Dynamics of groups of birational automorphisms of cubic surfaces and Fatou/Julia decomposition for Painlevé 6

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Dynamics on character varieties

Let A, B, C, and D be fixed complex parameters.

$$S_{A,B,C,D} = \{(x,y,z) \in \mathbb{C}^3 \,:\, x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D\}.$$

Switching intersections of $S_{A,B,C,D}$ with lines parallel to the x-axis yields

$$s_x \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x - yz + A \\ y \\ z \end{pmatrix}.$$

Analogously defined: $s_y : S_{A,B,C,D} \to S_{A,B,C,D}$ and $s_z : S_{A,B,C,D} \to S_{A,B,C,D}$

Dynamics on character varieties

Consider the group

$$\Gamma^* = \Gamma^*_{A,B,C,D} = \langle s_x, s_y, s_z \rangle \leq \mathrm{Aut}(S_{A,B,C,D}), \tag{1}$$

Rk. Generic parameters: $\Gamma_{A,B,C,D} = \mathrm{Aut}(S_{A,B,C,D})$. In general, $\Gamma_{A,B,C,D}$ is a subgroup of $\mathrm{Aut}(S_{A,B,C,D})$ of index at most 24.
Consider also the subgroup

$$\Gamma = \Gamma_{A,B,C,D} = \langle g_x, g_y, g_z \rangle < \Gamma^*,$$

where g_x, g_y, and g_z are defined as follows

$$g_x = s_z \circ s_y, \qquad g_y = s_x \circ s_z, \qquad \text{and} \qquad g_z = s_y \circ s_x.$$

and satisfy

$$g_x \circ g_z \circ g_y = \mathrm{id}\,.$$

Maps g

We have:

$$g_x \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y - xz + B \\ xy + (x^2 - 1)z + C - Bx \end{pmatrix},$$

$$g_y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (y^2 - 1)x + yz + A - Cy \\ y \\ -yx - z + C \end{pmatrix}, \quad \text{and}$$

$$g_z \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x - yz + A \\ zx + (z^2 - 1)y + B - Az \\ z \end{pmatrix}.$$

Pointwise dynamics

Purpose: to study the pointwise dynamics of Γ and Γ^*

Relatively few works about these dynamics:

- W. Goldman - ergodic theory of the dynamics in the real slice of "torus parameters" $(A = B = C = 0)$;
- Cantat-Loray - Non-existence of invariant affine/foliated structures;
- Dynamics of individual (hyperbolic) elements: Cantat, Iwasaki-Uehara.

Yet, there are many motivations: from Number theory (Markoff triplets), Teichmuller theory, and Painlevé 6.

Character varieties

To simplify: consider the case “Torus parameters” ($A = B = C = 0$) and let Π be the fundamental group of the punctured torus.

- Representations $\rho : \Pi \to \mathrm{SL}(2, \mathbb{C})$ (6-parameter space).
- Up to conjugation: $\rho \simeq \rho'$ if $\rho' = g \circ \rho \circ g^{-1}$, $g \in \mathrm{SL}(2, \mathbb{C})$.
- (Categorical, see GIT) quotient: character variety.
- $\mathrm{Aut}\,(\Pi)$ acts on space of representations by pre-composition:

$$(\gamma, \rho) \longmapsto \rho \circ \gamma : \Pi \to \mathrm{SL}(2, \mathbb{C}) \,,$$

$\gamma \in \mathrm{Aut}\,(\Pi)$, $\rho : \Pi \to \mathrm{SL}(2, \mathbb{C})$.

- Descends to character variety.
- Inner automorphism action becomes trivial.

Modular action

The action of $\mathrm{Aut}\,(\Pi)$ factors through

$$\mathrm{Out}\,(\Pi) = \mathrm{Aut}\,(\Pi)/\mathrm{Inn}\,(\Pi)\,.$$

In turn: $\mathrm{Out}\,(\Pi)$ is identified with the extended *Mapping Class Group* of the punctured torus.

Fricke coordinates: Character variety identified with $\mathbb{C}^3$.
More GIT: The action of $\mathrm{Out}\,(\Pi)$ is the action of Γ^* on $\mathbb{C}^3$.

Who is Painlevé 6?

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx} + \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right), \quad (2)$$

where $\alpha,\ \beta,\ \gamma, \delta \in \mathbb{C}$.

Vector field formulation on $\mathbb{C}^3$ ($z = dy/dx$, x-space coordinate identified with time).

$$Z_{\mathrm{VI}} = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \mathcal{H}_{\alpha,\beta,\gamma,\delta}(x,y,z)\frac{\partial}{\partial z}, \quad (3)$$

Fibrations, foliations, and Okamoto

Compactify $\mathbb{C}^3$ into $\mathbb{CP}^2 \times \mathbb{CP}^1$ and denote by $\mathcal{F}$ the resulting singular foliation.

- Invariant fibers $x = 0$, $x = 1$, $x = \infty$.
- Plenty of singularities and reasonably bad behavior (=lack of transversality).
- Okamoto: get rid of "poles" (=dicritical singularities) with blow-ups.
- He went all the way: nicer picture emerges - foliation tranverse to a fibration.
- BUT fibers are open (complement of Okamoto divisor).
- Painlevé property comes to the rescue.

Global holonomy representation and Riemann-Hilbert transform

Conclusion: Holonomy representation is "complete", i.e., it yields a representation ρ from $\Pi(S^2 \setminus \{0, 1, \infty\})$ to $\mathrm{Aut}\,(\mathrm{Fiber})$, where Fiber is OPEN.

(M. Inaba, K. Iwasaki, M. Saito) Riemann-Hilbert transform: conjugate the above action to the action of Γ on $S_{A,B,C,D}$ (explicit correspondence of parameters).

Rk: Riemann-Hilbert transform is highly transcendental - not liked by many Physicists.

Locally non-discrete groups

Some credits:

- Originally introduced on $\mathrm{Diff}\,(S^1)$ - highly motivated by previous works by Shcherbakov, Nakai, and Ghys.
- Applications to ergodic theory and to rigidity phenomena.
- Complex dynamics variant first considered on $\mathrm{Diff}\,(\mathbb{C}^n, 0)$, joint with F. Loray.
- Their "complement", i.e., locally discrete actions on S^1 where intensively and detailed studied by Deroin, Kleptsyn, Navas, Triestino, and their collaborators
- (Coarse) Classification of locally discrete groups (beyond Fuchsian ones).

Set: M complex manifold (possibly open) and G a group of holomorphic diffeomorphisms of M.

G is said to be *locally non-discrete* on an open set $U \subset M$ if there is a sequence of maps $\{f_n\}_{n=0}^{\infty} \in G$ satisfying the following conditions:

1. For every n, f_n is different from the identity.
2. The sequence of maps f_n converges uniformly to the identity on compact subsets of U.

If there is no such sequence f_n on U we say that G is *locally discrete* on U.

Rk: For finite dimensional Lie groups, local non-discreteness implies that the corresponding sequence of converges *globally* to the identity on M. However, in our context the non-linearity of the mappings allow for local non-discreteness to occur on a proper open subset $U \subset M$.

(Non) - discreteness locus

Fixed (A, B, C, D), let

$$\mathcal{N}_{A,B,C,D} = \{p \in S_{A,B,C,D} \, : \, \Gamma_{A,B,C,D} \text{ locally non-discrete on nghd of } p\},$$

and let

$$\mathcal{D}_{A,B,C,D} = S_{A,B,C,D} \setminus \mathcal{N}_{A,B,C,D}.$$

$\mathcal{N}_{A,B,C,D}$ is the "locally non-discrete locus".
$\mathcal{D}_{A,B,C,D}$ is the "locally discrete locus".

By definition, $\mathcal{N}_{A,B,C,D}$ is open, $\mathcal{D}_{A,B,C,D}$ is closed, and both of them are invariant under $\Gamma_{A,B,C,D}$.

Fatou - Julia

The *Fatou set* of the group action Γ is defined as

$$\mathcal{F}_{A,B,C,D} = \{p \in S_{A,B,C,D} \ : \ \Gamma \text{ normal family in nghd of } p\}.$$

The *Julia set* is

$$\mathcal{J}_{A,B,C,D} = S_{A,B,C,D} \setminus \mathcal{F}_{A,B,C,D}.$$

$\mathcal{F}_{A,B,C,D}$ is open while $\mathcal{J}_{A,B,C,D}$ is closed.
Both sets are invariant under Γ.

General Julia sets

Theorem A. For any parameters (A, B, C, D) there is a dense orbit of Γ in the Julia set $\mathcal{J}_{A,B,C,D}$.
Moreover:

- Denseness of points with shear stabilizer

$$D\gamma(p) \qquad \text{conjugate to} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- The Julia set is connected.

Theorem B. For the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ we have:

(i) $\mathcal{J}_{0,0,0,4} = S_{0,0,0,4}$ and consequently $\mathcal{F}_{0,0,0,4} = \emptyset$,

(ii) The action of $\Gamma_{0,0,0,4}$ is locally discrete on any open subset of $S_{0,0,0,4}$, and

(iii) The closure of the set of points $\mathcal{J}^*_{0,0,0,4}$ that have hyperbolic stabilizers is contained in $S_{0,0,0,4} \cap [-2, 2]^3$ and hence is a proper subset of $\mathcal{J}_{0,0,0,4} = S_{0,0,0,4}$.

However:

(1) Punctured Torus Parameters: For any complex D not equal to 4 the Fatou set $\mathcal{F}_{0,0,0,D}$ is non-empty.

(2) Dubrovin-Mazzocco Parameters: For any $a \in (-2, 2)$ the Fatou set $\mathcal{F}_{A(a),B(a),C(a),D(a)}$ is non-empty.

Moreover, the result carries over to an open neighborhood in $\mathbb{C}^4$ of any such parameter.

where

Dubrovin-Mazzocco Parameters: Real 1-parameter family studied by Dubrovin and Mazzocco (Physically relevant). The Dubrovin-Mazzocco parameters correspond to

$$A(a) = B(a) = C(a) = 2a + 4, \quad \text{and} \quad D(a) = -(a^2 + 8a + 8) \qquad (4)$$

for $a \in (-2, 2)$.

Now:

Theorem C. There is neighborhood of the Markoff Parameters $(0, 0, 0, 0)$ and of each of the Dubrovin-Mazzocco Parameters $(A(a), B(a), C(a), D(a))$, where $a \in (-2, 2)$, such that the following holds: the following property.
There are disjoint opens sets $U, V_\infty \subset S_{A,B,C,D}$ such that:

1. The action of $\Gamma_{A,B,C,D}$ is locally non-discrete on U; i.e. $U \subset \mathcal{N}_{A,B,C,D}$.
2. The action of $\Gamma_{A,B,C,D}$ is locally discrete on nghd of any point from V_∞, i.e. $V_\infty \subset \mathcal{D}_{A,B,C,D}$. Indeed, the action of $\Gamma_{A,B,C,D}$ on V_∞ is properly discontinuous.

Moreover, V_∞ is the previously described Fatou component.

Also

Theorem D. For the previous neighborhoods and up to deleting a countable union of real analytic sets, we have:

$$U \subset \mathcal{J}_{A,B,C,D} \qquad \text{and} \qquad V_\infty \subset \mathcal{F}_{A,B,C,D}.$$

Here, U and V_∞ are the open subsets from the statement of Theorem C.

Consequence of Theorems A, C, and D.

- There are points with orbits dense in open sets.
- There is a set

$$K_{A,B,C,D} \subset \partial \mathcal{N}_{A,B,C,D} = \partial \mathcal{D}_{A,B,C,D}$$

with topological dimension equal to three and invariant under $\Gamma_{A,B,C,D}$.

Remark. The boundary of Bers slice in Teichmuller theory allows one to produce a similar example, though the existence of sets $K_{A,B,C,D}$ as above goes beyond the parameters arising from Bers slice. Would this be a meaningful generalization of Bers slice?
In any case, it would be interesting to study the geometry of the sets $K_{A,B,C,D}$ which appear to be fractal.