Hilbert correspondence

$$\mathcal{M}^{\alpha,\circ}_{C,D,\Lambda} \xrightarrow{1:1} \mathcal{R}(S,(n_i))$$

we get a smooth wild character variety $\mathcal{R}^*(S,(n_i),(\lambda^{(i)}_{-1}))$.
- Projectivized version: $\bar{\mathcal{R}}^*(S,(n_i),(\lambda^{(i)}_{-1}))$.

Fix S, (n_i) and the **residues** $(\lambda_{-1}^{(i)})$.
(Residues are defined up to $\times(-1)$; we make a choice).

$$\begin{tikzcd}
\mathcal{P}^\circ(S,(n_i),(\lambda_{-1}^{(i)})) \arrow[r] \arrow[dr, "\mathrm{Mon}_{S,(n_i),(\lambda_{-1}^{(i)})}"'] & \mathcal{M}(S,(n_i),(\lambda_{-1}^{(i)})) = \bar{\mathcal{R}}^*(S,(n_i),(\lambda_{-1}^{(i)})) \times T \arrow[d] \\
& \bar{\mathcal{R}}^*(S,(n_i),(\lambda_{-1}^{(i)}))
\end{tikzcd}$$

➥ The monodromy map is holomorphic (Allegretti/Bridgeland 2020).

A generalization of Hejhal's theorem

Theorem (T.S.)

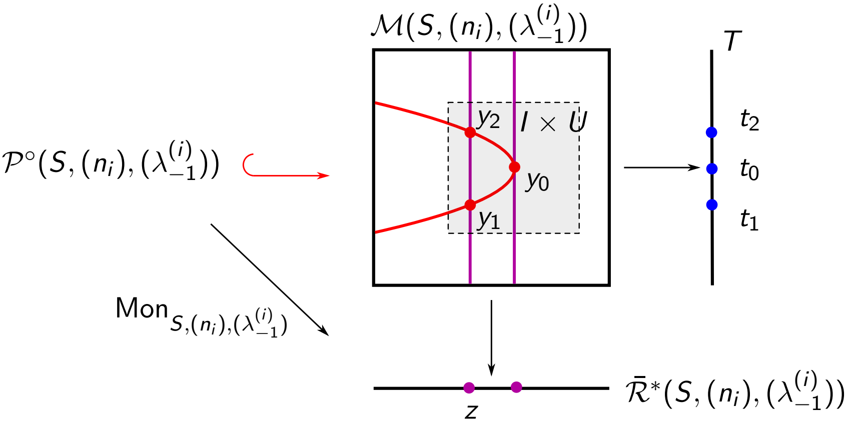
Assume that we are not in one of the special cases listed below. Then, the monodromy map

$$\mathrm{Mon}_{S,(n_i),(\lambda_{-1}^{(i)})} : \mathcal{P}^{\circ}(S,(n_i),(\lambda_{-1}^{(i)})) \longrightarrow \bar{\mathcal{R}}^{*}(S,(n_i),(\lambda_{-1}^{(i)}))$$

is a **local biholomorphism**.

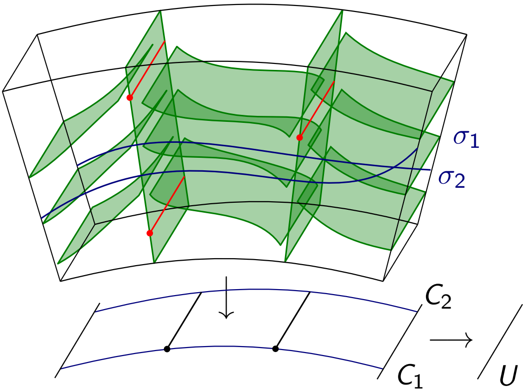
Special cases:

- $g = 0$, $(n_i) = (1)$, (2), (3), (4), $(1,1)$, $(1,2)$, $(2,2)$, $(1,3)$ or $(2,3)$ (up to a permutation of its entries)
- $g = 1$, $(n_i) = \emptyset$



Local injectivity $\Leftrightarrow$ transversality.

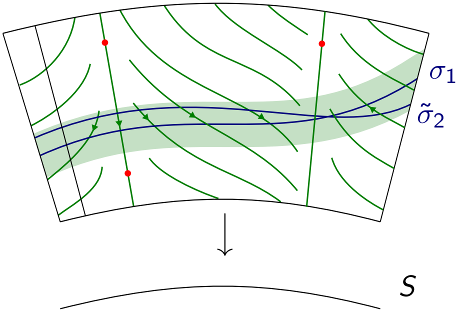
Local injectivity of the monodromy map



Isomonodromic deformations are induced by **a codimension one foliation** (Heu, 2010).

This allows to lift the $\mathcal{C}^\infty$ trivialization of the family of curves (Ehresmann).

Local injectivity of the monodromy map



Existing works

This theorem was already proven in special cases via different techniques:

- all poles have order ≤ 2 with non-trivial and non-parabolic local monodromy (Luo 1993);
- all poles have order ≤ 2 with parabolic local monodromy and some specific residues (Hussenot Desenonges 2019);
- all poles have order ≥ 3 (Gupta/Mj 2021).

Thank you for your attention!