# Several dynamics for the Painlevé V foliation

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joint work with J.P. Ramis

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Web seminar on Painlevé equations and related topics

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We present here the main tools in the case of the Painlevé V foliation.

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• the hamiltonian property: the Painlevé vector fields can be written on $T \times C^2$ under a hamiltonian form [Okamoto]:

$$\begin{cases} \dot{p} = -\dfrac{\partial H_\bullet}{\partial q} \\ \dot{q} = \dfrac{\partial H_\bullet}{\partial p} \end{cases}$$

with $H_\bullet = H(p, q, t, \theta)$, $t \in T = \mathbb{P}^1 \setminus \{\text{fixed sing.}\}, \theta :$ parameters.

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Dynamics through $RH_{VI}$ induced by $Aut(\pi_1^{VI})$: generated by 3 braids. In trace coordinates:

$$h_{1,2} : \begin{cases} x_1' = -x_1 - x_2x_3 + b_1 \\ x_2' = -x_2 + x_1x_3 + x_2x_3^2 - b_1x_3 + b_2 \\ x_3' = x_3 \end{cases}$$

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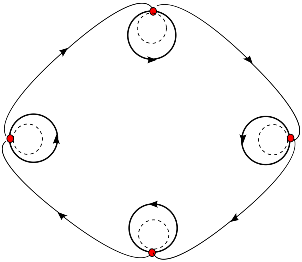

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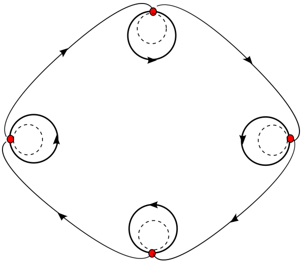

Linear representation $\rho$ of the groupoid:
- objects (base point) $s \to \rho(s) = 2$ dim vector space $V_s$ (RH: local matrix solution $Y_s$);
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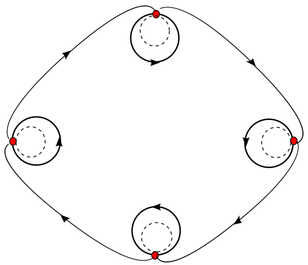

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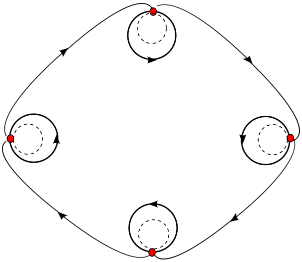

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The character variety $\chi_{VI}$ is the variety of equivalent representations in $SL_2(\mathbb{C})$ of $\pi_1^{VI}(X,S)$.

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Some intersections of these lines correspond to special solutions studied by K. Kaneko.

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Around $z = x^{-1} = 0$, formally,

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Formal solutions around $\infty$:

$$\widehat{Y}(z) = \widehat{F}(z) z^L \exp Q/z.$$

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under the action of $D$ (diagonal matrices) and $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Invariant coordinates :

$\alpha_0 = det(A_0), \; \alpha_1 = det(A_1), \; \alpha_\infty = det(L) = (a_0 + a_1)^2$; ("local" coordinates);

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The method of H. Chiba in order to compactify the variety on which live the Painlevé foliations can also be used starting from the spaces of linear connections.

# The wild character variety $\chi_V$

- The *monodromy operators* are defined by the analytic continuation of the matrix solutions along paths. They define a representation of the group $\pi_1(P_1 \setminus S, x_0)$ or of a fundamental groupoid with 3 base points.
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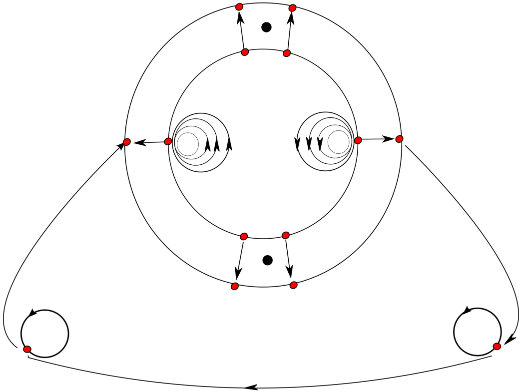
- The *Stokes operators* are defined by the comparison between two resummations of a formal matrix solution $\widehat{Y}$ on the left and rightside of a singular direction.
- The *exponential torus* is an action of the algebraic group $(\mathbb{C}^*, \times)$ on the local formal solutions.

Any element of $\mathcal{M}_V$ defines a representation of the following extended groupoid:

# The wild fundamental groupoid $\pi_1^V(X, S)$



with topological relations and additional wild relations
$t_{i,i}(\kappa\kappa') = t_{i,i}(\kappa) \cdot t_{i,i}(\kappa')$ i=0,1;
$[\widehat{\gamma}_{i,i}, t_{i,i}(\kappa)] = \star_i$
$[[\sigma_i, t_{i,i}(\kappa)], \sigma_i] = \star_i$ ($\sigma_i$: Stokes loops based in $s_i$).

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- Such representation satsfies the property $(\star)$: there exists a Borel-Cartan configuration $(B^-, C, B^+)$ such that:

$$\rho(t_{1,1}(\kappa)) = C,\ \rho(\sigma_1) \in U^-,\ \rho(\widehat{\gamma}^I_{1,2} \cdot \sigma_2 \cdot \widehat{\gamma}^I_{2,1}) \in U^+.$$

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The character variety $\chi_V$ is the variety of equivalent representations in $SL_2(\mathbb{C})$ of $\pi_1^V(X,S)$ which satisfy $(\star)$.
A class of such representations is "quasi" characterized by the data

$$\rho(\sigma_1) = U_1 = \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}, \rho(\sigma_2) = U_2 = \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \rho(\widehat{\gamma}_{1,1}) = M_0 = \begin{pmatrix} e_0 & 0 \\ 0 & e_0^{-1} \end{pmatrix}$$

and $\rho(\gamma_{3,3}) = M_3,\ \rho(\gamma_{4,4}) = M_4$.

# The wild character variety $\chi_V$

- Any connection in $\mathcal{M}_V$ induces a linear representation of the groupoid $\pi_1^V(X,S)$ : the paths are represented either by analytic continuation of local solution or by resummation process for a ray.
- Such representation satsfies the property $(\star)$: there exists a Borel-Cartan configuration $(B^-, C, B^+)$ such that:

$$\rho(t_{1,1}(\kappa)) = C,\ \rho(\sigma_1) \in U^-,\ \rho(\widehat{\gamma}^l_{1,2} \cdot \sigma_2 \cdot \widehat{\gamma}^l_{2,1}) \in U^+.$$

The character variety $\chi_V$ is the variety of equivalent representations in $SL_2(\mathbb{C})$ of $\pi_1^V(X,S)$ which satisfy $(\star)$.
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Indeed from the local relations we have either $\rho(t_{1,1}(\kappa)) = diag(\kappa, \kappa^{-1})$ or $\rho(t_{1,1}(\kappa)) = diag(\kappa^{-1}, \kappa)$, defining $\chi_V = \chi_V^+ \cup \chi_V^-$.

## "Trace" coordinates on $\chi_V(a)$.

The local data of $\rho$ is defined by $a^+ = (e_0, a_3 = tr(M_3), a_4 = tr(M_4))$ or $a^- = (e_0^{-1}, a_3 = tr(M_3), a_4 = tr(M_4))$.

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The coordinates $(a^+, x^+)$ define a map $Tr_V^+$, invertible for a generic $a$, from $\chi_V^+$ to the family affine cubic surface $\mathcal{C}_V(b^+)$ defined by

$$F_V(b^+, x) = x_1 x_2 x_3 + x_1^2 + x_2^2 - b_1^+ x_1 - b_2^+ x_2 - b_3^+ x_3 + b_4^+ = 0,$$

where $b_1^+ = a_3 + e_0 a_4,\ b_2^+ = a_4 + e_0 a_3,\ b_3^+ = e_0,\ b_4^+ = e_0^2 + e_0 a_3 a_4 + 1$.

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We also have a similar map $Tr_V^- : \chi_V^- \to \mathcal{C}_V(b^-)$ defined by $x_1^- = M_3[1,1]$, $x_2^- = M_4[1,1]$, $x_3^+ = x_3^-$.

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Expected results: for $J = VI$, $V$, etc... :

- $RH_J$ is a symplectic morphism between $(\mathcal{M}_J(\alpha), \omega_J(\alpha))$ and $(\chi_J(a), 2i\pi\omega_J(a))$;

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**Proposition** [Ramis, P.], [Klimes]

$$h_b : \begin{cases} x_1' = -x_1 + x_2 x_3 + x_1 x_3^2 - \theta_2 x_3 + \theta_1 \\ x_2' = -x_2 - x_1 x_3 + \theta_2 \\ x_3' = x_3 \end{cases}$$

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**Remark.** The tame dynamics are always *polynomial* dynamics on $\chi$ in trace coordinates.

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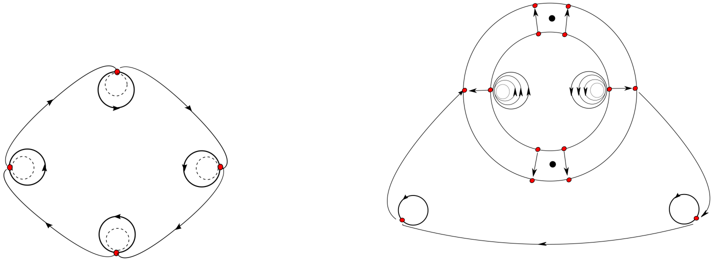
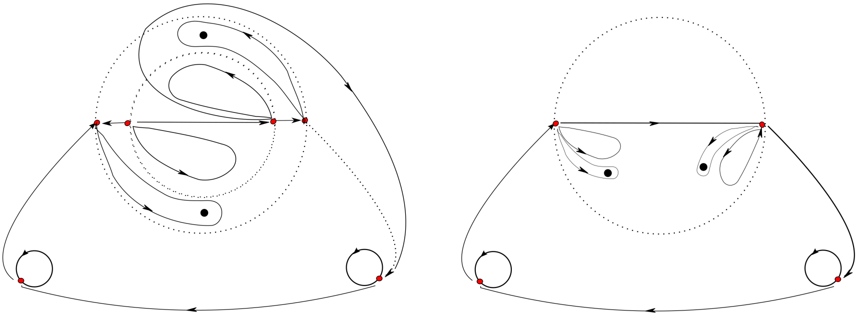


$$\varphi(\kappa): \ \pi_1^{VI}(X, S_{VI}) \to \pi_1^{V,\kappa}(X, S_V)$$

# The confluent dynamics on $\chi_V(a)$: $\varphi(\kappa)$.

Another presentation for $\pi_1^{V,\kappa}(X,S)$:



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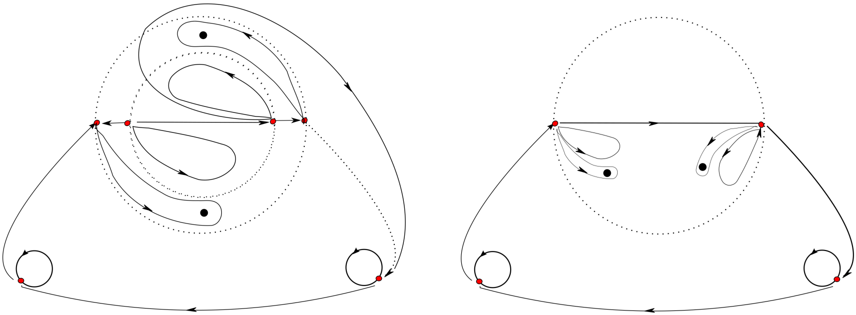

$\varphi_\kappa(\gamma_{1,1}) = \sigma_1 \cdot t_{1,1}(\kappa)$,
$\varphi_\kappa(\gamma_{2,2}) = \sigma_2 \cdot t_{2,2}(\kappa)$,
For the other generators : $\varphi_\kappa(\gamma_{i,j})$ is defined by the figure above.

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**Theorem.** *The morphisms* $\Phi(\kappa)^{\pm}: \ \chi_V^{\pm}(a) \to \chi_{VI}(a_\kappa)$*:* $\rho \to \rho \circ \varphi_\kappa$ *are generically invertible (on a Zariski open set).*
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**Idea of the proof.** We want to construct an inverse of

$$\Phi(\kappa): \ [U_1, M_0, U_2, M_3, M_4] \mapsto [M_{1,\kappa}, M_{2,\kappa}, M_3, M_4] = [U_1 D_\kappa, D_\kappa^{-1} M_0 U_2, M_3, M_4].$$

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We use the LDU decomposition in $SL_2$: If $a \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \cdot \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : e = a, l = c/a, u = b/a.$$

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$U_1 M_0 U_2$ is a representation through $\varphi_\kappa$ of a loop around the 2 confluent singularities $s_1$ et $s_2$. Therefore:
- $U_1 M_0 U_2$ is the LDU decomposition of a matrix *conjugated to* $M_{1,\kappa} M_{2,\kappa}$.
- its diagonal component $D$ is $\begin{pmatrix} e_{1,\kappa} e_{2,\kappa} & 0 \\ 0 & e_{1,\kappa}^{-1} e_{2,\kappa}^{-1} \end{pmatrix}$.

**Lemma**. *Let* $M_1$, $M_2$, $\neq \pm I$, *with eigenvalues* $(e_1, e_1^{-1})$ *et* $(e_2, e_2^{-1})$.
*Suppose that the eigenvectors related to* $(M_1, e_1)$ *et* $(M_2, e_2^{-1})$ *are independent.*
*There exists a unique matrix* $M = P^{-1} M_1 M_2 P$ *whose LDU decomposition satisfies* $D = diag(e_{1,\kappa} e_{2,\kappa}, e_{1,\kappa}^{-1} e_{2,\kappa}^{-1})$.

($P$ is obtained by using the "mixed" basis induced by the hypothesis.)

# The confluent morphisms.

In trace coordinates, we recover the formulas of M. Klimes:

$\Phi_\kappa = \varphi_\kappa^* : \ \chi_V(a) \to \chi_{VI}(a_\kappa)$ is given by

$$\begin{cases} x_{1,\kappa} = e_0^{-1}\kappa x_1 + \kappa^{-1} x_2 \\ x_{2,\kappa} = -e_0^{-1}\kappa x_1 x_3 + \kappa^{-1} x_1 - e_0^{-1}\kappa x_2 + a_3\kappa + a_4 e_0^{-1}\kappa \\ x_{3,\kappa} = x_3. \end{cases}$$

$\Phi_\kappa$ is invertible outside the line $L_{e_{1,\kappa},e_{2,\kappa}}$ and $\Phi_\kappa^{-1}$ is given by

$$\begin{cases} x_1 = (-\kappa x_{1,\kappa} - e_0\kappa^{-1} x_{2,\kappa} + a_3 e_0 + a_4)(x_{3,\kappa} - c_{e_{1,\kappa},e_{2,\kappa}})^{-1} \\ x_2 = (\kappa x_{1,\kappa} x_{3,\kappa} - e_0\kappa^{-1} x_{1,\kappa} + \kappa x_{2,\kappa} - a_3\kappa^2 - a_4\kappa^2 e_0^{-1})(x_{3,\kappa} - c_{e_{1,\kappa},e_{2,\kappa}})^{-1} \\ x_3 = x_{3,\kappa}. \end{cases}$$

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We transfer the dynamics $< h_{i,j} >$ of Painlevé VI on $\chi_V$:
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$$\begin{cases} X_1 = \dfrac{e_0}{x_2} \\ X_2 = x_2 + \dfrac{\kappa^2}{x_2} - e_0^{-1}\kappa^2 x_1 \\ X_3 = -\kappa^2 x_2^2 x_3 + (e_0^{-2}\kappa^2 - \kappa^{-2}) x_1 x_2 - 2e_0^{-1} x_2^2 + \\ \quad + (e_0^{-1} b_2 \kappa^{-2} b_1) x_2 - (e_0^{-1}\kappa^2 + e_0 \kappa^{-2}) \end{cases}$$

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We transfer the dynamics $< h_{i,j} >$ of Painlevé VI on $\chi_V$:
$g_{i,j}(\kappa) := \Phi(\kappa)^{-1} \circ h_{i,j} \circ \Phi(\kappa)$. We obtain:

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**Remark.** The family $t_{2,3}(\kappa)$ defined by $g_{2,3}(\kappa) = g_{2,3}(1) \circ t_{2,3}(\kappa)$ is multiplicative: $t_{2,3}(\kappa\kappa') = t_{2,3}(\kappa) \circ t_{2,3}(\kappa')$, and can be defined as the flow of a complete vector field.

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The dots terms are obtained by using the following "exchange" relations:
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**Remark.** $z_1z_2z_3 = 0$ is the equation of 12 lines in $\mathcal{C}_V(b)$.

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**Proposition.** The log-canonical sequence satisfies the Laurent property.

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• $B_1^\natural =< U_1, T_1 >$. $B_1^\natural$ is also a meta-abelian group which contains $U_1$, but it contains only a maximal torus of rank 1. $B_1^\natural$ is a Borel of rank 1.

# The log-canonical dynamics on $\mathcal{C}_V(b)$.

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• We consider a log-canonical triple $(z, y, z')$.
The pull-back of $T_1$ by $(z, y)$ is equal the pull-back of $T_2$ by $(y, z')$ and defines the subgroup $T_y$ of $Symp(\mathcal{C}_V(b))$: the exponential torus related to $y$.

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**Remark.** $Dyn(\mathcal{C}_V(b))$ contains all the $T_y$ and all the $s_z$.

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If we extend both $Conf$ and $Dyn$ by this element $b_{y_2}$, $Conf^\sharp = Dyn^\sharp$.

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- for generic parameters, the Malgrange groupoid of the Painlevé foliations is maximal.

Thank you for your attention.

HAPPY NEW YEAR!