Quantisation via the Q-top recursion and the Nekrasov-Shatashivili Limit.

main reference arXiv: 2305.02494 ... ⛤

<u>Note</u>: some formulae are too lengthy to write down here, so I often give eq. # in ⛤

- see also 2204:12431 w/ O. Kidwai
- Advertisement: I will post one more paper to arXiv soon

Where is the Q-top recursion coming from?

Story of Topological recursion & its refinement.

Topological recursion of Chekhov-Eynard-Orantin '05 '06 '07. $\longleftrightarrow$ Virasoro algebra with $c = 1$.

$\Big\downarrow$ β-deformation

Refined TR. (CE '06, Kidwai-O '22, O '23) $\longleftrightarrow$ β-deformed Virasoro algebra

$$c = 1 - 6Q^2, \quad Q = \beta^{\frac{1}{2}} - \beta^{-\frac{1}{2}}$$

↑ computations are hard

self-dual limit $\beta = 1$; $Q = 0$

$Q \to \infty$ $(\beta \to \infty, 0)$

Q-top recursion. $\longleftrightarrow$ Nekrasov-Shatashvili limit

What can we do with the Q-top rec.?

if you want to compute the Nek. Sha. effective twisted superpotential W

① $Z_{Nek}(\varepsilon_1,\varepsilon_2)$ from $\mathcal{M}_{n,r=2}$ = moduli space of instantons

$$W^{eff}_{4D} := \lim_{\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \cdot \log Z_{Nek}(\varepsilon_1,\varepsilon_2).$$

AGT correspondence

② 4pt function $\langle \Phi(0)\,\Phi(1)\,\Phi(t)\,\Phi(\infty)\rangle =: Z_{CB}(\varepsilon_1,\varepsilon_2)$ from conformal blocks (CB)

$$W^{eff}_{CB} := \lim_{\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \cdot \log Z_{CB}(\varepsilon_1,\varepsilon_2)$$

?

③ $F_g^{Q\text{-top}}$: free energy from the Q-top rec. on

S (refined spectral curve) $= \{\Sigma, x, y, \ldots\}$

$$W^{eff}_{Q\text{-top}} := \sum_{g\in\frac{1}{2}\mathbb{Z}_{\geq 0}} \varepsilon_1^{2g} F_g^{Q\text{-top}}$$

Relation to Painlevé I.

$y^2 = 4x^3 + 2tx + u$

$g(\Sigma)=1$

(refined) spectral curve $S = \{ \Sigma, x, y, \ldots \}$

$\downarrow$

$\boxed{\text{(R) TR}}$

refinement parameter $\downarrow$ $Q=0$ CEO TR.

Q-top.

Iwaki '19

$$\left(\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \left(4x^3 + 2tx + \frac{\partial F^{TR}}{\partial t}\right)\right)\psi^{TR} = 0$$

$$\left(\frac{\partial}{\partial t} \cdot - \quad \right) \cdot \psi^{TR} = 0.$$

Z^{TR} : partition function of TR is related to the τ-function of Painlevé I.

$$W^{\text{eff}}_{Q\text{-top}}$$

$$\parallel$$

$$\left(\varepsilon_1^2 \frac{\partial^2}{\partial x^2} - \left(4x^3 + 2tx + \frac{\partial F^{Q\text{-top}}}{\partial t}\right)\right)\psi^{Q\text{-top}} = 0$$

To appear $\rightarrow$

- Thm if u is chosen s.t. $\Sigma = \mathbb{P}^1$ i.e. singular limit.
- Conj. for a generic u.
- No $\varepsilon_1^2 \frac{\partial}{\partial t}$, expected from CB perspective.

Definitions

$C = (\Sigma, x, y)$: normalised & compactified hyperelliptic curve

i.e. Σ : compact RS of genus g

(x, y) : mero. func. on Σ s.t.

$$y^2 = Q(x) \quad \text{for some } Q \in \mathbb{C}(x)$$

$R = \{$ set of ramification pts of $x : \Sigma \to \mathbb{P}^1$

σ = hyperelliptic invol. i.e. $\sigma : y \mapsto -y, \; x \mapsto x$.

Def : A (hyperelliptic) refined spectral curve $S_{\kappa,\mu}$ consists of the following three data.

- (Σ, x, y) : hyperelliptic curve.

- choice of $\{A_i, B_i\}_{i\in\{1,\dots,\tilde{g}\}} \in H_1(\Sigma,\mathbb{Z})$ & for each i $\kappa_i \in \mathbb{C}$

- choice of $\mathcal{P} = \mathcal{P}_+ \cup \sigma(\mathcal{P}_+)$ & $p \in \mathcal{P}_+$ $\mu_p \in \mathbb{C}$

  where $\mathcal{P}$ = set of un-ramified zeroes & poles of $y\,dx$.

<u>Def</u>. Fix $Q \in \mathbb{C}$. Given $S_{\kappa,\mu}$, the (hyperelliptic) refined TR is a recursive definition of an infinite sequence of mero. multidifferentials $\{\omega_{g,n}\}$ on $(\Sigma)^n$ labelled by $n \in \mathbb{Z}_{\geq 0}$, $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ by the following formulae.

- $\omega_{0,1} := y\,dx$

- $\omega_{0,2}(p_0,p_1) := -B(\sigma(p_0),p_1)$

  B: fundamental bidifferential of the 2^{nd} kind (Sec 2.1.3 in my paper).

- $$\omega_{\frac{1}{2},1}(p_0) := \frac{Q}{2}\left(-\frac{dy(p_0)}{y(p_0)} + \sum_{p\in\mathcal{P}_+} \mu_p \cdot \int_{\sigma(p)}^{p} \omega_{0,2}(p_0,\cdot) + \sum_{i=1}^{\tilde{g}} \kappa_i \int_{B_i} \omega_{0,2}(p_0,\cdot)\right)$$

For $2g-2+n \geq 0$, $\omega_{g,n+1}$ is defined as

$$\omega_{g,n+1}(p_0, \overset{(p_1,\cdots,p_n)}{\overset{\Downarrow}{J}}) = \frac{1}{2\pi i}\left(\underset{\underset{\sim}{p\in}}{\int_{C_+}} - \underset{\underset{\sim}{p\in}}{\int_{C_-}}\right) \cdot \frac{\int_{\sigma(p)}^{p} \omega_{0,2}(p_0,\cdot)}{4\omega_{0,1}(p)} \cdot \underset{\substack{\uparrow \\ \text{Quadratic differential in } p}}{\mathrm{Rec}_{g,n+1}(p,J)}$$

$\boxed{\text{Def 2.9}}$ in my paper.

some specific contour defined in Sec 2 in my paper.

$$\mathrm{Rec}^{\mathcal{Q}}_{g,n+1}(p_0;J) := \sum^{*}_{\substack{g_1+g_2=g \\ J_1\sqcup J_2=J}} \omega_{g_1,n_1+1}(p_0,J_1)\cdot\omega_{g_2,n_2+1}(p_0,J_2) + \sum_{t\sqcup I=J} \frac{dx(p_0)\cdot dx(t)}{(x(p_0)-x(t))^2}\cdot\omega_{g,n}(p_0,I)$$

$$+\,\omega_{g-1,n+2}(p_0,p_0,J) + \mathcal{Q}\cdot dx\cdot d_0\frac{\omega_{g-\frac{1}{2},n+1}(p_0,J)}{dx(p_0)},$$

explicit.

This formula is recursive in $2g-2+n = -\chi$

Prop (O23, Kidwai-O23).

I: $\omega_{g,n}|_{Q=0} = \omega_{g,n}^{\mathrm{CEO}}$

II: $\omega_{\frac{1}{2},2}$ is sym. bidiff, no residue, $\int_{A_i} \omega_{\frac{1}{2},2}(p_0,\bullet) = 0.$

III: When $\Sigma = \mathbb{P}^1$, then all $\omega_{g,n}$ are sym., no residue

Conj. III holds for any Σ.

Important Observation.

$\omega_{g,n}$ polynomially depend on Q up to deg $2g$.

$$\omega_{g,n} =: \sum_{k=0}^{2g} Q^k \, \omega_{g,n}^{(k)}$$

$$\varpi_{g,n} := \omega_{g,n}^{(2g)}$$ Q-top deg. part of $\omega_{g,n}$

the recursion for $\varpi_{g,n}$ is self-closed,

$\varpi_{g,n}$ can be recursively determined without info of $\omega_{g,n}$ $(k<2g)$.

In particular, $\varpi_{g,n}$ are given by

$$\varpi_{g,n+1} = \frac{1}{2\pi i}\left(\int_{C_+} - \int_{C}\right) \cdot \frac{\int_{\sigma(p)}^{p} \omega_{0,2}(p,\cdot)}{4\,\omega_{0,1}(p)} \cdot \mathrm{Rec}^{\mathcal{Q}\text{-top}}$$

$$\mathrm{Rec}^{\mathcal{Q}\text{-top}}_{g,n+1}(p_0;J) := \sum^{*}_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=J}} \varpi_{g_1,n_1+1}(p_0,J_1)\cdot\varpi_{g_2,n_2+1}(p_0,J_2)$$

$$+\sum_{t\sqcup I=J}\frac{dx(p_0)\cdot dx(t)}{(x(p_0)-x(t))^2}\cdot\varpi_{g,n}(p_0,I) + dx\cdot d_0\frac{\varpi_{g-\frac{1}{2},n+1}(p_0,J)}{dx(p_0)}.$$

$$\mathrm{Rec}^{\mathcal{Q}}_{g,n+1}(p_0;J) := \sum^{*}_{\substack{g_1+g_2=g\\ J_1\sqcup J_2=J}} \omega_{g_1,n_1+1}(p_0,J_1)\cdot\omega_{g_2,n_2+1}(p_0,J_2) + \sum_{t\sqcup I=J}\frac{dx(p_0)\cdot dx(t)}{(x(p_0)-x(t))^2}\cdot\omega_{g,n}(p_0,I)$$

$$+\,\omega_{g-1,n+2}(p_0,p_0,J) + \mathcal{Q}\cdot dx\cdot d_0\frac{\omega_{g-\frac{1}{2},n+1}(p_0,J)}{dx(p_0)},$$

Because there is NO $\varpi_{g-1,n+2}$, the $\varpi_{g,n}$-recursion is recursively separately in g & n. Q-top recursion

$\Rightarrow \{\varpi_{g,1}\}_{g\in\frac{1}{2}\mathbb{Z}_{\geq 0}}$ is determined w/o $\{\omega_{g,n+2}\}$.

Thm (0 23) · $\{\varpi_{g,1}\}$ for all g is residue-free, $\oint_{A_i}\varpi_{g,1}=0$ $g\geq 1$.

& , $\exists$ 2^{nd} order-diff. equ. s.t. Q-top Quantum curve

$$\left(\varepsilon_1^2\frac{d^2}{dx^2}-Q_0(x)-\sum_{k\geq 1}\varepsilon_k Q_k(x)\right)\cdot\psi^{Q\text{-top}}=0$$

in CEO formalism

where
$$d\log\psi^{Q\text{-top}}=\sum_{g\in\frac{1}{2}\mathbb{Z}}\varepsilon^{2g}\varpi_{g,1}\longleftrightarrow\sum\frac{\hbar^{2g-2+n}}{n!}\int\cdots\int\omega_{g,n}$$

Q_k explicitly defined.

if one applies to $y^2 = 4x^3 + 2tx + 8q_0^3$ $\quad q_0 = \sqrt{-\frac{t}{6}}$

$\Rightarrow$ $Q_k(x)$ is related to $F_g^{Q\text{-top}}$ s.t.

$$Q_k(x)\Big|_{\mu=\mu_0} = \frac{\partial}{\partial t} F^{Q\text{-top}}_{\frac{k}{2}}.$$

$\rightarrow$ consistent with results from Conformal blocks.