q-connection problems on hypergeometric and Painlevé equations

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1. Introduction

1. Study q-connection problems on hypergeometric equations
2. Study q-connection problems on linearized equations of the Painlevé equations
3. Consider q-connection problems on the Painlevé equations

[Differential case]
In differential cases, we know many **connection formulae** on hypergeometric. general linear equations and Painlevé equations.

[Difference cases]
In difference cases, we know a little on connection formula.

q-hypergeometric: We know well now
q-linear : See my talk on July (just started. q-Painlevé VI)
q-Painlevé : Mano [Ma] 2010

2. Connection formula

We consider q-difference equations. $0 < |q| < 1$

$$Y(xq) = A(x)Y(x).$$

$A(x)$ is a $n \times n$ matrix; matrix elements are rational functions on x

Local fundamental solutions

$Y_0(x)$: Local fundamental solution around $x = 0$

$Y_\infty(x)$: Local fundamental solution around $x = \infty$

Connection matrix

$$Y_\infty(x) = Y_0(x)P(x), \qquad P(xq) = P(x)$$

We do not have enough examples of connection formulae on q-difference equations.

Modern works

After 1990s, J.-P. Ramis, C. Zhang, J. Sauloy are constructing modern theory on q-asymptotics [RSZ]

q-Borel-Laplace method

2.1 Differential case

Hypergeometric equations ${}_nF_{n-1}$ (Gauss, Thomae, Orr)

other **rigid systems**, middle convolutions

(Kac, Simpson, Dettweiler-Reiter, Crawley-Boevey, ..)

irregular cases ${}_rF_s$ (Barnes, Meijer, Duval-Mitchi,...)

$\rightarrow$ the Stokes phenomenon

Known examples of q-connection See Slater's book [Sl]

J. Thomae (1869) ${}_2\phi_1(a, b; c; q, x)$

J. Thomae (1870) ${}_r\phi_{r-1}(\boldsymbol{a}; \boldsymbol{b}; q, x)$

G. N. Watson (1910) [cannot treat divergentseries]

${}_r\phi_{r-1}(\boldsymbol{a}_r; \boldsymbol{b}_s, \boldsymbol{0}; q, x)$

${}_{s+1}\phi_s(\boldsymbol{a}_r, \boldsymbol{0}; \boldsymbol{b}_s; q, x)$

${}_r\phi_s(\boldsymbol{a}_r; \boldsymbol{b}_s; q, x)$ $(s + 1 > r)$ (not complete)

Second order q-hypergeometric equations : done [Zhang, Morita, Ohyama]

${}_2\phi_1(a, 0; c; q, x)$, ${}_2\phi_1(a, b; 0; q, x)$, ${}_1\phi_1(a; c; q, x)$ (q-confluent)

${}_2\phi_1(0, 0; c; q, x)$ (q-Bessel $J_\nu^{(1)}$), ${}_0\phi_1(-; c; q, x)$ (q-Bessel $J_\nu^{(2)}$)

${}_1\phi_1(0; c; q, x)$ (q-Bessel $J_\nu^{(3)}$), ${}_1\phi_1(a; 0; q, x)$ (q-Hermite-Weber)

${}_2\phi_0(a, b; -; q, x)$, ${}_2\phi_0(a, 0; -; q, x)$, ${}_2\phi_0(0, 0; -; q, x)$ (q-Airy)

2.2 From hypergeometric to Painlevé

Monodromy/connection/Stokes data are

determined exactly for **rigid systems**.

rigid systems : no accessary parameters

Ex. hypergeometric, Simpson's even/odd system

Application to global analysis on q-Painlevé equations

For Painlevé, the Lax pair has accessary parameters (two in Painlevé)
We **cannot determine connection data** exactly.
But we can determine the space of connection

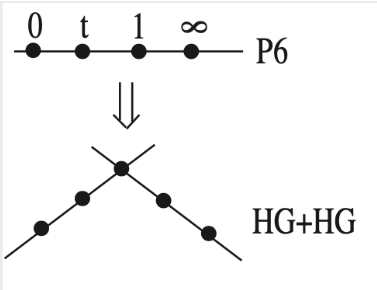
$\mathcal{RM}$: Moduli of connections $\to$ monodromy space
(Okamoto IVS) (Fricke cubic for PVI)

The connection space of q-Painlevé VI : Ramis, Sauloy, O
[My talk July at this seminar]

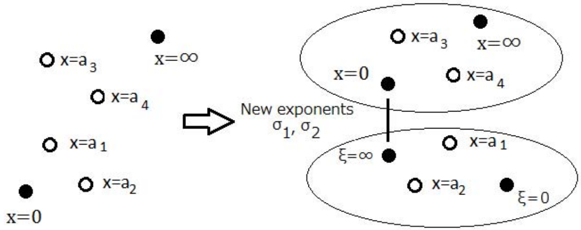
In order to study other q-Painlevé, we need **q-confluent cases**

2.3. Hypergeometric connection to Painlvé connection

Differential P6



q-P6



Painlevé $(2 \times 2$ matrix$)$

$$\frac{dY}{dx} = \left[\frac{A_0}{x} + \frac{A_t}{x-t} + \frac{A_1}{x-1}\right] Y(x)$$

$$Y(xq) = \left[A_0 + A_1 x + A_2 x^2\right] Y(x)$$

$$\det A(x) = c \prod_{j=1}^{4} (x - a_j)$$

Hypergeometric

$$\frac{dY}{dx} = \left[\frac{A_0 + A_t}{x} + \frac{A_1}{x-1}\right] Y(x)$$

$$Y(xq) = x\left[\Lambda + A_2 x\right] Y(x)$$

$$\frac{dY}{d\xi} = \left[\frac{A_0}{\xi} + \frac{A_t + A_1}{\xi - 1}\right] Y(\xi)$$

$$Y(\xi q) = \left[A_0' + \Lambda \xi\right] Y(\xi)$$

3. Single q-difference equation

A q-difference linear equation with polynomial coefficients:

$$\sum_{j=0}^{n} a_j(x)u(q^j x) = 0$$

When $a_0(0)a_n(0) \neq 0$, $x = 0$ is called a **regular singular point**.

characteristic equation at the origin

$$a_n(0)\lambda^n + a_{n-1}(0)\lambda^{n-1} + \cdots + a_1(0)\lambda + a_0(0) = 0$$

Let $\lambda_j = q^{c_j}$ be a solution of the characteristic equation.

Proposition (Adams 1929 [A]) The q-difference equation has solutions represented by convergent power series:

$$u_j = x^{c_j} \sum_{n=0}^{\infty} u_{j,k} x^k \quad (j = 1, ..., n),$$

when $c_j - c_m \notin \mathbb{Z}$.

3.1 Basic notations

0) **q-shifted factorial**:

$$(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{n} (a_i; q)_n, \quad (a; q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a).$$

1) **q-shift operator**

$$\sigma_q[f(x)] = f(xq)$$

2) **generalized q-hypergeometric series**:

$$\begin{aligned} &{}_r\varphi_s\left(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z\right) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n. \end{aligned}$$

3) **bilateral q-hypergeometric series**:

$$\begin{aligned} &{}_r\psi_s\left(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z\right) \\ &\qquad = \sum_{n \in \mathbb{Z}} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n. \end{aligned}$$

3) **Theta function**:

$$\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_\infty.$$

$$e_a(x) := \frac{\theta(x)}{\theta(ax)}, \text{ for } a \in \mathbb{C}^\times.$$

$$x\sigma_q[\theta_q(x)] = \theta_q(x), \quad \sigma_q[e_a(x)] = ae_a(x);$$

Remark. For $u_c(x) = x^c$, $\sigma_q[u_c(x)] = q^c u_c(x)$.
$u_c(x)$ and $e_a(x)$ satisfy the same q-difference equation if $q^c = a$.

We take $e_a(x)$ instead of x^c.

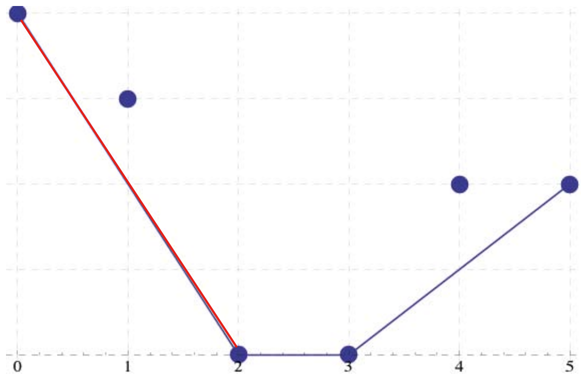
Since $e_a(x)$ is **single-valued**, all solutions are also single-valued.
Therefore the connection matrix $P(x)$ is elliptic on $\mathbb{C}^\times/q^{\mathbb{Z}}$:

$$Y_\infty(x) = Y_0(x)P(x)$$

$$P(xq) = P(x), \qquad P(xe^{2\pi i}) = P(x)$$

The Newton-Puiseux diagram of a linear q-difference equation at the origin is a **lower convex hull** of

$$\{(j, \mathrm{ord}\ a_j(x)) \in \mathbb{R}^2 \,|\, 0 \leq j \leq n\}$$



segment : a line jointed with $(j, \mathrm{ord}\ a_j(x))$ and $(k, \mathrm{ord}\ a_k(x))$ $(j < k)$
slope of s segment:

$$\mu = \frac{\mathrm{ord}\ a_k(x) - \mathrm{ord}\ a_j(x)}{k - j}$$

length of s segment: $m = k - j$

General Theorem (Adams [A])

1) If partial characteristic exponents λ_j are non-resonant for any **segment with the slope μ and the length m**, the q-difference equation has m formal solutions

$$u_j(x) = \theta(x)^{\mu} \frac{\theta_q(x)}{\theta_q(\lambda_j x)} \sum_{k \geq 0} u_k x^k,$$

for each segment

2) When the slope of a segment is not integer $\mu = r/s$ ($s > 0$), formal solutions are given by formal power series of $x^{1/s}$.

3) When the segment contains $(0, \operatorname{ord} a_0(x))$, the power series are convergent.

Remark

Around $x = \infty$ the **Newton-Puiseux diagram** is defined by the upper convex hull of

$$\{(j, \deg a_j(x)) \in \mathbb{R}^2 \,|\, 0 \leq j \leq n\}$$

When the segment contains $(n, \deg a_n(x))$, the power series are convergent.

3.4 Second order Hypergeometric series

$${}_2\phi_1\binom{a\ \ b}{c}$$

q-confluent: ${}_2\phi_1\binom{a\ \ 0}{c} \sim {}_1\phi_1\binom{a}{c}$ ${}_2\phi_1\binom{a\ \ b}{0}$ ${}_2\phi_0\binom{a\ \ b}{-}$

divergent: ${}_2\phi_0\binom{a\ \ b}{-}$, ${}_2\phi_0\binom{a\ \ 0}{-}$, ${}_2\phi_0\binom{0\ \ 0}{-}$

First order: ${}_1\phi_0\binom{a}{-}$ (q-binomial), ${}_1\phi_0\binom{0}{-}$ ($\exp_q$), ${}_0\phi_0\binom{-}{-}$ (Exp_q)

Jackson[(1)] ${}_2\phi_1\binom{0\ \ 0}{c} \sim$ Jackson[(2)] ${}_0\phi_1\binom{-}{c}$

Hahn-Exton ${}_1\phi_1\binom{0}{c}$ $\cup$

q-Hermite: ${}_1\phi_1\binom{a}{0}$ ${}_2\phi_0\binom{a\ \ 0}{-}$

q-Airy: ${}_1\phi_1\binom{0}{-q}$ **Ramanujan:** ${}_0\phi_1\binom{-}{0}$ ${}_2\phi_0\binom{0\ \ 0}{-}$

$${}_2F_1\binom{a,\ b}{c}$$

Kummer: ${}_1F_1\binom{a}{c}$ ${}_2F_0\binom{a,\ b}{-}$

Bessel ${}_0F_1\binom{-}{c}$ Weber ${}_1F_1\binom{a}{1/2}$

Airy ${}_0F_1\binom{-}{2/3}$

First Order

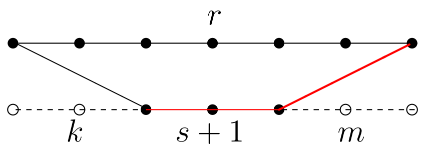
$${}_1F_0\left(\begin{matrix}a\\ -\end{matrix};x\right) = (1-x)^{-a}$$

$${}_0F_0\left(\begin{matrix}-\\ -\end{matrix};x\right) = \exp x$$

4. Basic hypergeometric equations (with Zhang)

$0 \leqq s+1 \leqq r$, $k+s+1 \leqq r$; $k+s+1+m=r$. $b_0 = q$, $a_j \neq 0$:

$$\left[x \prod_{j=1}^{r}(1-a_j\sigma_q) - \left(-\frac{\sigma_q}{q}\right)^k \prod_{j=0}^{s}\left(1-\frac{b_j}{q}\sigma_q\right)\right] y(x) = 0,$$



I) Around $x=\infty$, we have r convergent hypergeometric solutions

II) Around $x=0$, we have

(1) k convergent non-hypergeometric solutions with ramification

(2) $s+1$ divergent hypergeometric solutions

(3) m divergent non-hypergeometric solutions with ramification

The cases $k=0$, $s=-1$ or $m=0$ may appear.

If $k=0$, there exist $s+1$ convergent solutions around $x=0$.

If $k=0$ and $s=-1$, there exist $m(=r)$ convergent solutions around $x=0$.

Local solutions around the infinity $(j = 1, \dots, r)$

$$y_{j,\infty}(x) = \frac{\theta(-a_j x)}{\theta(-x)} {}_{k+s+1}\phi_{r-1}\left(\begin{matrix} qa_j/b_0, qa_j/b_1, \dots, qa_j/b_s, \mathbf{0}_k \\ qa_j/a_1, \dots, qa_j/a_{j-1}, qa_j/a_{j+1}, \dots, qa_j/a_r, \end{matrix}; q, \frac{q^m a_j^m b_0 b_1 \cdots b_s}{a_1 a_2 \cdots a_r x}\right)$$

Ramified convergent solutions at the origin $(j = 1, \dots, k)$

$x = z^k$ and $q = {p_0}^k$. $\omega^k = 1$.

$$y_{j,a}(z) = \frac{1}{\theta_{p_0}(-{p_0}^{(m+1)/2}\omega^j z)} w_j(z), \;\; w_j(z) = \sum_{n=0}^{\infty} w_n^{(k)} z^n.$$

Formal divergent solutions at the origin $(j = 0, \dots, s)$

$$y_{j,b}(x) = \frac{\theta(-x)}{\theta(-qx/b_j)} {}_r\phi_{s+m}\left(\begin{matrix} qa_1/b_j, qa_2/b_j, \dots, qa_r/b_j \\ qb_1/b_j, \dots, qb_{j-1}/b_j, qb_{j+1}/b_j, \dots, qb_s/b_j, \mathbf{0}_m \end{matrix}; q, x\right)$$

Ramified divergent solutions at the origin $(j = 1, \dots, m)$

$x = z^m$ and $q = p^m$. $\omega^m = 1$.

$$y_{j,c}(z) = \theta_p(-cp^{(m+1)/2}\omega^j z) u_j(z), \;\; v_j(z) = \sum_{n=0}^{\infty} u_n^{(k)} z^n.$$

4.2. Transformations

0) **Ramification**: $x = z^n, \quad q = p^n$.

1) Gauge transform by θ-functions:

$$x^m \sigma_q^n \left[\theta(cx)^{\pm 1} f(x)\right] = c^{\mp n} q^{\mp n(n-1)/2} x^{m \mp \mathbf{n}} \theta(cx)^{\pm 1} \sigma_q^n f(x).$$

2) q-Borel transformation $\mathcal{B}_q^{\pm} : \mathbb{C}[[t]] \to \mathbb{C}[[\tau]]$:

$$\mathcal{B}_q^{\pm}\left[\sum_{n=0}^{\infty} a_n t^n\right] := \sum_{n=0}^{\infty} a_n q^{\pm n(n-1)/2} \tau^n.$$

The q-Borel transformation $\mathcal{B}_q^{\pm}$ satisfies

$$\mathcal{B}_q^{\pm}(t^m \sigma_q^n f) = q^{\pm m(m-1)/2} \tau^m \sigma_q^{n \pm \mathbf{m}} \mathcal{B}_q^{\pm}(f).$$

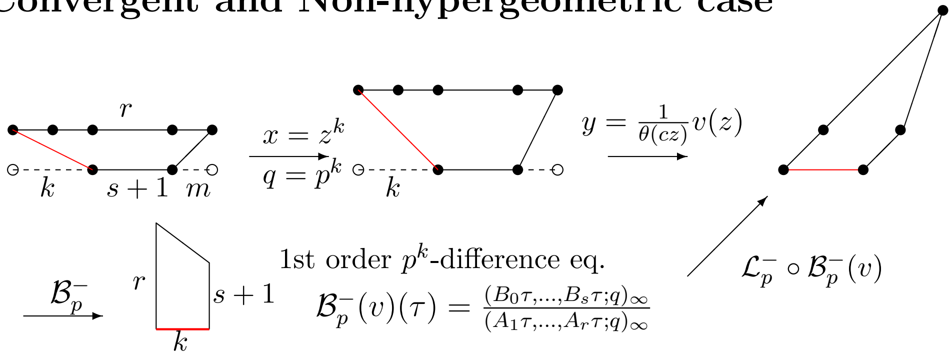
3) The q-Laplace transform $\mathcal{L}_{q;1}^{[\lambda]}$ (**a formal inverse of** $\mathcal{B}_q^{+}$)

$$\mathcal{L}_{q;1}^{[\lambda]}(\varphi)(x) = \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\tau)}{\theta_q(\tau/x)} \frac{d_q \tau}{\tau} = \sum_{n \in \mathbb{Z}} \frac{\varphi(q^n \lambda)}{\theta_q(q^n \lambda / x)}.$$

3') The q-Laplace transform $\mathcal{L}_q^{-}$ (**an inverse of** $\mathcal{B}_q^{-}$)

$$\mathcal{L}_q^{-}(\varphi)(x) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \varphi(\tau) \theta_p(x/\tau) \frac{d\tau}{\tau}$$

5. Convergent and Non-hypergeometric case



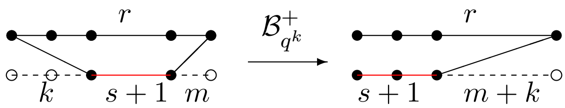
Lemma 1. $m > 0$, $p^m = q$. *We assume that* $1 + s + m \leqq r$.

$$I = \int_{|\tau|=\varepsilon} \frac{\prod_{j=0}^{s}(B_j\tau;q)_\infty}{\prod_{k=1}^{r}(A_k\tau;q)_\infty}\theta_p(z/\tau)\,\frac{d\tau}{\tau}$$

$$= \frac{(B_0/A_1,...,B_s/A_1;q)_\infty}{(q,A_2/A_1,...A_r/A_1;q)_\infty}\theta_p(A_1 z)$$

$$\times\ {}_{1+s+m}\phi_{r-1}\left(\begin{matrix} qA_1/B_0,...,qA_1/B_s,\mathbf{0}_m \\ qA_1/A_2,...,qA_1/A_r\end{matrix};q,\frac{(-1)^r q^{r-s-(1+m)/2}B_0\cdots B_s}{A_1^{m-r+1}A_2\cdots A_r z^m}\right)$$

$$+\ \mathrm{idem}(A_1;A_2,...,A_r).$$



The case $k = 0$ is shown by **Watson** 1910.

Theorem 2 (Watson)**.** *We assume that* $0 \leqq s < r$. *Then*

$$
{}_r\phi_{r-1}\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s, 0, \ldots, 0 \end{matrix}; q, z\right) = \frac{(a_2, \ldots, a_{r+1}, b_s/a_1, \ldots b_s/a_1; q)_\infty}{(b_1, \ldots, b_s, a_2/a_1, \ldots, a_r/a_1; q)_\infty}
$$

$$
\times \frac{\theta_q(-a_1 z)}{\theta_q(-z)} {}_{s+1}\phi_{r-1}\left(\begin{matrix} a_1, qa_1/b_1, \ldots, qa_1/b_s \\ qa_1/a_2, \ldots, qa_1/a_r \end{matrix}; q, (a_1 q)^{r-s-1} \frac{qb_1 \cdots b_s}{a_1 \cdots a_r z}\right)
$$

$$
+\mathrm{idem}(a_1; a_2, \ldots, a_r).
$$

The right handside is convergent when $|z| < 1$ *and RHS is convergent when* $s < r - 1$ *or* $s = r - 1$ *and* $|qb_1 \cdots b_s/a_1 \cdots a_r z| < 1$.

6.1 q^k-Laplace transform

We can apply q^k-**Laplace transform** to Watson's connection formula:

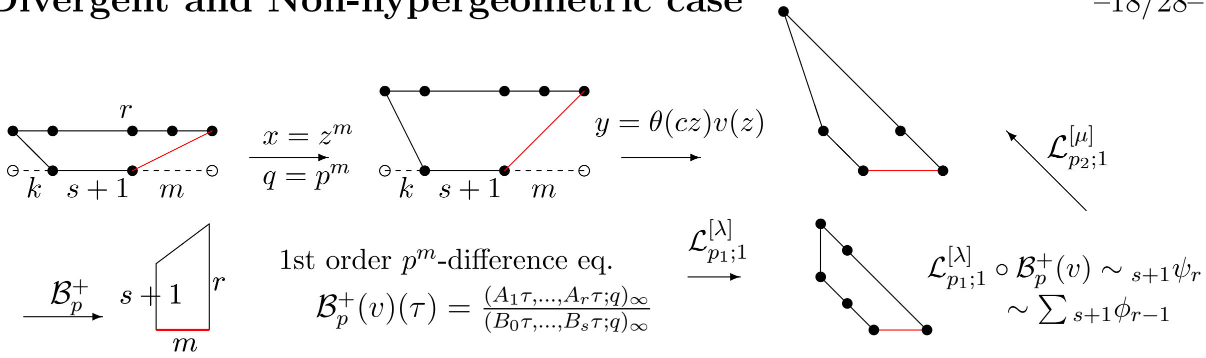
Lemma 3. *We take a positive integer m. We consider*

$$\varphi(\xi) = \frac{\theta_q(a\xi)}{\theta_q(b\xi)} \sum_{n \geq 0} c_n \xi^{-n}.$$

Then we obtain

$$\mathcal{L}^{[\lambda]}_{q^k;1}\varphi(x) = \frac{\theta_q(a\lambda)\theta_{q^k}(q^k a^k x/b^k \lambda)}{\theta_q(b\lambda)\theta_{q^k}(q^k x/\lambda)} \sum_{n \geq 0} c_n (q^k)^{-\frac{n(n-1)}{2}} \left(\frac{b^k}{a^k q^k x}\right)^n.$$

7. Divergent and Non-hypergeometric case



We set $\varphi(\tau) = \mathcal{B}^+_p(v)(\tau)$.

$$\varphi(\tau) = \frac{(A_1\tau, ..., A_r\tau; q)_\infty}{(B_0\tau, ..., B_s\tau; q)_\infty}$$

We set $p_1 = p^{m/m+k} = q^{1/m+k}$. $(m + k = r - s - 1)$

$$\mathcal{L}^{[\lambda]}_{p_1;1}(\phi)(\xi) \sim \frac{(A_1\lambda, \ldots, A_r\lambda; q)_\infty}{(B_0\lambda, \ldots, B_s\lambda; q)_\infty} {}_{s+1}\psi_r \left[\begin{matrix} B_0\lambda, \ldots, B_s\lambda \\ A_1\lambda, \ldots, A_r\lambda \end{matrix} ; q, \frac{c}{\xi^{m+k}} \right]$$

Theorem 4 (Slater [Sl]). *We set* $d = a_1 a_2 \dots a_r / c_1 c_2 \dots c_r$. *For* $\left|\dfrac{b_1 b_2 \cdots b_s}{a_1 a_2 \cdots a_r}\right| < |z| < 1$, *we have*

$$\frac{(b_1, b_2, \dots, b_r, q/a_1, q/a_2, \dots, q/a_r, dz, q/dz; q)_\infty}{(c_1, c_2, \dots, c_r, q/c_1, q/c_2, \dots, q/c_r; q)_\infty} {}_r\psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right]$$
$$= \frac{q}{c_1} \frac{(c_1/a_1, c_1/a_2, \dots, c_1/a_r, qb_1/c_1, qb_2/c_1, \dots, qb_r/c_1, dc_1 z/q, q^2/dc_1 z; q)_\infty}{(c_1, q/c_1, c_1/c_2, \dots, c_1/c_r, qc_2/c_1, \dots qc_r/c_1; q)_\infty}$$
$$\times {}_r\psi_r \left[\begin{matrix} qa_1/c_1, qa_2/c_1, \dots, qa_r/c_1 \\ qb_1/c_1, qb_2/c_1, \dots, qb_r/c_1 \end{matrix}; q, z\right] + \mathrm{idem}(c_1; c_2, \dots, c_r).$$

Theorem 5. *We set* $d' = a_1 a_2 \dots a_s / b_1 b_2 \dots b_r$. *When* $s < r$, *we have*

$$\frac{(q/a_1, q/a_2, \dots, q/a_s; q)_\infty}{(q/b_1, q/b_2, \dots, q/b_r; q)_\infty} {}_s\psi_r \left[\begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_r \end{matrix}; q, x\right]$$
$$= \frac{q}{b_1} \frac{(b_1/a_1, b_1/a_2, \dots, b_1/a_s, q; q)_\infty}{(b_1, q/b_1, b_1/b_2, \dots, b_1/b_r; q)_\infty} \cdot \frac{\theta(-d' b_1 x/q)}{\theta(-d' x)}$$
$$\times {}_s\phi_{r-1} \left[\begin{matrix} qa_1/b_1, qa_2/b_1, \dots, qa_s/b_1 \\ qb_2/b_1, \dots, qb_r/b_1 \end{matrix}; q, \left(\frac{q}{b_1}\right)^{r-s} x\right] + \mathrm{idem}(b_1; b_2, \dots, b_r),$$

7.2 q-Laplace transform to series on x^m

We take a positive integer m. We consider

$$\varphi(\xi) = \frac{\theta_q(a\xi)}{\prod_{j=1}^m \theta_q(b_j\xi)} \sum_{n\geq 0} c_n \xi^{-mn}.$$

Then we obtain

$$\mathcal{L}_{q;1}^{[\lambda]}\varphi(x) = \frac{\theta_q(a\lambda)\theta_{q^m}(q^m ax/\lambda^m \prod_{j=1}^m b_j)}{\theta_q(qx/\lambda)\prod_{j=1}^m \theta_q(b_j\lambda)} \sum_{n\geq 0} c_n (q^m)^{-\frac{n(n-1)}{2}} \left(\frac{\prod_{j=1}^m b_j}{aq^m x}\right)^n.$$

We set $p_2 = p^{k/m+k}$. $p_1 \cdot p_2 = p$.

We apply p_2-Laplace transformation to ${}_s\phi_{r-1}(1/x^{m+k})$, then we have a power series on x^m.

8. q-Painlevé equations: q-P_{VI}, q-P_{V}

$y=y(t), z=z(t),\ \bar{y}=y(qt), \bar{z}=z(qt).$

$a_1,a_2,a_3,a_4,b_1,\ b_2,\ b_3,\ b_4$: parameter

$$P(A_3)\sim q\text{-}P_{\mathrm{VI}}:\ \frac{y\bar{y}}{a_3a_4}=\frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{(\bar{z}-b_3)(\bar{z}-b_4)},\quad \frac{z\bar{z}}{b_3b_4}=\frac{(y-a_1t)(y-a_2t)}{(y-a_3)(y-a_4)},$$

$$\frac{b_1b_2}{b_3b_4}=q\frac{a_1a_2}{a_3a_4}.$$

$$P(A_4)\sim q\text{-}P_{\mathrm{V}}:\ \frac{y\bar{y}}{a_3a_4}=-\frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{\bar{z}-b_3},\quad \frac{z\bar{z}}{b_3}=-\frac{(y-a_1t)(y-a_2t)}{a_4(y-a_3)},$$

$$\frac{b_1b_2}{b_3}=q\frac{a_1a_2}{a_3a_4}.$$

$$P(A_5)^\sharp\sim q\text{-}P_{\mathrm{III}}(D_6):\ \frac{y\bar{y}}{a_3a_4}=-\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3},\ \frac{z\bar{z}}{b_3}=-\frac{y(y-a_1t)}{a_4(y-a_3)},\ \frac{b_1b_2}{b_3}=q\frac{a_1}{a_3a_4}.$$

$$P(A_6)^\sharp\sim q\text{-}P_{\mathrm{III}}(D_7):\ \frac{y\bar{y}}{a_3a_4}=-\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3},\ \frac{z\bar{z}}{b_3}=-\frac{y^2}{a_4(y-a_3)},\ \frac{b_1b_2}{b_3}=q\frac{1}{a_3a_4}.$$

9. Connection Preserving Deformation

by Jimbo-Sakai, $P(A_3)$, by M. Murata (other) [Mu]

$$Y(qx,t) = A(x,t)Y(x,t),$$
$$Y(x,qt) = B(x,t)Y(x,t).$$

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2,$$
$$B(x,t) = f(x,t)(xI + B_0(t)),$$

$$f(x,t) = \begin{cases} \dfrac{x}{(x-a_1qt)(x-a_2qt)} & P(A_3), P(A_4) \\ \dfrac{1}{x-a_1qt} & P(A_5)^\sharp \\ \dfrac{1}{x} & P(A_6)^\sharp \end{cases}$$

The **compatibility condition** leads to q-Painlevé equations

$$A(x,qt)B(x,t) = B(qx,t)A(x,t).$$

10. Connection formula

1. The first limit

We take a limit $t \to 0$. Then $A(x,t)$ goes to $x\mathbf{\Lambda} + x^2 A_2$

2. The second limit

We set $x = \xi t$. We take a new connection:

$$\tilde{A}(\xi, t) = t^{-1} t^{-\log_q \Lambda} A(t\xi, t) t^{\log_q \Lambda}$$

We take a limit $t \to 0$. Then $\tilde{A}(\xi, t)$ goes to $M + \xi\mathbf{\Lambda}$. $M \sim A_0/t$

The limit equations

$$Y_1(xq) = [x(\Lambda + xA_2)]Y_1(x),\ Y_2(\xi q) = (M + \xi\Lambda)Y_2(\xi)$$

are reduced to hyperegeometric.

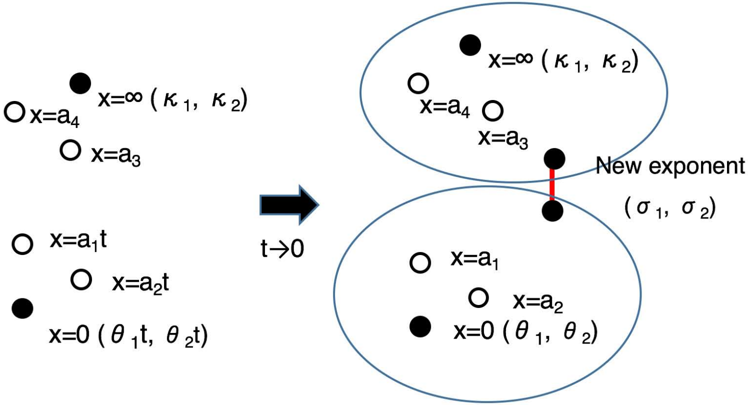
We set eigenvalues of Λ as σ_1, σ_2. (assume that $\sigma_1\sigma_2 \neq 0$)

$$C^{-1}\Lambda C = \mathrm{diag}(\sigma_1, \sigma_2).$$

We write $D = \mathrm{diag}(\log_q \sigma_1, \log_q \sigma_2)$. Then $t^{\log_q \Lambda} = C^{-1} t^D C$.

11. Naive proof

In $P(A_3)$ case:



$$\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4).$$

$$\to \det A_1(x,t) = \kappa_1\kappa_2 x^2(x-a_3)(x-a_4)$$

$$\to \det A_2(x,t) = \sigma_1\sigma_2(x-a_1)(x-a_2)$$

Theorem 6.

$$P(x) = P_2(x/t)P_1(x)$$

12. Higher order case

We consider

$$Y(xq,t) = A(x,t)Y(x,t)$$

$$A(x,t) = A_\infty + \frac{A_1}{1 - x/c_1 t} + \frac{A_2}{1 - x/c_2}$$

$$A_\infty = \begin{bmatrix} \kappa_1 & a_{12} & a_{23} \\ 0 & \kappa_2 & a_{23} \\ 0 & 0 & \kappa_3 \end{bmatrix}, \ A_j = \begin{bmatrix} 1 \\ p_2^j \\ p_3^j \end{bmatrix} \begin{bmatrix} q_1^j & q_2^j & q_3^j \end{bmatrix} \ (j = 1, 2)$$

$$A_0 = \begin{bmatrix} \theta_1 & 0 & 0 \\ a_{21} & \theta_2 & 0 \\ a_{31} & a_{32} & \theta_3 \end{bmatrix} = A_\infty + A_1 + A_2$$

$$\det A(x,t) = \kappa_1 \kappa_2 \kappa_3 \frac{(x - c_3 t)(x - c_4)}{(x - c_1 t)(x - c_2)}.$$

q-Fuchs' relation:

$$c_1 c_2 \theta_1 \theta_2 \theta_3 = \kappa_1 \kappa_2 \kappa_3 c_3 c_4$$

Unknown functions of the equations

$$a_{ij}\ (i \neq j)\ [6], \quad p_i^j, q_2^j\ [10]$$

The parameter of equation [9]

$$\theta_1, \theta_2, \theta_3; \kappa_1, \kappa_2, \kappa_3; c_1, c_2, c_3, c_4$$

c_3, c_4 are new parameters [2]

The condition of A_0 [9]

Up to the gauge transformation [2]

$$A(x,t) \to G^{-1}A(x,t)G, \quad G: \text{diagonal}$$

The number of accessary parameters:

$$16 - 9 - 2 - 2 + 1 = 4$$

Deformation equation :

$$Y(x,tq) = B(x,t)Y(x,t)$$

$$B(x,t) = \frac{x(x^2I + xB_1 + B_2)}{(x-c_1t)^2(x-c_3t)}$$

The compatibility condition gives a q-analogue of Fuji-Suzuki system.
The spectral type is

$$(1^3; 1^3; 2, 1, 2, 1)$$

It is easy to generalize higher rank cases.

We can generalize degenerated (confluent) cases.

The **connection matrix** is a product of connection matrix of ${}_3\phi_2$

Question
Is this system is equivalent to a q-analogue of the Drinfeld-Sokolov hierarchy of type A? [Su]

13. Summary

- We can determine the connection matrix of basic hypergeometric equations of **the second order**.
- We can determine the connection matrix of basic hypergeometric equations of **any order** when **one singular point is regular**
- For q-Painlevé equations, linear connection problems are solved for $P(A_3)$, $P(A_4), P(A_5)^\sharp, P(A_6)^\sharp$, which correspond to q-P_{VI}, q-P_{V}, q-$P_{\mathrm{III}}(D_6)$, q-$P_{\mathrm{III}}(D_7)$. The connection matrix is a product of a connection matrix of basic hypergeometric functions:

  $P(A_3)$: Heine $\times$ Heine, $P(A_4)$: Heine $\times$ q-Kummer,

  $P(A_5)^\sharp$: q-Kum. $\times$ q-Kum., $P(A_5)^\sharp$: q-Kum. $\times$ Hahn-Exton

Future problems

- How about other q-Painlevé equations? (nonlinear irregular)
- Elliptic Painlevé case?
- The space of connection of other q-Painlevé equations?

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