Various forms of the Riemann-Hilbert correspondance for q-difference equations

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Summary

In 1913, Birkhoff proposed a new solution (different of that of Hilbert and Plemelj) to what he called the "Riemann problem" and used it to provide a unified framework for (linear complex analytic) differential, difference and q-difference equations. Starting from Birkhoff, I shall describe various ways of associating "intrinsic transcendental invariants" (as Birkhoff said) to q-difference equations: geometric, galoisian, cohomological ... up to a mysterious sheaf theoretical formulation by Kontsevitch and Soibelman. Most of that will rest on work by the Ramis school.

Plan

Birkhoff's formulation of "the Riemann problem"

Birkhoff 1913 (regular singular case only):
"The program of obtaining a characterization of a function in simple descriptive terms which are independent of the equations of definition of the function ..."
"...a certain number of characteristic constants - the monodromic group constants ..."
"Riemann also proposed the associated problem of assigning these constants at pleasure."

Birkhoff and Guenther 1941 (also tackling irregular case):
"...the explicit determination of the essential transcendental invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of $x = \infty$ and in the complete plane (case of rational coefficients), ..."

General Notations

All along, we let $q \in \mathbf{C}^*$ s.t. $|q| < 1$ and set $\sigma_q f(x) := f(qx)$, where *e.g.* $f \in K := \mathbf{C}(x)$, $\mathbf{C}(\{x\})$ or $\mathbf{C}((x))$.

For $K := \mathcal{M}(\mathbf{C}^*)$, the field of q-constants is $\mathcal{M}(\mathbf{C}^*)^{\sigma_q} = \mathcal{M}(\mathbf{E}_q)$, where $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$, the Tate elliptic curve (q-elliptic functions).

This extends coefficientwise to vectors and matrices.

We identify the q-difference system $\sigma_q X = AX$ with its matrix $A(x) \in \mathrm{GL}_n(K)$.

A *morphism* $A \in \mathrm{GL}_n(K) \to B \in \mathrm{GL}_p(K)$ is a $F \in \mathrm{Mat}_{p,n}(K)$ such that $(\sigma_q F)A = BF$.
It sends a solution X of A to the solution FX of B.

If $n = p$ and $F \in \mathrm{GL}_n(K)$, the isomorphism F is also called a *gauge equivalence*.
We then write $B = F[A] := (\sigma_q F)AF^{-1}$.

Birkhoff 1913

Birkhoff considers a system:

$$\sigma_q X = AX, \quad A = A_0 + \cdots + A_\mu x^\mu, \quad A_0, \ldots, A_\mu \in \mathrm{Mat}_n(\mathbf{C}),$$

with the *fuchsianity* condition at 0 and ∞:

$$\mathrm{Sp}(A_0) =: \{\rho_1, \ldots, \rho_n\} \subset \mathbf{C}^*, \quad \mathrm{Sp}(A_\mu) =: \{\sigma_1, \ldots, \sigma_n\} \subset \mathbf{C}^*,$$

and the *strong non-resonancy* condition:

$$\forall i, j = 1, \ldots, n \ , \ i \neq j \Longrightarrow \rho_i/\rho_j \notin q^{\mathbf{Z}} \text{ and } \sigma_i/\sigma_j \notin q^{\mathbf{Z}}.$$

We also require "fuchsianity at intermediate singularities":

$$\det A(x) = \sigma_1 \cdots \sigma_n (x - x_1) \cdots (x - x_N), \quad N := n\mu,$$

where:

$$i \neq j \Longrightarrow x_i/x_j \notin q^{\mathbf{Z}}.$$

Note *Fuchs relation*: $(-1)^N x_1 \cdots x_N = \rho_1 \cdots \rho_n / \sigma_1 \cdots \sigma_n$.

Construction of local solutions

First we need a function e_x s.t. $\sigma_q e_x = x e_x$ and "q-characters" e_c s.t. $\sigma_q e_c = c e_c$. Birkhoff takes them multivalued: $e_x := q^{(\log_q x)(\log_q x - 1)/2}$ and $e_c(x) := x^{\log_q c}$.

Following Ramis (1990), they could be taken uniform, using a Tate Theta function $\theta_q \in \mathcal{O}(\mathbf{C}^*)$ s.t. $\sigma_q \theta_q = x^{-1} \theta_q$.

Combining them, we build invertible matrices e_{A_0} and e_{A_∞} s.t. $\sigma_q e_{A_0} = A_0 e_{A_0}$ and $\sigma_q e_{A_\infty} = A_\infty e_{A_\infty}$, where $A_\infty := x^\mu A_\mu$.

Combining those, we build "local solutions":

$$\begin{cases} \mathcal{X}_0 := M_0 e_{A_0}, \text{ where } M_0 \in \mathrm{GL}_n(\mathbf{C}\{x\}), \\ \mathcal{X}_\infty := M_\infty e_{A_\infty}, \text{ where } M_\infty \in \mathrm{GL}_n(\mathbf{C}\{1/x\}). \end{cases}$$

Birkhoff then introduces the *connection matrix*:

$$P := \mathcal{X}_0^{-1} \mathcal{X}_\infty \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^*)).$$

Connection matrices, monodromy data

Since P connects two solutions, it is q-invariant, *i.e.* $\sigma_q P = P$. If it was uniform, it would be q-elliptic: $P \in \mathrm{GL}_n(\mathcal{M}(\mathbf{E}_q))$.

Birkhoff actually gets a *multivalued* connection matrix such that $P(qx) = P(x)$ but $P(xe^{2i\pi}) \neq P(x)$.

He characterizes the zeroes and poles of the coefficients $p_{i,j}(x)$ of $P(x)$ and writes, using Weierstraß sigma function:

$$p_{i,j}(q^t) = c_{i,j} e^{-(\eta\mu/2)t^2 + (\eta(\rho_j+\sigma_i) - \eta'\mu/2)t} \prod_{1 \leq k \leq \mu} \sigma\left(t - \alpha_{i,j}^{(k)}\right),$$

where all $c_{i,j}, \alpha_{i,j}^{(k)} \in \mathbf{C}$ and $\sum_{1 \leq k \leq \mu} \alpha_{i,j}^{(k)} = \sigma_i + \rho_j - \mu\tau'/2$

We shall rather use a *uniform*, whence q-elliptic connection matrix; and theta functions instead of the sigma function.

The joys of counting

I am great headman, Ghân-buri-Ghân. I count many things: stars in sky, leaves on trees, men in the dark. (The Lord of the Rings: The return of the King)

Birkhoff then counts the "characteristic constants" up to gauge freedoms. On the left (equations) as well as on the right (connection matrices), he finds $(n-1)(n\mu-2)$.

He then proves that every such connection matrix P actually comes from a system A.

For that, he uses the famous theorem of factorisation of analytic matrices ("Riemann problem") that allows for the unified point of view in his 1913 article.

A slight bit of modernisation

Without essentially changing Birkhoff's method of rational classification of fuchsian q-difference equations, we can:

1. Avoid the use of multivalued functions (using θ_q)
2. Allow for non generic cases (multiple exponents, q-logarithmic parts)
3. Work with rational systems $A(x) \in \mathrm{GL}_n(\mathbf{C}(x))$ (without chasing denominators)
4. State correspondance as an equivalence of categories
5. Replace the connection matrix $P = (M_0 e_{A_0})^{-1}(M_\infty e_{A_\infty})$ by its central part $M := M_0^{-1} M_\infty$.

Indeed, M does not take in account the "local q-monodromies" at 0 and ∞, which are encapsulated in the e_{A_0} and e_{A_∞} factors.

Plan

Galois group in the *regular* case

Let $A(x) \in \mathrm{GL}_n(\mathbf{C}(x))$ such that $A(0) = A(\infty) = I_n$.
Let $\mathrm{Sing}(A) :=$ poles of $A \cup$ poles of A^{-1}. Then:

$$\begin{cases} \mathcal{X}_0 = A^{-1}(x)A^{-1}(qx)\cdots \in \mathrm{GL}_n(\mathbf{C}(\{x\})), \\ \mathcal{X}_\infty = A(q^{-1}x)A(q^{-2}x)\cdots \in \mathrm{GL}_n(\mathbf{C}(\{1/x\})) \end{cases}$$

$$\Longrightarrow P = \cdots A(qx)A(x)A(q^{-1}x)A(q^{-2}x)\cdots \in \mathcal{M}(\mathbf{E}_q),$$

$$\mathrm{Sing}(P) = q^{\mathbf{Z}}\mathrm{Sing}(A)$$

Then using Picard-Vessiot theory, Etingof (1995) proves:

Theorem

The values $P(a)^{-1}P(b)$, $a, b \in \mathbf{C}^* \setminus \mathrm{Sing}(P)$, generate an algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$, the Galois group of A.

Galois group in the *fuchsian* case: first strike

We call $A(x) \in \mathrm{GL}_n(\mathbf{C}(x))$ *fuchsian* if $A(0), A(\infty) \in \mathrm{GL}_n(\mathbf{C})$ (maybe up to rational gauge equivalence $A \sim (\sigma_q F)AF^{-1}$, $F \in \mathrm{GL}_n(\mathbf{C}(x))$. Such systems form a tannakian category.

Fiber functors over $\mathbf{C}$ are difficult to produce because the natural field of constants is the field $\mathcal{M}(\mathbf{E}_q)$ of q-elliptic functions; same problem with Picard-Vessiot theory.

Van der Put and Singer (1999) bypass the problem by using *symbolic characters* e_c constrained to satisfy $e_c e_d = e_{cd}$ (which would be impossible using "true functions").

Using either Picard-Vessiot or tannakian theory, they get:

Theorem

The values $P(a)^{-1}P(b)$, $a, b \in \mathbf{C}^* \setminus \mathrm{Sing}(P)$, *together with explicitly described "local components" at* 0 *and* ∞, generate an algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$, the Galois group of A.

Galois group in the *fuchsian* case: second strike

In spite of the bad multiplicative behaviour of the q-characters, it is possible to define and describe the Galois group while using "true functions". The method is tannakian.

One thus obtains (S. 2003) a "universal fuchsian Galois group", a proalgebraic group over $\mathbf{C}$ of which all fuchsian systems "are" rational representations.

The local components at 0 and ∞ can be defined (see section on irregular case) and proved to be isomorphic to $\mathbf{C} \times \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$.

They can be connected by values of a twisted version $\check{P}$ of P (alternatively, by values of the central part M, without twisting).

Monodromy group in the fuchsian case

One can then extract from the local components isomorphic to $\mathbf{C} \times \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$ discrete Zariski-dense subgroups isomorphic to $\mathbf{Z}^3$, natural candidates to be "local monodromy groups" at $0, \infty$.

However, no *general* process has been found to extract local contributions of intermediate singularities to the *connection component* (generated by values $\check{P}(a)^{-1}\check{P}(b)$).

To the best of my knowledge, the only significant advance is by Roques (2011) for q-hypergeometric functions.

Related advances (not Galoisian though) are by Roques-S (2019) on rigidity index, see last section; and Ohyama-Ramis-S (2021), see next two slides.

A more geometric q-RHB correspondance

We return to Birkhoff's setting: $A = A_0 + \cdots + A_\mu x^\mu$, where:

$$R := \mathrm{Sp}(A_0), S := \mathrm{Sp}(A_\mu) \subset \mathbf{C}^* \text{ and } \underline{x} := \text{ zeroes of } \det A$$

satisfy the same non resonancy assumptions as before.

Fixing the "local data" $R, S, \underline{x}$, we want to define a *space* of such matrices A up to rational gauge equivalence:

$$\mathcal{E}_{R,S,\underline{x}} := \frac{E_{R,S,\underline{x}} := \text{ all such matrices } A(x)}{A \sim (\sigma_q F) A F^{-1} \text{ whenever } F \in \mathrm{GL}_n(\mathbf{C}(x))}.$$

The right hand side of q-RHB correspondance (the side of the connection matrix) is handy for that. We use the "central parts" M of Birkhoff's connection matrices P.

The set $F_{R,S,\underline{x}}$ of all possible such matrices M is an affine algebraic subset of a finite dimensional complex space.

A more geometric q-RHB correspondance

The rational gauge equivalence on $E_{R,S,\underline{x}}$ translates to a rational action on $F_{R,S,\underline{x}}$ by a linear algebraic group.

By q-RHB correspondance, there is in natural bijection:

$$\mathcal{E}_{R,S,\underline{x}} = \frac{E_{R,S,\underline{x}}}{\text{rat. equiv.}} \longleftrightarrow \mathcal{F}_{R,S,\underline{x}} = \frac{F_{R,S,\underline{x}}}{\text{alg. gr. action}}.$$

When $n = \mu = 2$ ("Jimbo-Sakai family"), it is shown by Ohyama-Ramis-S (2021) that $\mathcal{F}_{R,S,\underline{x}}$ is an algebraic surface with interesting properties.

This rests on a process of *localisation around pairs of intermediate singularities* ("Mano decomposition").

The study has been extended by Joshi-Roffelsen (2022) and Ramis-S (2022) . However, so far, $\mathcal{F}_{R,S,\underline{x}}$ has not been shown to be a moduli space.

Plan

Local fuchsian Galois group

In the course of a work on loop groups, Baranovsky and Ginzburg (1996) prove that the category of fuchsian q-difference systems over $\mathbf{C}((x))$ is equivalent to the category of flat vector bundles over the elliptic curve $\mathbf{E}_q$.

⚠ Caution ! here and in the sequel, "flat vector bundle" means "which can be equipped with a flat connection".
The connection is *not* part of the structure.

Using that equivalence, Kontsevitch (appendix to BG1996) shows that the corresponding universal Galois group is
$\mathbf{C} \times \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$.

We shall return (in the next section) to the appearance of vector bundles over $\mathbf{E}_q$.

Filtration by the slopes

Every q-difference system [with matrix] $A(x) \in \mathrm{GL}_n(\mathbf{C}(\{x\})$ is endowed with a formal invariant, its Newton polygon: rational slopes $\mu_1 > \cdots > \mu_k$ with multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$ s.t. $r_1 + \cdots + r_k = n$.

For simplicity we assume that $\mu_1, \ldots, \mu_k \in \mathbf{Z}$.

There is a *canonical filtration by slopes* and an *associated graduation*:
Up to formal, resp. to analytic equivalence, A can be put in block diagonal form A_0, resp. block triangular form A_U:

$$A_0 := \begin{pmatrix} x^{\mu_1} A_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x^{\mu_k} A_k \end{pmatrix} \quad A_U := \begin{pmatrix} x^{\mu_1} A_1 & \ldots & U_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x^{\mu_k} A_k \end{pmatrix},$$

where all $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$, all $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}[x, x^{-1}])$.

The graded form A_0 encodes the formal class of all A_U.

q-Stokes phenomenon, cohomological form

There is a unique formal isomorphism $\hat{F} : A_0 \to A_U$ having the form left below; and, for almost $\overline{c} \in \mathbf{E}_q$, a unique "summation of $\hat{F}_U$ in direction $\overline{c}$", a meromorphic isomorphism $S_{\overline{c}}\hat{F}_U : A_0 \to A_U$ having the form right below, subject to some precise polarity conditions on the $F_{i,j}$:

$$\hat{F}_U = \begin{pmatrix} I_{r_1} & \dots & \hat{F}_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{r_k} \end{pmatrix} \quad S_{\overline{c}}\hat{F}_U = \begin{pmatrix} I_{r_1} & \dots & F_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{r_k} \end{pmatrix}$$

The $S_{\overline{c},\overline{d}}\hat{F}_U := (S_{\overline{c}}\hat{F}_U)^{-1} S_{\overline{d}}\hat{F}_U$ form a cocycle of the sheaf $\Lambda_I(A_0)$ of meromorphic automorphisms of A_0 "tangent to the identity" (*i.e.* in the above unipotent triangular form), a *Stokes cocycle*.

Theorem (q-analogue of Birkhoff-Malgrange-Sibuya)

This induces a bijection from the set of *isoformal analytic classes in the formal class* A_0 onto $H^1(\mathbf{E}_q, \Lambda_I(A_0))$ (Ramis, S., Zhang, 2013).

q-Stokes phenomenon, galoisian form

The meromorphic isomorphisms $S_{\overline{c}}\hat{F}_U$ are Galoisian.

We fix an arbitrary base point $x_0 \in \mathbf{C}^*$ and consider the map $\overline{c} \mapsto \log S_{\overline{c}}\hat{F}_U(x_0)$. It is meromorphic with simple poles at the prohibited directions of summation. The corresponding residues $\dot{\Delta}_{\overline{c}}(A_U)$ belong to the Lie algebra $L(A_U)$ of the unipotent Stokes component $\mathfrak{S}(A_U)$ of $Gal(A_U)$. Precisely:

$$Gal(A_U) = \mathfrak{S}(A_U) \rtimes G_p(A_U) \text{ and } L(A_U) := \text{Lie}(\mathfrak{S}(A_U)),$$

where G_p the formal Galois group, which is semi-simple.

Theorem (Ramis, S., 2015)

(i) The spectral components of the $\dot{\Delta}_{\overline{c}}$ under the action of G_p generate a free graded Lie algebra L.
(ii) The q-analogue of the Wild Fundamental Group $\exp L \rtimes G_p$ is Zariski-dense in the Galois group.

Corollary

Solution of the local and global inverse problem (for integral slopes).

Plan

Vector bundles attached to q-difference equations

The categories of fuchsian q-difference systems over $\mathbf{C}((x))$ or over $\mathbf{C}(\{x\})$ are equivalent.
In the analytic setting, we have a sheaf theoretic interpretation of the construction of Baranovsky-Ginzburg.

Let $\pi : \mathbf{C}^* \to \mathbf{E}_q$ the natural projection, and let $A \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$.
For $U \subset \mathbf{E}_q$, the following sheaf is locally free:

$$\mathcal{F}_A(U) := \{\text{solutions of } \sigma_q X = AX \text{ holomorphic over } \pi^{-1}(U) \text{ near } 0\},$$

The corresponding vector bundle can be defined geometrically:

$$\mathcal{F}_A := \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(x, X) \sim (qx, A(x)X)}.$$

According to Praagman (1985), $\mathcal{F}_A$ is meromorphically free (*i.e.* tensoring it with the sheaf $\mathcal{M}_{\mathbf{E}_q}$ of meromorphic functions on $\mathbf{E}_q$ yields a free sheaf $\simeq \mathcal{M}_{\mathbf{E}_q}^n$). As a consequence, there always exists a (uniform) fundamental local solution $\mathcal{X}^{(0)}$.

Local fuchsian and pure Galois groups

We recover Baranovsky-Ginzburg-Kontsevich (1996):

Theorem

The functor $A \rightsquigarrow \mathcal{F}_A$ is a $\otimes$-equivalence from the category of fuchsian q-difference systems over $\mathbf{C}(\{x\})$ to the category of flat vector bundles over $\mathbf{E}_q$.

Corollary

The fuchsian local Galois group is $\mathbf{C} \times \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$.

Pure isoclinic systems are those with only one slope; pure systems are their direct sums, *i.e.* those graded by the slopes.

Corollary

The local Galois group of the category of pure systems with integral slopes is $\mathbf{C}^* \times \mathbf{C} \times \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$.

The Galois group for arbitrary pure systems is more complicated, it was determined by van der Put and Reversat (2007).

Global Galois group and vector bundles

For rational systems (equivalently: global over the Riemann sphere) $A \in \mathrm{GL}_n(\mathbf{C}(x))$, the two reductions at 0 and ∞ give rise to two vector bundles $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$.

Birkhoff's connection matrix (or rather its central part) takes the form of a *meromorphic isomorphism* $M : \mathcal{F}_A^{(0)} \to \mathcal{F}_A^{(\infty)}$

Theorem

The functor $A \rightsquigarrow (\mathcal{F}_A^{(0)}, M, \mathcal{F}_A^{(\infty)})$ is exact, faithful and $\otimes$-compatible. Restricted to fuchsian systems and flat bundles, it is a $\otimes$-equivalence.

In the irregular case, under the assumption of integral slopes at 0 and at ∞, Ramis and S. (2015) describe the Galois group and solve the inverse problem.

Here again, local wild monodromy groups at 0 and ∞ are known, but nothing of the sort at intermediate singularities.

Vector bundles for general irregular systems

The functor $A \rightsquigarrow \mathcal{F}_A$ is exact, faithful, $\otimes$-compatible.

⚠ It is not fully faithful. The function θ_q is a morphism $\mathcal{F}_x \to \mathcal{F}_1$ but it does not come from a morphism $x \to 1$.

The bundle associated to a pure module of integral slope μ is the tensor product of a line bundle of degree $-\mu$ by a flat bundle. In particular, it is semi-stable.

The slope filtration on A induces a filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k$ on $\mathcal{F}_A$ such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a bundle of rank r_i, slope $-\mu_i$.

⚠ This is not the Harder-Narasimhan filtration:
1) The slopes $-\mu_1 < \cdots < -\mu_k$ of the successive quotients are strictly increasing.
2) If $A := \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix}$ then $\mathcal{F}_A$ is stable but its filtration has two steps with quotients $\mathcal{F}_x$ and $\mathcal{F}_1$.

Filtration and q-Gevrey asymptotics

Proposition (S. 2009)

The functor $A \rightsquigarrow (\mathcal{F}_A, (\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k))$ is exact, fully faithful, $\otimes$-compatible.

I do not know its essential image.

It is related to the q-Gevrey asymptotics of Ramis,Zhang (2002) (also see Ramis,S.,Zhang (2013)). Actually, it corresponds to usual filtration by asymptotic growths.

For instance, for a pure isoclinic module of slope μ, a section, seen as a function on $(\mathbf{C}^*, 0)$, behaves like $Cx^k q^{-\mu(\log_q^2 x)/2}$ when $x \to 0$ (for some $C > 0$ and k).

Plan

Stokes data according to Kontsevich-Soibelman

In "Symplectic geometry of Riemann-Hilbert correspondences", accessible on Youtube, Yan Soibelman (2016) considers our $(\mathcal{F}_A, (\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k))$ as a q-analogue for Stokes data.

The framework is slightly different from ours. $\mathcal{D}_q$-modules are meant in the sense of Sabbah (1993).
And Soibelman always mentions coherent sheaves where we only found locally free sheaves.

He calls *anti-HN filtration* a filtration by coherent subsheaves $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \mathcal{F}_\infty = \mathcal{F}$ of the sheaf $\mathcal{F}$ (over $\mathbf{E}_q$) of solutions such that all $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable with *increasing* slopes and $\mathcal{F}_\infty/\mathcal{F}_k$ is a torsion sheaf (so rank $= 0$ and slope $= \infty$).

Theorem (or conjecture ?) Kontsevich-Soibelman (2016 ?)

The category of holonomic $\mathcal{D}_q$-modules on $\mathbf{C}^*$ is equivalent to the category of coherent sheaves on the elliptic curve $\mathbf{E}_q$, which are endowed with two anti-HN filtrations labeled by $\mathbf{Q} \cup \{\infty\}$.

Trying to understand the correspondence

Although details are missing, the following facts appear clearly.

- $\mathcal{F}$ is made of "global" solutions, so we should consider a rational system $A \in \mathrm{GL}_n(\mathbf{C}(x))$.
- The two anti-HN filtrations are (in essence) our filtrations by slopes at 0 and at ∞:

$$\mathcal{F}_1^{(0)} \subset \cdots \subset \mathcal{F}_k^{(0)} = \mathcal{F}_A^{(0)} \subset \mathcal{F}_\infty \supset \mathcal{F}_A^{(\infty)} = \mathcal{F}_\ell^{(\infty)} \supset \cdots \supset \mathcal{F}_1^{(\infty)}$$

- The reverse construction (from such data to $\mathcal{D}_q$-modules) uses q-Gevrey asymptotic behaviour of sections (although there seems to be a typographic error in the exponent for the factor $q^{-\mu_i(\log_q^2 x)/2}$).

Here, we only try to understand, within our framework, what is that sheaf $\mathcal{F}_\infty$ such that $\mathcal{F}_\infty/\mathcal{F}_k^{(0)}$ and $\mathcal{F}_\infty/\mathcal{F}_\ell^{(\infty)}$ are torsion sheaves.

Some sheaves of solutions

Let $A \in \mathrm{GL}_n(\mathbf{C}(x))$. We call *(intermediate) singularities* of A its poles and the poles of A^{-1} lying in $\mathbf{C}^*$. Their locus is $\mathrm{Sing}(A)$. Generically, A is non resonant, *i.e.* $\mathrm{Sing}(A) \cap q^{\mathbf{N}^*}\mathrm{Sing}(A) = \emptyset$; and this can always be achieved by a rational gauge transformation.

For reasons (apparently) unrelated to q-RHB, Roques and S. (2019) introduce the following sheaves of solutions (on $\mathbf{E}_q$) of $\sigma_q X = AX$; notation is $V \subset \mathbf{E}_q$, $U := \pi^{-1}(V) \subset \mathbf{C}^*$:

$$\begin{aligned}
\mathcal{F}_A(V) &:= \{\text{solutions holomorphic over } U\},\\
\mathcal{F}_A^{(0)}(V) &:= \{\text{solutions holomorphic over } U \text{ near } 0\},\\
\mathcal{F}_A^{(\infty)}(V) &:= \{\text{solutions holomorphic over } U \text{ near } \infty\},\\
\mathcal{F}'_A(V) &:= \{\text{solutions holomorphic over } U \text{ except possibly for a}\\
&\qquad \text{finite number of poles over any } q\text{-spiral } aq^{\mathbf{Z}} \subset U\}
\end{aligned}$$

Then:

- $\mathcal{F}_A \subset \mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)} = \mathcal{F}'_A$. If A is non resonant, $\mathcal{F}_A = \mathcal{F}'_A$.
- $\mathcal{F}_A^{(0)}$, $\mathcal{F}_A^{(\infty)}$ and $\mathcal{F}'_A$ are "intrinsic" w.r.t. rational gauge equivalence but $\mathcal{F}_A$ is not: $B = F[A] \not\Rightarrow \mathcal{F}_B = F\mathcal{F}_A$.

Torsion and singularities; and a proposal

We restrict to the generic nonresonant case.

Theorem

(i) $\mathcal{F}_A^{(0)}/\mathcal{F}_A$ and $\mathcal{F}_A^{(\infty)}/\mathcal{F}_A$ are skyscraper sheaves concentrated at $\pi(\mathrm{Sing}(A)) \subset \mathbf{E}_q$.
(ii) Their stalks at $\pi(a) \in \mathbf{E}_q$ can be computed from the elementary divisors of $A(x)$ at $a \in \mathrm{Sing}(A)$ (over the valuation ring $\mathcal{O}_{\mathbf{C}^*,a}$).

Note that $\dfrac{\mathcal{F}_A^{(0)}}{\mathcal{F}_A} = \dfrac{\mathcal{F}_A^{(0)}}{\mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)}} \simeq \dfrac{\mathcal{F}_A^{(0)} + \mathcal{F}_A^{(\infty)}}{\mathcal{F}_A^{(\infty)}}$ and similarly $\dfrac{\mathcal{F}_A^{(\infty)}}{\mathcal{F}_A} \simeq \dfrac{\mathcal{F}_A^{(0)} + \mathcal{F}_A^{(\infty)}}{\mathcal{F}_A^{(0)}}$.

Also, in the two anti-HN filtrations encountered above, the penultimate upper terms are $\mathcal{F}_k^{(0)} = \mathcal{F}_A^{(0)}$ and $\mathcal{F}_\ell^{(\infty)} = \mathcal{F}_A^{(\infty)}$

So it seems that one should take $\mathcal{F}_\infty := \mathcal{F}_A^{(0)} + \mathcal{F}_A^{(\infty)}$.
We do not (yet) know how to characterize it, nor have we achieved a proof of the correspondence in that form.

That's all folks.

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