Analytic classification of doubly resonant germs of analytic vector fields

and Applications to Painlevé equations

(Phd work of Amaury Bittmann)

Reference: Annales de l'Institut Fourier 68 (2018) no. 4

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Goal:

Give an **analytic classification** under the action of the germs of analytic diffeomorphisms in the neighborhood of a singular point, by exhibiting a complete system of invariants

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The work of Bittmann concerns the classification of a doubly resonant germ of vector field appearing in the compactification of (P_I) - Painlevé's first equation

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Stokes phenomena and quasi-linear connecting formulas (Kapaev, Kitaev, Costin,.)

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Theorem (Kapaev 2004)

For $\arg(t) \in \left[\pi, \frac{9\pi}{5}\right]$ and $|t| \to \infty$, if we note $\alpha = \frac{e^{i\pi/8}}{\sqrt{\pi}} 2^{-11/8} 3^{-1/8}$ then

$$\boxed{z_{1,-}(t) - z_{1,+}(t) = \alpha t^{-1/8} \exp\left(-\frac{1}{5} 2^{11/4} 3^{1/4} (e^{-7i\pi/5} t)^{5/4} \right) (1 + O(t^{-3/8}))}$$

We consider the differential system

$$\begin{cases} \frac{dz_1}{dt} &= z_2(t) \\ \frac{dz_2}{dt} &= 6z_1(t)^2 + t \end{cases}$$

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$$\frac{\partial}{\partial t} + z_2 \frac{\partial}{\partial z_1} + (6\, z_1^2 + t) \frac{\partial}{\partial z_2}$$

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$$Y_I = x^2\frac{\partial}{\partial x} + \left(-\lambda y_1 + \frac{1}{2}x y_1 + \frac{1}{25\lambda}x y_2 - \frac{\lambda}{96}x^2 - \frac{48}{25\lambda}\left(y_1 - \frac{2}{5\lambda}y_2\right)^2\right)\frac{\partial}{\partial y_1}$$

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where $\lambda = \dfrac{4 \cdot 2^{3/4} \cdot 3^{1/4}}{5} e^{i\pi/4}$.

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$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + F_1(x, \boldsymbol{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + F_2(x, \boldsymbol{y})) \frac{\partial}{\partial y_2}$$

where $\boldsymbol{y} = (y_1, y_2) \in (\mathbb{C}^2, 0)$, $\lambda \in \mathbb{C}^\star$, $F_1, F_2 \in \mathbb{C}\{x, \boldsymbol{y}\}$ have order at least 2.

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We say that Y is a $\mathbf{2-resonant}$ **diagonal saddle node**

Initially, we consider the *formal fibered* classification, i.e. the orbits under action of the group

$$\widehat{\mathrm{Diff}}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id}) = \{\hat{\Phi}\colon (x, \boldsymbol{y}) \to (x, y + \hat{\varphi}(x, \boldsymbol{y})) : \mathrm{ord}(\hat{\varphi}) \geqslant 2\}$$

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where $v = y_1 y_2$ and x are the so-called **resonant monomials** and g_1, g_2 are formal series.

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where $\alpha(x) \in \mathrm{Mat}_{2,1}(x^2\mathbb{C}[[x]])$, $\boldsymbol{A}(x) \in \mathrm{Mat}_{2,2}(\mathbb{C}[[x]])$ and $\boldsymbol{f}(x,\boldsymbol{y}) \in O(\|\boldsymbol{y}\|^2)$.

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In particular, $\widehat{\mathrm{SN}}_{\mathrm{diag,nd}}$ is invariant under the action of $\widehat{\mathrm{Diff}}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$

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Let $Y \in \widehat{\mathrm{SN}}_{\mathrm{diag,nd}}$. Then, there exists an unique $\hat{\Phi} \in \widehat{\mathrm{Diff}}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ and a unique $5-\mathrm{uple}$

- $\lambda \in \mathbb{C}^\star$
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In other words, we can identify the orbit space $\widehat{\mathrm{SN}}_{\mathrm{diag,nd}} / \widehat{\mathrm{Diff}}_{\mathrm{fib}}(\mathbb{C}^3, 0; \mathrm{Id})$ to the set

$$P_{\mathrm{fib}} = \{(\lambda, a_1, a_2, c_1, c_2) \in \mathbb{C}^\star \times (\mathbb{C}^2 \setminus \Delta) \times (v\mathbb{C}[[v]])^2\}$$

where $\Delta = \{(a_1, a_2) \in \mathbb{C}^2 \colon a_1 + a_2 \notin \mathbb{Q}_{\leqslant 0}\}$.

Remark: In the case where Y is preserves a **symplectic structure** (as is the case for P_I), we can also obtain a normalisation by a fibered diffeomorphism preserving such structure. In this case, we have $c_1 = c_2$ and $\mathrm{Res}(Y) = 1$.

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Remark

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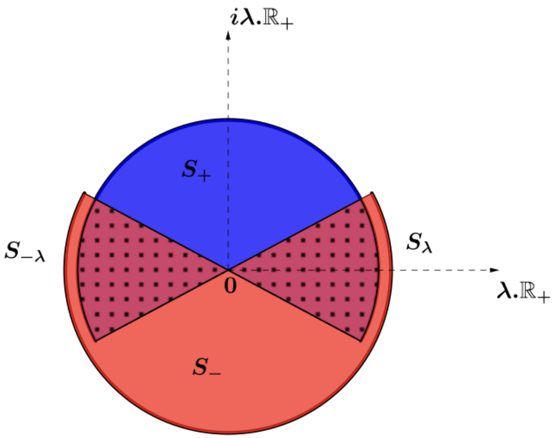

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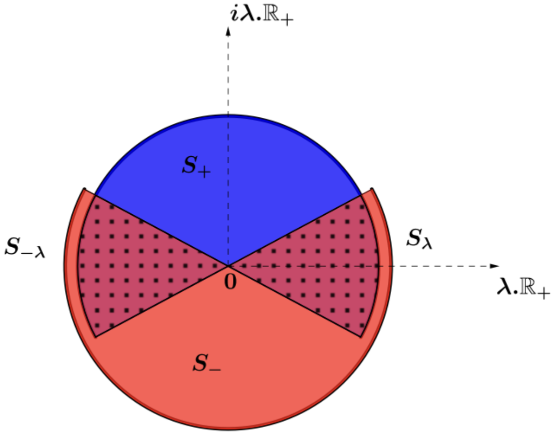

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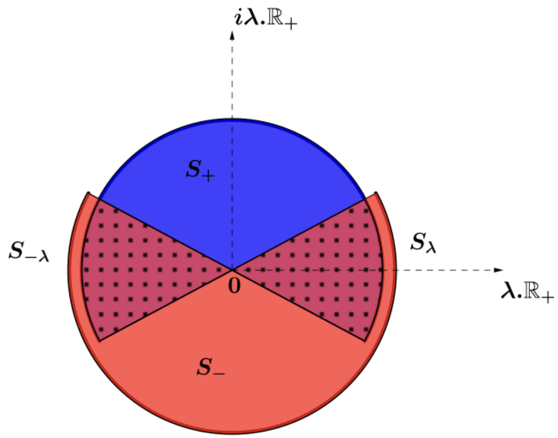

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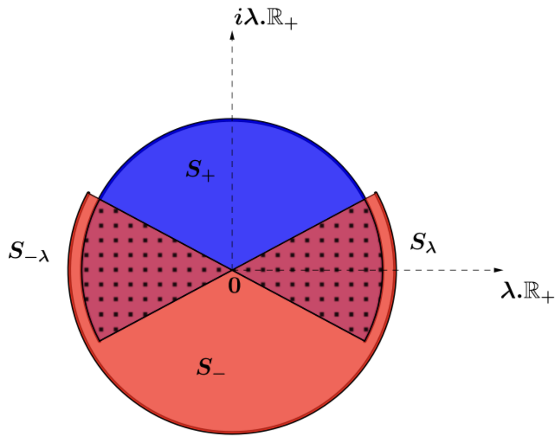

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Leaf space

The normal form

$$Y_{\mathrm{norm}} = x^2\frac{\partial}{\partial x} + (-\lambda + a_1 x + \boldsymbol{c}(\boldsymbol{v}))\, y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + \boldsymbol{c}(\boldsymbol{v}))\, y_2 \frac{\partial}{\partial y_2}$$

has two functionally independent sectorial first integrals

$$h_{1,\pm\lambda}(x,\boldsymbol{y}) = y_1 \exp\left(\frac{-\lambda}{x} + \frac{c_m (y_1 y_2)^m \log(x)}{x} + \frac{\tilde{c}(y_1 y_2)}{x}\right) x^{-a_1}$$

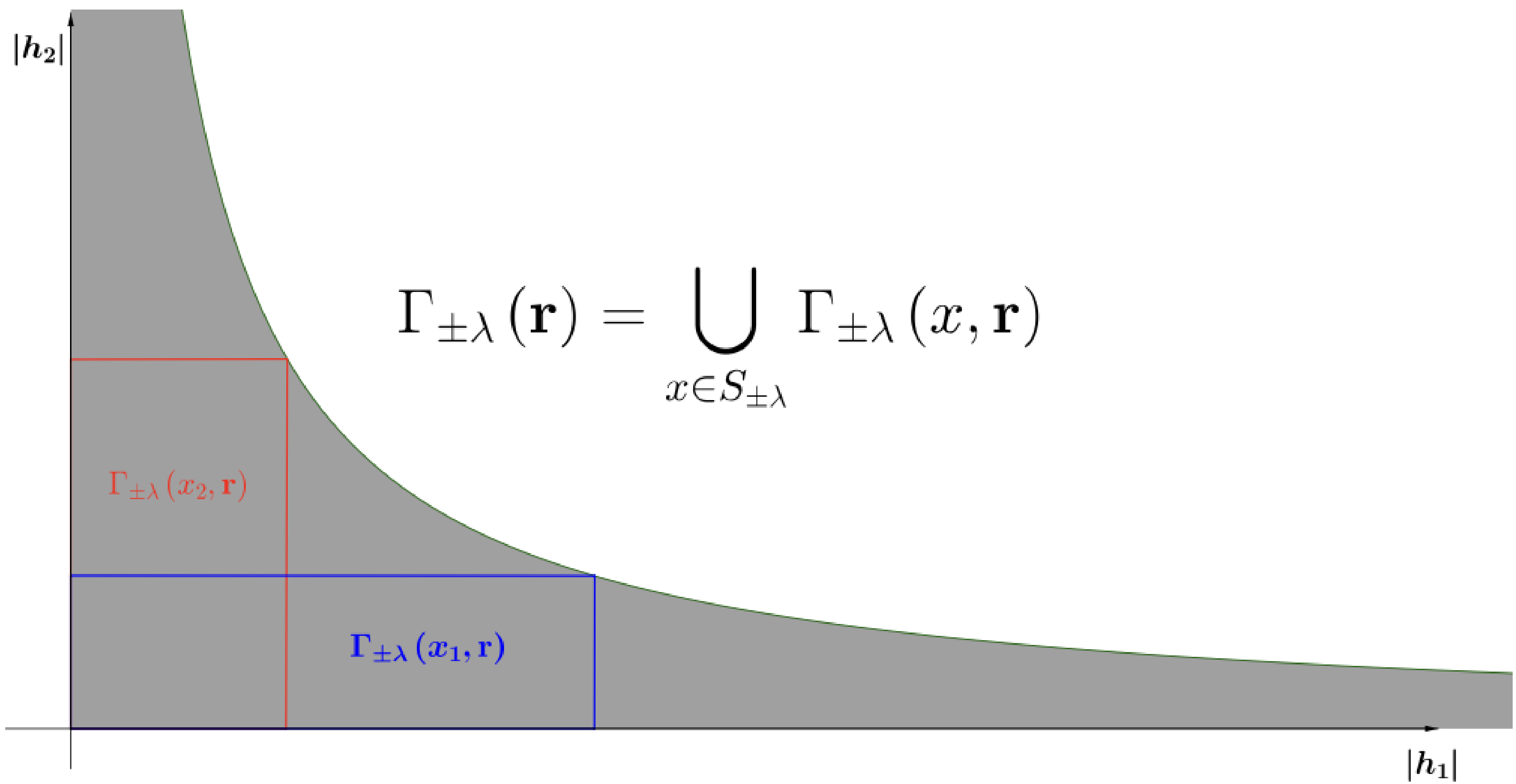
$$h_{2,\pm\lambda}(x,\boldsymbol{y}) = y_2 \exp\left(\frac{-\lambda}{x} + \frac{c_m (y_1 y_2)^m \log(x)}{x} + \frac{\tilde{c}(y_1 y_2)}{x}\right) x^{-a_2}$$

and we can consider the sectorial diffeomorphisms to the **leaf space** $\Gamma_{\pm\lambda}$

$$H_{\pm\lambda}\colon S_{\pm\lambda} \times (\mathbb{C}^2, 0) \longrightarrow S_{\pm\lambda} \times \Gamma_{\pm\lambda}$$

$$(x,\boldsymbol{y}) \longmapsto (x, h_{1,\pm\lambda}(x,\boldsymbol{y}), h_{2,\pm\lambda}(x,\boldsymbol{y}))$$

The leaf space for $c = 0$.



Stokes diffeomorphisms as maps on the leaf space

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$$
\begin{array}{ccc}
Y_{\mathrm{norm}}\,\mathrm{on}\,S_\lambda\times(\mathbb{C}^2,0) & \xrightarrow{\quad\Phi_\lambda\quad} & Y_{\mathrm{norm}}\,\mathrm{on}\,S_\lambda\times(\mathbb{C}^2 \\
\Big\downarrow H_\lambda & & \Big\downarrow H_\lambda \\
x^2\dfrac{\partial}{\partial x}\,\mathrm{on}\,S_\lambda\times\Gamma_\lambda & \xrightarrow{\quad\Psi_\lambda\quad} & x^2\dfrac{\partial}{\partial x}\,\mathrm{on}\,S_\lambda\times\Gamma_\lambda
\end{array}
$$

Stokes diffeomorphisms as maps on the leaf space

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where

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\end{array}
$$

where

$$\Psi_\lambda(h_1,h_2) = \left(h_1 + \sum_{n \geqslant 2} \Psi_{1,\lambda,n}(h_1 h_2)\, h_1^n,\, h_2 + \sum_{n \geqslant 2} \Psi_{2,\lambda,n}(h_1 h_2)\, h_2^n \right)$$

where each $\Psi_{i,\lambda,n}$ is an entire function satisfying an appropriate growth condition.

Stokes diffeomorphisms as maps on the leaf space

$$\begin{array}{ccc} Y_{\mathrm{norm}} \,\mathrm{on}\, S_\lambda \times (\mathbb{C}^2, 0) & \xrightarrow{\Phi_\lambda} & Y_{\mathrm{norm}} \,\mathrm{on}\, S_\lambda \times (\mathbb{C}^2 \\ H_\lambda \big\downarrow & & \big\downarrow H_\lambda \\ x^2 \dfrac{\partial}{\partial x} \,\mathrm{on}\, S_\lambda \times \Gamma_\lambda & \xrightarrow{\Psi_\lambda} & x^2 \dfrac{\partial}{\partial x} \,\mathrm{on}\, S_\lambda \times \Gamma_\lambda \end{array}$$

where

$$\Psi_\lambda(h_1, h_2) = \left(h_1 + \sum_{n \geqslant 2} \Psi_{1,\lambda,n}(h_1 h_2)\, h_1^n, h_2 + \sum_{n \geqslant 2} \Psi_{2,\lambda,n}(h_1 h_2)\, h_2^n \right)$$

where each $\Psi_{i,\lambda,n}$ is an entire function satisfying an appropriate growth condition.

By using Kapaev'04, we can compute the first coefficients

$$\boxed{\Psi_{2,\lambda,0}(0) = i\, \Psi_{1,-\lambda,0}(0) = \frac{e^{i\pi/8}}{\sqrt{\pi}} 2^{3/8} 3^{1/8}}$$

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Thanks!