WKB Filtrations and the Singularly Perturbed Riccati Equation

based on arXiv:1909.04011, arXiv:2008.06492 and work in progress

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- Consider a meromorphic $\hbar$-connection $(\mathcal{E}, \nabla)$ on a compact Riemann surface X
- Let $(E, \phi) := \lim_{\hbar \to 0} (\mathcal{E}, \nabla)$, the limiting meromorphic Higgs bundle on X
- Generically and locally, get eigendecomposition $(E, \phi) = \bigoplus (L_i, \eta_i)$

Theorem [N]: WKB Filtrations (rough statement)

Generically (and at least if $\operatorname{rank}(\mathcal{E}) = 2$), the vector bundle $\mathcal{E}$ has a canonical ∇-invariant piecewise filtration $\mathcal{E}^\bullet = (\mathcal{E}^1 \subset \mathcal{E}^2 \subset \cdots \subset \mathcal{E})$ such that

$$\lim_{\hbar \to 0} \left(\mathrm{gr}\, \mathcal{E}^\bullet, \mathrm{gr}\, \nabla \right) \xrightarrow{\sim} \bigoplus (L_i, \eta_i) .$$

- Generically, **WKB filtrations are transverse**, yielding piecewise decompositions

$$(\mathcal{E}, \nabla) \xrightarrow{\sim} \bigoplus (\mathcal{L}_i, \partial_i) \qquad \text{with the property} \qquad \lim_{\hbar \to 0} (\mathcal{L}_i, \partial_i) \xrightarrow{\sim} (L_i, \eta_i) .$$

- Generically, **WKB filtrations restrict to Levelt filtrations** for fixed nonzero $\hbar$.
- **Important special case (the Schrödinger equation)**: if $(\mathcal{E}, \nabla)$ is $\mathfrak{sl}_2$-$\hbar$-oper, (that is, $\mathcal{E}$ is 1-jet of anticanonical line bundle, ∇ is 1-jet of $\hbar^2 \partial_x + Q(x, \hbar)$) then $\mathcal{E}^1 \subset \mathcal{E}$ is generated by 1-jet of the exact WKB solutions.
- Construction of **WKB filtrations is a generalisation of the exact WKB analysis** from $\mathfrak{sl}_2$-$\hbar$-opers to general $\hbar$-connections of rank 2 [*rank n is work in progress*].

Consider a singularly perturbed linear differential system:

$$\nabla := \hbar\,\mathrm{d} + A(x,\hbar)\,\mathrm{d}x \qquad \text{i.e.} \qquad \nabla_{\partial_x}\psi = \left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + A(x,\hbar)\right)\psi(x,\hbar) = 0 \;,$$

where $A(x,\hbar) = n\times n$ matrix, meromorphic in x and holomorphic* at $\hbar = 0$.

- We say ∇ is ***filtered*** if it is gauge equivalent to a triangular system:

$$\nabla \simeq \tilde{\nabla} = \hbar\,\mathrm{d} + \tilde{A}\,\mathrm{d}x = \hbar\,\mathrm{d} + \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\mathrm{d}x \;.$$

- A filtration on ∇ is induced by the standard canonical filtration on $\tilde{\nabla}$:

$$\left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}\right\rangle \subset \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\rangle \subset \cdots \subset \underline{\mathbb{C}^n}$$

- This filtration is ***canonical*** if it is independent of x and preserved by automorphisms.
- **Difficulty:** gauge transformations $\nabla \mapsto \tilde{\nabla}$ are generically <u>not</u> holomorphic at $\hbar = 0$.

Proposition [N] (roughly stated):

At least for $n = 2$, there is a large class of systems ∇ which are canonically filtered via gauge transformations that admit asymptotic expansions as $\hbar \to 0$ in a halfplane.

Filtering a System. Step 1: diagonalise the leading-order

- Put $n = 2$, and restrict $(x, \hbar)$ to some $\mathsf{U} \times \mathbb{D} \subset \mathbb{C}^2_{x\hbar}$ where A is holomorphic.
- In fact, assume $\mathbb{D}$ = infinitesimal disc, so $A \in \mathrm{Mat}_2\big(\mathcal{O}_{\mathsf{U}}\,\{\hbar\}\,\big)$.

- **Standard fact:** generically, $A_0(x) := A(x, 0)$ is diagonalisable via holomorphic gauge transformations, locally away from ***turning points*** := zeros of the discriminant Δ_0.

Assumption 0: no turning points

$\mathsf{U} \subset \mathbb{C}_x$ contains no turning points and supports a univalued branch of $\sqrt{\Delta_0}$.

- Then A_0 is holomorphically equivalent over U to a diagonal matrix of its eigenvalues:

$$P A_0 P^{-1} = \Lambda_0 = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \qquad \begin{cases} \lambda_i = \lambda_i(x) & \in \mathcal{O}(\mathsf{U}), \\ P = P(x) & \in \mathsf{GL}_2\big(\mathcal{O}(\mathsf{U})\big) \end{cases}$$

- The assignment $\nabla \mapsto \Lambda_0 \, \mathrm{d}x$ is canonical up to permutation and coordinate change.
- Then P gives a holomorphically equivalent differential system

$$\nabla' := P \nabla P^{-1} = \hbar\,\mathrm{d} + \big(\Lambda_0 + \hbar B\big)\,\mathrm{d}x = \hbar\,\mathrm{d} + \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \mathrm{d}x + \hbar \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \mathrm{d}x \,.$$

- To put the system ∇' into a triangular form, we search for a gauge transformation $G = G(x, \hbar)$ in the following unipotent form:

$$G = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \qquad \text{where } s = s(x,\hbar) \text{ is to be solved for.}$$

- Transforming ∇' by G we get:

$$\tilde{\nabla} := G\nabla' G^{-1} = \hbar\,\mathrm{d} + \begin{bmatrix} \lambda_1 + \hbar(b_{11} - s b_{12}) & \hbar b_{12} \\ * & \lambda_2 + \hbar(b_{22} + s b_{12}) \end{bmatrix} \mathrm{d}x$$

where $* = -\hbar\partial_x s + (\lambda_1 - \lambda_2)s + \hbar\Big(- b_{12}s^2 + (b_{11} - b_{22})s + b_{21}\Big)$.

- Thus, $\tilde{\nabla}$ is upper-triangular $\Leftrightarrow$ s satisfies a ***singularly perturbed Riccati equation***:

$$\hbar\partial_x s = \sqrt{\Delta_0}\,s + \hbar\Big(a_2 s^2 + a_1 s + a_0\Big)$$

- Standard filtration $\left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle \subset \underline{\mathbb{C}^2}$ yields filtration $\left\langle P^{-1}G^{-1}\begin{bmatrix}1\\0\end{bmatrix} = P^{-1}\begin{bmatrix}1\\-s\end{bmatrix}\right\rangle \subset \underline{\mathbb{C}^2}$ on ∇ .
- Moreover, G admits asymptotics as $\hbar \to 0$ $\quad\Leftrightarrow\quad$ s admits asymptotics as $\hbar \to 0$.
- In particular, $s_0(x) := \lim s(x,\hbar)$ as $\hbar \to 0$ must exist and equals 0.
- Canonicity of this filtration is equivalent to finding a canonical solution s.

$$\hbar \partial_x s = \sqrt{\Delta_0} s + \hbar \big(a_2 s^2 + a_1 s + a_0 \big)$$

Upshot: this Riccati equation has a canonical *exact* solution s defined for $x \in \mathsf{U}$ and $\hbar \in \mathsf{S}$ a halfplane sectorial domain (or germ), provided U is the complete forward flow of a certain vector field and the a_0, a_1, a_2 are appropriately bounded along this flow.

- Consider the holomorphic vector field $L := \frac{1}{\sqrt{\Delta_0}} \partial_x$ on U.
- The ***real forward flow*** of L is the flow of $\mathrm{Re}(L)$ for positive time.
- Concretely, the flow line through $x_0 \in \mathsf{U}$ is given by $\quad \mathrm{Im} \displaystyle\int_{x_0}^{x} \sqrt{\Delta_0(t)}\, \mathrm{d}t = 0.$

Assumption 1: completeness

The forward flow of every point in U is complete; i.e., exists for all positive time.

Assumptions 2: regularity

The coefficients a_0, a_1, a_2 are bounded by $\sqrt{\Delta_0}$ along the forward flow.

Main Technical Lemma [N]: E&U of exact Riccati solutions

The Riccati equation has a unique holomorphic solution s on $\mathsf{U} \times \mathsf{S}$ which admits Gevrey asymptotics as $\hbar \to 0$ along $[-\pi/2, +\pi/2]$ with leading-order $s_0 = 0$.

Theorem [N]: Existence of local WKB Filtrations

Under assumptions 0,1,2 (no turning points, completeness, regularity):

1. The system ∇, restricted to $\mathsf{U} \times \mathsf{S}$, has a canonical filtration whose associated graded converges as $\hbar \to 0$ to the eigendecomposition of $\phi = A_0 \, \mathrm{d}x$.
2. If the U flows into a pole p of ∇, then for every fixed nonzero $\hbar \in \mathsf{S}$ close to the positive real direction, this local WKB filtration restricts to the Levelt filtration associated with p or with the corresponding (anti-)Stokes sector at p.
3. If U is complete for both forward and backward flows, then ∇ restricted to $\mathsf{U} \times \mathsf{S}$ has two such filtrations (one for each flow direction) which are transverse.
   $\implies$ canonical diagonalisation of ∇ over $\mathsf{U} \times \mathsf{S}$.

- Can prove a similar E&U theorem for a very general Riccati equation

$$\hbar\partial_x s = as^2 + bs + c$$

where a, b, c are holomorphic functions of $(x, \hbar) \in \mathsf{U} \times \mathsf{S}$ which only admit Gevrey asymptotics as $\hbar \to 0$ in a halfplane.

[*proof inspired by ideas of Koike-Schäfke*]

- $\implies$ existence of (local) WKB filtration for a much larger class of systems ∇ defined over $\mathsf{U} \times \mathsf{S}$ which only admit Gevrey asymptotics as $\hbar \to 0$ in a halfplane.

- $\implies$ obtain a general existence and uniqueness of exact WKB solutions and Borel summability of formal WKB solutions for
  1. a large class of 2^{nd}-order ODEs on $\mathsf{U} \times \mathsf{S}$ with Gevrey asymptotics as $\hbar \to 0$ in a halfplane:

  $$\hbar^2\partial_x^2\psi + p(x,\hbar)\hbar\partial_x\psi + q(x,\hbar)\psi = 0$$

  2. more invariantly, a large class of meromorphic singularly perturbed 2^{nd}-order differential operators on an arbitrary line bundle $\mathcal{L}$ over a Riemann surface.

- Let $(\mathsf{X}, \mathsf{D}) :=$ compact Riemann surface + effective divisor.
- **Working Definition:** An *$\hbar$-connection* on (X, D) is an $\hbar$-family $(\mathcal{E}, \nabla)$ of vector bundles $\mathcal{E}$ on U and morphisms

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \omega_{\mathsf{X}}(\mathsf{D})$$

  satisfying the $\hbar$-twisted Leibniz rule: $\nabla(fe) = f\nabla(e) + e \otimes \hbar\, \mathrm{d}f$.
- Then $(E, \phi) := \lim(\mathcal{E}, \nabla)$ as $\hbar \to 0$ is the limiting Higgs bundle.
- If $\operatorname{rank}(\mathcal{E}) = 2$, the Higgs field ϕ has characteristic polynomial

$$\chi_\phi(\eta) = \eta^2 - \operatorname{tr}(\phi)\eta + \det(\phi)$$

  whose discriminant $\Delta_\phi := \operatorname{tr}(\phi)^{\otimes 2} - 4\det(\phi)$ is a quadratic differential on X.
- In a local coordinate, $\Delta_\phi = \Delta_0(x)\, \mathrm{d}x^2$.
- Horizontal foliation of $\Delta_\phi \quad \leftrightarrow \quad$ real flow lines of $L = \frac{1}{\sqrt{\Delta_0}}\partial_x$.

- **General Fact about Quadratic Differentials:**
  If $|\mathsf{D}| \geqslant 1$ (or $|\mathsf{D}| \geqslant 3$ if $\mathsf{X} \cong \mathbb{P}^1$), the generic situation is:
  1. the horizontal foliation of Δ_ϕ covers $\mathsf{X} \setminus \{\text{turning points}\}$ by complete maximal forward and backward flow domains (*cells*).
  2. Double intersections of cells consist of *strips* that cover $\mathsf{X} \setminus \{\text{critical graph}\}$, and their flows are complete in both directions.

  If $\mathsf{D} = \varnothing$, consider Δ_ϕ Strebel $\implies$ $\mathsf{X} \setminus \{\text{critical graph}\}$ is decomposed into cylinders.

Theorem [N]: WKB Filtrations

Suppose $(\mathcal{E}, \nabla)$ is a rank-two $\hbar$-connection on (X, D) with Higgs field (E, ϕ) such that

1. the discriminant Δ_ϕ induces one of the above generic situations;
2. $\forall \mathsf{p} \in \mathsf{D}$, eigenvalues of ∇_p are bounded by corresponding eigenvalues of ϕ_p.

Then $(\mathcal{E}, \nabla)$, restricted to $\mathsf{X} \times \mathsf{S}$, has a canonical flat piecewise filtration $\mathcal{E}^\bullet$ over $\mathsf{X} \setminus \{\text{turning points}\}$, comprised of local WKB filtrations over all the cells or cylinders.

- Over each cell, as $\hbar \to 0$ in S, its associated graded converges (in a canonical way) to the local Higgs eigendecomposition: $\lim(\mathsf{gr}\,\mathcal{E}^\bullet, \mathsf{gr}\,\nabla) \cong (L_1 \oplus L_2, \eta_1 \oplus \eta_2)$.
- Over each strip or cylinder, $(\mathcal{E}, \nabla)$ has canonical decomposition $(\mathcal{L}_1, \partial_1) \oplus (\mathcal{L}_2, \partial_2)$, and
$$\lim_{\hbar \to 0} (\mathcal{L}_i, \partial_i) = (L_i, \eta_i)\ .$$

:) Thank you for your attention! :)