Multi-particle Painlevé Hamiltonians: Noncommutativity, Reduction & Duality

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Web-seminar on Painlevé Equations and related topics
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Plan of the talk

- Painlevé equations : isomonodromic deformations and confluence.
- Calogero–Painlevé correspondence.
- Multi–particles systems and their isomonodromic formulation.
- The case of PII : Stokes data and applications.
- Ruijsenaars duality revisited.
- Beyond - Ruijsenaars–Painlevé correspondence.

Collaboration with M. Bertola, M. Cafasso,(Comm. Math. Phys. 363 (2018), no. 2, 503 - 530.) and with Ilia Gaiur (Revista Matemática Iberoamericana, accepted, april 2024)

Painlevé equations

Painlevé property : The only movable singularities are poles

$$(PVI)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\frac{\gamma(t-1)}{(\lambda-1)^2}+\frac{\delta t(t-1)}{(\lambda-t)^2}\right).$$

$$(PV)\quad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\lambda(\lambda-1)^2}{t^2}\left(\alpha+\frac{\beta}{\lambda^2}+\frac{\gamma t}{(\lambda-1)^2}+\frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right).$$

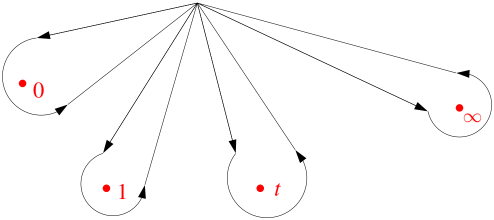
$$(PIV)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha)\lambda + \frac{\beta}{\lambda}.$$

$$(PIII)\quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\lambda^2}{4t^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\gamma\lambda+\frac{\delta t^2}{4\lambda^3}\right).$$

$$(PII)\quad \frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha.$$

$$(PI)\quad \frac{d^2\lambda}{dt^2} = 6\lambda^2 + t.$$

Isomonodromic deformations



$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}(\theta_\infty, -\theta_\infty).$$

Isomonodromic deformations $\longleftrightarrow$ Schlesinger equations :

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}.$$

$$\Updownarrow$$

Painlevè VI for $\lambda =$ (simple) zero of $(\mathcal{A})_{12}$.

Okamoto's Hamiltonians for Painlevé equations

$$(\mathrm{P_{VI}}) \qquad H = \frac{q(q-1)(q-t)}{t(t-1)}\left[p^2 - \left(\frac{\kappa_0}{q} + \frac{\kappa_1}{q-1} + \frac{\theta-1}{q-t}\right)p + \frac{\kappa}{q(q-1)}\right]$$

$$(\mathrm{P_{V}}) \qquad H = \frac{q(q-1)^2}{t}\left[p^2 - \left(\frac{\kappa_0}{q} + \frac{\theta_1}{q-1} - \frac{\eta_1 t}{(q-1)^2}\right)p + \frac{\kappa}{q(q-1)}\right]$$

$$(\mathrm{P_{IV}}) \qquad H = 2q\left[p^2 - \left(\frac{q}{2} + t + \frac{\kappa_0}{q}\right)p + \frac{\theta_\infty}{2}\right]$$

$$(\mathrm{P_{III}}) \qquad H = \frac{q^2}{t}\left[p^2 - \left(\eta_\infty + \frac{\theta_0}{q} - \frac{\eta_0 t}{q^2}\right)p + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2q}\right]$$

$$(\mathrm{P_{II}}) \qquad H = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

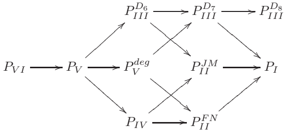
$$(\mathrm{P_{I}}) \qquad H = \frac{p^2}{2} - 2q^3 - tq$$

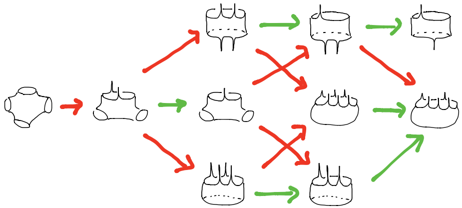
$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Confluence and Lax systems

Each of the Painlevé equations can be written as a Lax system (of isomonodromic type)

$$\begin{cases} \dfrac{\partial}{\partial z}\Psi(t;z) &= U(t;z)\Psi(t;z), \\ \dfrac{\partial}{\partial t}\Psi(t;z) &= V(t;z)\Psi(t;z). \end{cases} \implies \frac{\partial V}{\partial z} - \frac{\partial U}{\partial t} = [U,V].$$





Confluence (P. Gavrylenko & O. Lisovyy, *Commun. Math. Phys. 363*)

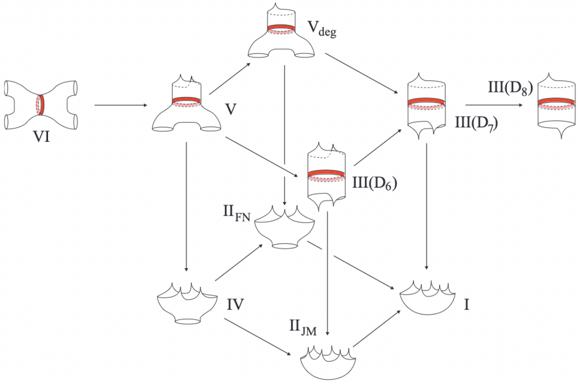


Figure 3: CMR confluence diagram for Painlevé equations.

Calogero–Painlevé correspondence for PVI

Take the elliptic curve $y^2 = z(z-1)(z-t)$ and the associated Weierstrass $\wp$ function

$$\wp(u;1,\tau) := \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(u+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right).$$

Define $e_n := \wp(\omega_n), \quad \omega_1 := 1/2,\ \omega_2 := -(1+\tau)/2,\ \omega_3 := \tau/2.$

Theorem (Fuchs, Painlevé, Lamé, Manin) :

Let q be implicitely defined by

$$\lambda = \frac{\wp(q) - e_1}{e_2 - e_1} \quad \text{and} \quad \dot{q} := \frac{dq}{d\tau}.$$

Then the PVI equation is equivalent to the Hamiltonian system

$$\dot{q} = \frac{1}{2\pi i}\frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{1}{2\pi i}\frac{\partial H}{\partial q}, \quad H(p,q,\tau) := \frac{p^2}{2} - \sum_{n=0}^{3} g_n \wp(q+\omega_n)$$

with

$$g_0 = \alpha,\ g_1 = -\beta,\ g_2 = \gamma,\ g_3 = -\delta + \frac{1}{2}.$$

Why "Calogero – Painlevé" ?

Remark :

Levin et Olshanetsky observed that Manin's Hamiltonian

$$H(p,q,\tau) := \frac{p^2}{2} - \sum_{n=0}^{3} g_n \wp(q+\omega_n)$$

is the rank–one case of a system of n particles $q_1,\ldots,q_n$ introduced by Inozemtsev

$$H_{VI} = \sum_{j=1}^{n}\left(\frac{p_j^2}{2} + \sum_{n=0}^{3} g_n^2 \wp(q_j+\omega_n)\right) + g_4^2 \sum_{j\neq k}\left(\wp(q_j-q_k)+\wp(q_j+q_k)\right)$$

and generalising the elliptic Calogero–Moser system.

Manin's system is non–autonomous, while Inozemtsev's system is an (integrable) autonomous Hamiltonian system.

Takasaki : “Painlevé–Calogero revisted” (2000)

Theorem (Takasaki) :
Each of the Painlevé equation can be written, using the confluence scheme, as an Hamiltonian system with Hamiltonian of the type

$$H(q,p;t) = \frac{p^2}{2} - V(q;t),$$

and there is a canonical transformation between these Hamiltonians and the Okamoto’s (polynomial) ones.
The correspondence extends to the case of many particles.

Remark

The Calogero–Painlevé VI version of the model had also showed up in a completely different context in the work of P. Etingof, W. L. Gan and A. Oblomkov (2005) in connection with their study of the generalized double affine algebras of higher rank.

$$H_{VI}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}+\sum_{n=0}^{3} g_n^2 \wp(q_j+\omega_n)\right)+g_4^2\sum_{j\neq k}\left(\wp(q_j-q_k)+\wp(q_j+q_k)\right).$$

$$H_{V}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{\sinh^2(q_j/2)}-\frac{\beta}{\cosh^2(q_j/2)}+\frac{\gamma t}{2}\cosh(q_j)+\frac{\delta t^2}{8}\cosh(2q_j)\right)+$$
$$+g_4^2\sum_{j\neq k}\left(\frac{1}{\sinh^2((q_j-q_k)/2)}+\frac{1}{\sinh^2((q_j+q_k)/2)}\right).$$

$$H_{IV}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(\frac{q_j}{2}\right)^6-2t\left(\frac{q_j}{2}\right)^4-2(t^2-\alpha)\left(\frac{q_j}{2}\right)^2+\beta\left(\frac{q_j}{2}\right)^{-2}\right)+$$
$$+g_4^2\sum_{j\neq k}\left(\frac{1}{(q_j-q_k)^2}+\frac{1}{(q_j+q_k)^2}\right).$$

$$H_{III}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{4}e^{q_j}+\frac{\beta t}{4}e^{-q_j}-\frac{\gamma}{8}e^{2q_j}+\frac{\delta t^2}{8}e^{-2q_j}\right)+g_4^2\sum_{j\neq k}\frac{1}{\sinh^2((q_j-q_k)/2)}.$$

$$H_{II}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(q_j^2+\frac{t}{2}\right)^2-\alpha q_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

$$H_{I}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-2q_j^3-tq_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

Our aim :

Describing an isomonodromic formulation of multi–particles Calogero–Painlevé systems.

- *A central issue will be to find an isomonodromic description of the multicomponent Painlevé equations. If such an isomonodromic description does exist, it should be related to a new geometric structure* (Takasaki).
- Applying the classical tool of isomonodromic deformations to such systems.
- Quantization.
- Ruijsenaars - type Duality

Our procedure :

Applying a Hamiltonian reduction à la Kazhdan-Konstant-Sternberg on a matrix-valued version of Painlevé equations.

The simplest example : PI

A matrix–valued Lax pair for the first Painlevé equation :

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial z}\Psi(t;z) & = & \begin{pmatrix} \mathbf{p} & z-\mathbf{q} \\ z^2+z\mathbf{q}+\mathbf{q}^2+\frac{t}{2} & -\mathbf{p} \end{pmatrix}\Psi(t;z) \\ \frac{\partial}{\partial t}\Psi(t;z) & = & \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{z}{2}+\mathbf{q} & 0 \end{pmatrix}\Psi(t;z) \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} \dot{\mathbf{q}} & = & \mathbf{p} \\ \dot{\mathbf{p}} & = & \frac{3}{2}\mathbf{q}^2+\frac{t}{4}. \end{array} \right.$$

Lemma I :
The Lax equations are Hamiltonian on $\mathcal{M} := T^*\mathfrak{gl}_n$ with respect to the standard symplectic structure $\omega := \mathrm{Tr}[d\mathbf{q} \wedge d\mathbf{p}]$ and

$$H := \mathrm{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - t\frac{\mathbf{q}}{4}\right).$$

Moreover, the commutator $[\mathbf{p}, \mathbf{q}]$ is conserved along the flow.

Reduction à la Kazhdan-Konstant-Sternberg

$$\mathcal{M}_{g_4} := \left\{ (\mathbf{q}, \mathbf{p}) \in \mathcal{M} \text{ s.t. } [\mathbf{p}, \mathbf{q}] = i g_4 (\mathbf{1}_n - v^T v) \right\}, \quad v := (1, \dots, 1).$$

Lemma II :

Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{M}_{g_4}$ with $\mathbf{q}$ diagonalizable. Then it exists G such that

$$G^{-1} \mathbf{q} G = X = \mathrm{diag}(q_1, \dots, q_n),$$

and, if $Y = G^{-1} \mathbf{p} G$, then

$$[Y, X] = i g_4 (\mathbf{1}_n - v^T v).$$

Corollary :

$$Y_{ij} = -\frac{i g_4}{q_i - q_j}, \ i \neq j = 1, \dots, n.$$

The variables $p_i := Y_{ii}, i = 1, \dots, n$ are the conjugated variables of $\{q_1, \dots, q_n\}$ for the reduced system.

More precisely : $\mu : \mathcal{M} \to \mathfrak{gl}_\ell^*$, $\mu(\mathbf{q}, \mathbf{p}) := [\mathbf{p}, \mathbf{q}]$ is the moment map and $\{q_i, p_j\}$ are the symplectic coordinates on the quotient $\mu^{-1}(\mathcal{O}) / PGL_n(\mathbb{C})$, with $\mathcal{O}$ orbit of $\mathrm{diag}(n-1, -1, \dots, -1)$.

Reduction à la Kazhdan-Konstant-Sternberg II

The gauge–transformed eigenfunction

$$\Phi(t;z) := (G^{-1}(t) \otimes \mathbf{1}_2)\Psi(t;z) = \mathbf{G}^{-1}(t)\Psi(t;z)$$

satisfies the Lax pair

$$\begin{cases} \dfrac{\partial}{\partial z}\Phi(t;z) & = & \tilde{U}(t;z)\Phi(t;z) \\[2ex] \dfrac{\partial}{\partial t}\Phi(t;z) & = & \tilde{V}(t;z)\Phi(t;z) \end{cases}$$

$$\tilde{U} := \mathbf{G}^{-1}(t)\begin{pmatrix} \mathbf{p} & z-\mathbf{q} \\ z^2+z\mathbf{q}+\mathbf{q}^2+\frac{t}{2} & -\mathbf{p} \end{pmatrix}\mathbf{G}(t) = \begin{pmatrix} Y & z-X \\ z^2+zX+X^2+\frac{t}{2} & -Y \end{pmatrix}$$

$$\tilde{V} := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{z}{2}+X & 0 \end{pmatrix} - \mathbf{G}^{-1}(t)\dot{\mathbf{G}}(t) = \begin{pmatrix} -F & \frac{1}{2} \\ \frac{z}{2}+X & -F \end{pmatrix}, \quad F(t) := G^{-1}(t)\dot{G}(t)$$

Reduction à la Kazhdan-Konstant-Sternberg III

Compatibility conditions yields

$$\begin{cases} \dot{X} &= Y - [F, X], \\ \dot{Y} &= \frac{3}{2}X^2 + \frac{t}{4} - [F, Y], \end{cases}$$

$$\Longrightarrow F_{j,k} = -\frac{ig_4}{(x_j - x_k)^2},\ j \neq k, \quad \ddot{q}_j = \frac{3}{2}q_j^2 + \frac{tq_j}{4} - \sum_{k \neq j} \frac{g_4^2}{(x_j - x_k)^3}$$

On the other hand

$$H(\mathbf{q}, \mathbf{p}) = H(X, Y) = \mathrm{Tr}\left(\frac{Y^2}{2} - \frac{X^3}{2} - \frac{tX}{4}\right) = \sum_{j=1}^{n}\left(\frac{p_j^2}{2} - \frac{q_j^3}{2} - \frac{tq_j}{4}\right) + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}.$$

Spectral type of Fuchsian System-I

Consider Fuchsian system

$$\Psi_z = \sum_{i=1}^{n} \frac{A_i}{z-t_i}\Psi, \quad A_j \in \mathrm{Mat}(m,\mathbb{C}), \quad A_\infty = -\sum_{i=1}^{n} A_i$$

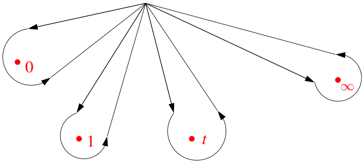
$$\mathrm{Spec}(A_j) = \left\{\theta_1^{(j)}, \theta_2^{(j)}, \ldots, \theta_{l_j}^{(j)}\right\} = \underbrace{\theta_1^{(j)}, \theta_1^{(j)}, \ldots \theta_1^{(j)}}_{m_1^j}, \underbrace{\theta_2^{(j)}, \theta_2^{(j)}, \ldots \theta_2^{(j)}}_{m_2^j}, \ldots, \underbrace{\theta_{l_j}^{(j)}, \theta_{l_j}^{(j)}, \ldots \theta_{l_j}^{(j)}}_{m_{l_j}^j}$$

$$\left\{\begin{array}{c} A_1; t_1 \\ A_2; t_2 \\ \ldots \\ A_n; t_n \\ A_\infty; \infty \end{array}\right\} \to \begin{array}{l} m_1^1 m_2^1 \ldots m_{l_1}^1 \\ m_1^2 m_2^2 \ldots m_{l_2}^2 \\ \ldots \\ m_1^n m_2^n \ldots m_{l_n}^n \\ m_1^\infty \ldots m_{l_\infty}^\infty \end{array} \to \underbrace{m_1^1 m_2^1 \ldots m_{l_1}^1 | m_1^2 m_2^2 \ldots m_{l_2}^2 | \ldots, m_1^n m_2^n \ldots m_{l_n}^n | m_1^\infty \ldots m_{l_\infty}^\infty}_{\text{Spectral type}}$$

Number of accessory parameters

$$N = 2 + (n-1)m^2 - \sum_{i=1}^{n}\sum_{j=1}^{l_j} (m_j^i)^2 - \sum_{j=1}^{l_\infty} (m_j^\infty)^2$$

Spectral type and Fuchsian systems-II



$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_\infty := -A_0 - A_t - A_1 = \mathrm{diag}(\theta_\infty, -\theta_\infty).$$

A_k with eigenvalues $(\theta_k, -\theta_k)$, $\quad k = 0, 1, t$.

$$\Downarrow$$

Painlevé VI = Fuchsian system of spectral type $11, 11, 11, 11$.
This is, essentially, the only Fuchsian system with phase space of dimension two ("accessory parameters") and one–dimensional deformation.

Spectral type of Painlevé VI

$$11, 11, 11, 11$$

Number of accessory parameters

$$N = 2 + (3-1)4 - 6 - 2 = 2$$

Given a Fuchsian system of rank m with n singular points, its spectral type is given by n partitions $Y_1, \ldots, Y_n$ of m and the dimension of its phase space is given by (Katz)

$$2 + (n-2)m^2 - \sum (Y_{j,\ell})^2.$$

Spectral type and Fuchsian systems - III

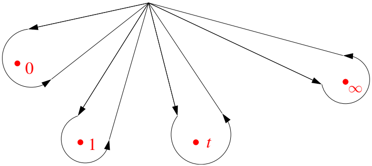
Proposition (Oshima) :

There are (essentially) just 4 Fuchsian systems whose phase space has dimension 4.
One is $11, 11, 11, 11, 11$, giving the Garnier system in two variables. The other three, which admits a one–dimensional deformation, are

$$21, 21, 111, 111, \quad 31, 22, 22, 1111, \quad 22, 22, 22, 211.$$

Kawakami : The Fuchsian system $n\,n, n\,n, n\,n, n\,n-1\,1$ gives a “matrix version” of the Painlevé VI equation, and its degenerations (confluence) yields a matrix version of all the other equations.

Matrix Painlevé VI equation



$$\frac{\partial}{\partial z}\Psi = \mathcal{A}(z)\Psi, \quad \mathcal{A}(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

$$A_k \sim \begin{pmatrix} 0_n & 0_n \\ 0_n & \theta^k \mathrm{I}_n \end{pmatrix} \quad k = 0, 1, t$$

$$A_\infty = \mathrm{diag}(\theta_1^\infty, \ldots, \theta_1^\infty, \theta_2^\infty, \ldots, \theta_2^\infty, \theta_3^\infty).$$

Theorem (Kawakami) :
$\mathcal{A}(z) = \mathcal{A}(t, \mathbf{q}, \mathbf{p}; z)$ in such a way that

$$[\mathbf{p}, \mathbf{q}] = (\theta^0 + \theta^1 + \theta^t + \theta_1^\infty)\mathrm{I} + \mathrm{diag}(\theta_2^\infty, \ldots, \theta_2^\infty, \theta_3^\infty)$$

and the isomonodromic deformation of the system is governed by the Hamiltonian ($\theta := \theta^0 + \theta^1 + \theta^t$).

$$t(t-1)H_{VI} = \mathrm{Tr}\Big[\mathbf{q}(\mathbf{q}-1)(\mathbf{q}-t)\mathbf{p}^2 + \\ + \Big((\theta^0 + 1 - [\mathbf{p}, \mathbf{q}])\mathbf{q}(\mathbf{q}-1) + \theta^t(\mathbf{q}-1)(\mathbf{q}-t) + (\theta + 2\theta_1^\infty - 1)\mathbf{q}(\mathbf{q}-t)\Big)\mathbf{p} + \\ + (\theta + \theta_1^\infty)(\theta^0 + \theta^t + \theta_1^\infty)\mathbf{q}\Big]$$

Hamiltonians by confluence

$$
\begin{aligned}
tH_V &= \operatorname{Tr}\Big[\mathbf{p}(\mathbf{p}+t)\mathbf{q}(\mathbf{q}-1)+\beta\mathbf{p}\mathbf{q}+\gamma\mathbf{p}-(\alpha+\gamma)t\mathbf{q}\Big],\\
tH_{IV} &= \operatorname{Tr}\Big[\mathbf{p}\mathbf{q}(\mathbf{p}-\mathbf{q}-t)+\beta\mathbf{p}+\alpha\mathbf{q}\Big],\\
tH_{III(D6)} &= \operatorname{Tr}\Big[\mathbf{p}^2\mathbf{q}^2-(\mathbf{q}^2-\beta\mathbf{q}-t)\mathbf{p}-\alpha\mathbf{q}\Big],\\
tH_{III(D7)} &= \operatorname{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\alpha\mathbf{p}\mathbf{q}+t\mathbf{p}+\mathbf{q}\Big],\\
tH_{III(D8)} &= \operatorname{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\mathbf{p}\mathbf{q}-\mathbf{q}-t\mathbf{q}^{-1}\Big],\\
tH_{II} &= \operatorname{Tr}\Big[\mathbf{p}^2-(\mathbf{q}^2+t)\mathbf{p}-\alpha\mathbf{q}\Big],\\
tH_{I} &= \operatorname{Tr}\Big[\mathbf{p}^2-\mathbf{q}^3-t\mathbf{q}\Big].
\end{aligned}
$$

$$[\mathbf{p},\mathbf{q}] = \mathrm{const}$$

Final result : multi–component Painlevé Hamiltonians of Okamoto type

$\Downarrow$

Calogero–Painlevé systems.

(Using Takasaki's canonical transformations.)

Monodromy : the case of PII

$$\frac{d}{dz}\Psi(\mathbf{t};z) = A(\mathbf{t};z)\Psi(\mathbf{t};z); \quad A(\mathbf{t};z) := \begin{pmatrix} i\frac{z^2}{2} + i\mathbf{q}^2 + i\frac{\mathbf{t}}{2} & z\mathbf{q} - i\mathbf{p} - \frac{\theta}{z} \\ z\mathbf{q} + i\mathbf{p} - \frac{\theta}{z} & -i\frac{z^2}{2} - i\mathbf{q}^2 - i\frac{\mathbf{t}}{2} \end{pmatrix}.$$

Slight generalisation :

$$t \longmapsto \mathbf{t} := \mathrm{diag}(t_1, \ldots, t_n), \quad \frac{d}{dt} \longmapsto \frac{d}{d\mathbf{t}} := \sum_{i=1}^{n} \frac{d}{dt_i}$$

$$\Downarrow$$

$$\ddot{\mathbf{q}} = 2\mathbf{q}^3 + \frac{1}{2}[\mathbf{t}, \mathbf{q}]_+ + \theta$$

(Retakh - V. R., Bertola–Cafasso)

Theorem : Given the equation

$$\frac{d}{dz}\Psi(\mathbf{t};z) = A(\mathbf{t};z)\Psi(\mathbf{t};z),$$

There exists a unique piecewise analytic solution $\Psi = \{\Psi_\nu,\ \nu = 0,\ldots,7\}$ satisfying

$$\Psi(\mathbf{t};z) \sim \left(\mathbf{1} + \frac{\alpha_1 \otimes \sigma_3 - \mathbf{q}\otimes\sigma_2}{z} + \mathcal{O}(z^{-2})\right) \mathrm{e}^{(\ln z + i\pi\epsilon)[\mathbf{q},\mathbf{p}]\otimes \mathbf{1}} \mathrm{e}^{\frac{i}{2}\left(\frac{z^3}{3}+\mathbf{t}z\right)\widehat{\sigma}_3},$$

The corresponding (matrix) Stokes operator $\mathbf{X},\mathbf{Y},\mathbf{Z}$ satisfy the relations

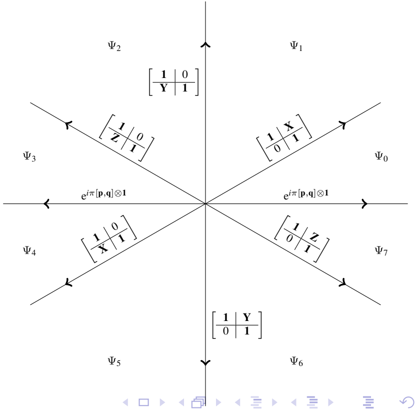
$$(\mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{XYZ})Q = -2i\cos(\pi\theta)$$

$$(\mathbf{XY}+\mathbf{1})Q - Q^{-1}(\mathbf{YX}+\mathbf{1}) = 0$$

$$\mathbf{ZX}Q - Q^{-1}\mathbf{XZ} + Q - Q^{-1} = 0$$

$$(\mathbf{YZ}+\mathbf{1})Q - Q^{-1}(\mathbf{ZY}+\mathbf{1}) = 0$$

$$Q := \mathrm{e}^{i\pi[\mathbf{p},\mathbf{q}]}.$$



"Classical" PII cubic I

Flaschka-Newell ('82) :

$$A(\lambda) = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & -4w \\ -4w & 0 \end{pmatrix}\lambda + \\ + \begin{pmatrix} -2w^2 - z & 2w_z \\ -2w_z & 2w^2 + z \end{pmatrix} - \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\frac{1}{\lambda}$$

Monodromy data :

$$M_0 = C\begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{-2i\pi\alpha} \end{pmatrix} C^{-1},$$

$$S_{2j} = \begin{pmatrix} 1 & x_{2j} \\ 0 & 1 \end{pmatrix}, \qquad S_{2j-1} = \begin{pmatrix} 1 & 0 \\ x_{2j-1} & 1 \end{pmatrix}, \qquad j = 1, 2, 3.$$

$x_{i+3} = x_i,$ and $x_1x_2x_3 + x_1 + x_2 + x_3 = \pm 2\cos(\pi\alpha)$

“Classical” and deformed cubics I

If $Q = \mathrm{e}^{i\pi[\mathbf{p},\mathbf{q}]} = \pm 1$ then

$$[\mathbf{X},\mathbf{Y}] = [\mathbf{X},\mathbf{Z}] = [\mathbf{Y},\mathbf{Z}] = 0, \quad \mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{XYZ} = \mathrm{const},$$

as in the classical case.

Example (Bertola, Cafasso) :

$C = \left(c_{ij}\right)_{i,j=1}^{n}$ Hermitian,

$$\mathcal{A}i_{\mathbf{t}} : \left(L^2(\mathbb{R}_+)\otimes\mathbb{R}^n\right)^{\circlearrowleft}, \ (\mathcal{A}i_{\mathbf{t}}\vec{f})_i(x) := \int_{\mathbb{R}_+} c_{i,j}\mathrm{Ai}(x+y+t_i+t_j)f_j(y)dy.$$

$$-\frac{\partial^2}{\partial \mathbf{t}^2}\log\det(\mathrm{I}-\mathcal{A}i_{\mathbf{t}}^2) = \mathrm{Tr}(\mathbf{q}^2),$$

where $\mathbf{q}$ is the unique solution with asymptotics

$$\mathbf{q}_{ij}(\mathbf{t}) = c_{ij}\mathrm{Ai}(t_i+t_j) + \mathcal{O}\left(\sqrt{T}\mathrm{e}^{-\frac{4}{3}(2T-2m)^{3/2}}\right)$$

$$T := \frac{1}{n}\sum t_j, \ m := \max_j(t_i - T), \ T\to\infty.$$

“Classical” and deformed cubics II

Suppose $[\mathbf{p},\mathbf{q}] = i\hbar$ multiple of the identity. Then

- Upon identification $\mathbf{p} = i\hbar\frac{\partial}{\partial\mathbf{q}}$, the term α_1 in

$$\Psi(\mathbf{t};z) \sim \left(\mathbf{1} + \frac{\alpha_1\otimes\sigma_3 - \mathbf{q}\otimes\sigma_2}{z} + \mathcal{O}(z^{-2})\right) \mathrm{e}^{(\ln z + i\pi\epsilon)[\mathbf{q},\mathbf{p}]\otimes\mathbf{1}} \mathrm{e}^{\frac{i}{2}\left(\frac{z^3}{3}+\mathbf{t}z\right)\widehat{\sigma}_3},$$

  gives the quantum Hamiltonian of Painlevé II.
- The Stokes relations read

$$(\mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{XYZ})Q = -2i\cos(\pi\theta)$$

$$Q\mathbf{XY} - Q^{-1}\mathbf{YX} = Q^{-1} - Q$$

$$Q\mathbf{ZX} - Q^{-1}\mathbf{XZ} = Q^{-1} - Q$$

$$Q\mathbf{YZ} - Q^{-1}\mathbf{ZY} = Q^{-1} - Q.$$

  These relations are the same obtained by Mazzocco and V.R. (2012) as a result of the quantisation of the Poisson structure of the classical cubic of the monodromy surface of Painlevé II.

Output ?

Alexander Its :

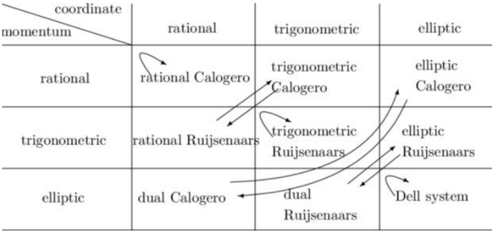
"Alll Calogero–Painlevé systems are Lax integrable, and hence their solutions admit a Riemann-Hilbert representation. This important observation has opened the door to rigorous asymptotic analysis of the Calogero–Painlevé equations which in turn yields the possibility of rigorous evaluation of the asymptotic behavior of the Tracy-Widom distributions for the values of beta beyond the classical $\beta = 1, 2, 4$."

A. Its and A. Prokhorov (2020) shall start an asymptotic analysis of the Calogero–Painlevé system with a special focus on the Calogero–Painlevé system corresponding to $\beta = 6$ Tracy-Widom distribution function.

Duality

- Duality for a system of free particles : Fourier transform (the exchange between coordinates and momenta) which are the integrals of motion themselves.
- Two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.
- First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).
- Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero- Sutherland type systems and their "relativistic" deformations.

Calogero-Moser-Ruijsenaars duality



Action of the coordinate-momentum duality on the Calogero-Ruijsenaars-Dell family. Hooked arrows mark self-dual systems. The duality leaves the coupling constant g intact

The simplest example-1

Rational Calogero system :

$$H_{Cal}(q,p) = \frac{1}{2}\left(\sum_{k=1}^{n} p_k^2 + \sum_{j\neq k} \frac{g^2}{(q_k - q_j)^2}\right)$$

Symplectic reduction :

Consider phase space $T^*(iu(n)) = iu(n) \times iu(n) := \{(Q,P)\}$ with two families of "free" Hamiltonians $\{\mathrm{tr}(Q_k)\}$ and $\{\mathrm{tr}(P_k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(g) = ig\sum_{j\neq k} E_{jk}.$$

This yields the self-dual Calogero system : gauge slice (i) :
$Q = q := \mathrm{diag}(q_1,\ldots,q_n), \quad q_1 > \ldots q_n,$ with $p := \mathrm{diag}(p_1,\ldots,p_n)$

$$P = p + ig\sum_{j\neq k} \frac{E_{ik}}{q_j - q_k} = L_{Cal}(q,p).$$

$$\mathrm{tr}(dP \wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k.$$

The simplest example-2

gauge slice (ii) : $P = \hat{p} := \mathrm{diag}(\hat{p}_1, \ldots, \hat{p}_n), \quad \hat{p}_1 > \ldots \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1, \ldots, \hat{q}_n)$

$$Q = \hat{q} + ig \sum_{j \neq k} \frac{E_{ik}}{\hat{p}_j - \hat{p}_k} = -L_{Cal}(\hat{p}, \hat{q}).$$

dual Lax matrix,

$$\mathrm{tr}(dP \wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the "action-angle map" for the two Calogero systems : alias the duality map.

Two interesting questions

- "NC Ruijsenaars duality" :
  We can also use "dual coordinates" imposing
  $$X := \mathrm{diag}(p_1, \ldots, p_n), \quad Y := \mathrm{diag}(q_1, \ldots, q_n) - \left(\frac{ig_4}{p_i - p_j}\right)_{i \neq j}.$$
  and, in these coordinates, the "dual" Hamiltonian reads
  $$H_I^{(\mathrm{dual})} := \sum_{i=1}^{n} \left(\frac{p_i^2}{2} - \frac{q_i^3}{2} - \frac{tq_i}{4}\right) - \sum_{j<k} \frac{g_4^2(q_j + q_k)}{(p_j - p_k)^2}. \tag{1}$$
- What about other Calogero-Painlevé ?
- What about $q-$ Painlevé ?

Dual form of Calogero-Painlevé II

Taking coordinates

$$Y = \mathrm{diag}(p_1, p_2, \ldots p_n), \quad X = \mathrm{diag}(q_1, q_2, \ldots q_n) - \left(\frac{ig}{p_i - p_j}\right)_{i \neq j} \tag{2}$$

Hamiltonian reads

$$H_{II}^{(\mathrm{dual})} = \sum_{i=1}^{n} \left[\frac{p_i^2}{2} - \frac{1}{2}(q_i^2 + \frac{t}{2})^2 - \theta q_i\right] + 2g^2 \sum_{i<j} \frac{q_i^2 + q_i q_j + q_j^2 + \frac{t}{2}}{(p_i - p_j)^2} -$$
$$-g^4 \left(\sum_{i<j} \frac{1}{(p_i - p_j)^4} + \sum_{i<j<k} \frac{2}{(p_i - p_j)^2 (p_j - p_k)^2}\right) +$$
$$+ \left(\sum_{i<j<k<l} \frac{4g^4}{(p_i - p_j)(p_j - p_k)(p_k - p_l)(p_l - p_i)}\right) \tag{3}$$

Four particle interaction

Selfdual form of Calogero-Painlevé IV

Taking coordinates

$$Q = \operatorname{diag}(q_1, \dots, q_n), \quad P = \operatorname{diag}(p_1, \dots, p_n) - \Big(\frac{ig}{q_k - q_j}\Big)_{k \neq j}.$$

Hamiltonian of dual Calogero-Painlevé IV reads :

$$H_{IV}^{(\mathrm{dual})} = \sum_k [q_k p_k^2 - p_k q_k^2 - t p_k q_k + \theta_0 q_k - (\theta_0 + \theta_1) p_k] + g^2 \sum_{k<j} \frac{q_k + q_j}{(q_k - q_j)^2}$$

Change of variables

$$p_k \to -q_k, \quad q_k \to -p_k, \quad \theta_0 \to \theta_1, \quad \theta_1 \to \theta_0 - \theta_1$$

transforms $H_{PIV}^{(\mathrm{dual})}$ to

$$H_{IV}^{(\mathrm{dual})} = \sum_k [q_k p_k^2 - p_k q_k^2 - t p_k q_k] + \theta_0 p_k - (\theta_0 + \theta_1) q_k] - g^2 \sum_{k<j} \frac{p_k + p_j}{(p_k - p_j)^2}$$

Theorem

The confluence map from the matrix Painlevé IV system to the matrix Painlevé II system holds for the Calogero-Painlevé multi-particle systems, but fails for the dual systems with $H_{IV}^{(\mathrm{dual})}$ and $H_{II}^{(\mathrm{dual})}$.

Painlevé-Ruijsenaars systems (after I. Gaiur)

Lemma

Zariski open set $X \subset T^\star \mathfrak{gl}_n$ *which is given by the equation*

$$\det(Q) \neq 0,$$

i.e. $Q \in GL_n$*, is symplectomoprhic to the* $T^\star GL_n$*. The symplectomorphism is given by*

$$\begin{cases} P = g^{-1}p \\ Q = g \end{cases} \qquad (Q, P) \in X, \quad (g, p) \in T^\star GL_n \cong GL_n \times \mathfrak{gl}_n, \tag{4}$$

where the $T^\star GL_n$ *is associated the the right invariant vector fields on* GL_n*.*

The symplectic form the $T^\star GL_n$ is given by

$$\omega_G = d\theta_G, \quad \theta_G = \mathrm{Tr}(p(\mathrm{d}\, g)g^{-1}).$$

The reduction procedure at $T^\star G$ gives not the Darboux cooedinates for the finall multi-particle system, but the log-symplectic coordinates

$$\omega = \sum_{i=1}^{n} \frac{dp_i \wedge dq_i}{q_i}.$$

Reductions of the matrix Painlevé III_{D8}.

The Hamiltonian for the matrix Painlevé III_{D8} is

$$tH_{D8} = \mathrm{Tr}\left(PQPQ + PQ - Q - tQ^{-1}\right), \tag{5}$$

with the symplectic form

$$\omega = \mathrm{Tr}(\mathrm{d}\,P \wedge \mathrm{d}\,Q).$$

The Hamiltonian contains the inverse of the matrix Q, so its phase space is the same as X in Lemma 2. The symplectomorphims (4) sends the matrix Painlevé III_{D8} Hamiltonian to the

$$tH_{D8} = \mathrm{Tr}\left(p^2 + p - g - tg^{-1}\right), \quad (g,p) \in T^{\star}GL_n, \tag{6}$$

and the moment map is given by

$$\mu = p - gpg^{-1}. \tag{7}$$

Shifting p by a scalar matrix $p = \tilde{p} - \frac{1}{2}$ and reducing from the Calogero-Moser space for the moment map μ diagonalizing g, we obtain the following parametrization

$$g_{ij} = \delta_{ij}q_i, \quad \tilde{p}_{ij} = \delta_{ij}p_i + (1-\delta_{ij})\frac{\sqrt{-1}h}{1 - q_i/q_j},$$

where h is an interaction constant.

Reductions of the matrix Painlevé III_{D8}-2.

The Hamiltonian turns to

$$tH = \sum_{i=1}^{n}\left(p_i^2 - q_i - \frac{t}{q_i}\right) + 2h^2\sum_{i<j}\frac{q_i q_j}{(q_i - q_j)^2}.$$

The reduced symplectic form is a log-symplectic form, i.e.

$$\omega = \sum_{i=1}^{n}\frac{dp_i \wedge dq_i}{q_i}.$$

Finally, applying the transformation $\ln(q_i) = q_i, \quad \ln(t) = T$, which sends the log-coordinates to the Darboux ones, we obtain the trigonometric Calogero-Painlevé system

$$H = \sum_{i=1}^{n}\left(p_i^2 - e^{\tilde{q}_i} - e^{T-\tilde{q}_i}\right) + \frac{h^2}{2}\sum_{i<j}\frac{1}{\sinh^2[(\tilde{q}_i - \tilde{q}_j)/2]}. \tag{8}$$

Reductions of the matrix Painlevé III_{D8}-3.

To obtain Ruijsenaars-like system, we first apply the following transformation to the (6)

$$g = \sqrt{t}\tilde{g}.$$

Since the transformation depends on the time, it should be considered as a transformation on the extended phase space

$$\Omega = \omega + dt \wedge dH.$$

Since that the Hamiltonian has to be shifted by $-\sqrt{t}p/2$ and writes as

$$tH_{D8} = \mathrm{Tr}\left[p^2 + \left(1 - \frac{\sqrt{t}}{2}\right)p - \sqrt{t}\left(\tilde{g} + \tilde{g}^{-1}\right)\right] = \mathrm{Tr}\left[p^2 + \left(1 - \frac{\sqrt{t}}{2}\right)p\right] - \sqrt{t}H_{\mathrm{Ruijsenaars}}. \tag{9}$$

Reducing from the Calogero-Moser space by diagonalization p now, we see that the first part of the Hamiltonian which depends only on p is diagonal, so that it will give some polynomial of p_i. The second part of the Hamiltonian is multiplied by a square root of the time free-particle Hamiltonian which reduction gives rational Ruijsenaars system.

We have obtained non-autonomous version of the rational Ruijsenaars system, which admits an isomonodromic representation and dual to the trigonometric Calogero-Painlevé system. We call such a system as the Painlevé-Ruijsenaars III_{D8} system.

Reductions of the matrix Painlevé III_{D7}-1.

The matrix Painlevé III_{D7} may be written as the following Hamiltonian system

$$tH = \mathrm{Tr}\left(PQPQ - \theta PQ + tP + Q\right). \tag{10}$$

In order to obtain physical Hamiltonian we doing the following time-dependent change of variables

$$p = \tilde{p} - \frac{1}{2}(tg^{-1} - \theta),$$

Reducing at the rank 1 level set of the moment map

$$\mu = p - gpg^{-1}$$

at the point where g is a diagonal matrix and sending log-symplectic form to the Darboux one by the map

$$\ln(t) = T, \quad \ln(g_i) = q_i$$

we automatically obtain the Painlevé-Calogero systems of the $D7$ type

$$\tilde{H}_{III_{D7}} = \sum_{i=1}^{n}\left(p_i^2 + \mathrm{e}^{q_i} + \frac{(\theta-1)\mathrm{e}^T}{2}\mathrm{e}^{-q_i} - \frac{1}{4}\mathrm{e}^{2T-2q_i}\right) + \frac{h^2}{2}\sum_{j<k}\frac{1}{\sinh^2\left((q_j-q_k)/2\right)}.$$

Reducing at the point where p is diagonal gives us some new Painlevé-Ruijsenaars system.

Dualities and Phase spaces

Observation (Fock-Gorsky-Nekrasov-R. : there are various phase spaces and reduction procedures "giving" **the same** integrable systems.

- For Painlevé I,II and IV the rationality of potential is an implication of $T^*\mathfrak{g}$ as a natural phase space
- For Painlevé III a natural phase spaces is T^*G - trigonometric Calogero / rational Ruijsenaars duality
- Painlevé VI has an elliptic potential. **We expect** that the natural phase space might be a reduction of the Hitchin-like equations related to $T^*(\mathcal{E})$ (Gorsky-Nekrasov-R.) or $T^*\hat{\mathfrak{g}}$ where $\hat{\mathfrak{g}}$ is an extension of the "elliptic loop" algebra $=\mathrm{Map}(\mathcal{E},\mathfrak{g})$ (Gorsky-Nekrasov).
- Should be closely related to the global description of the De-Rham moduli space

Many thanks for your attention !