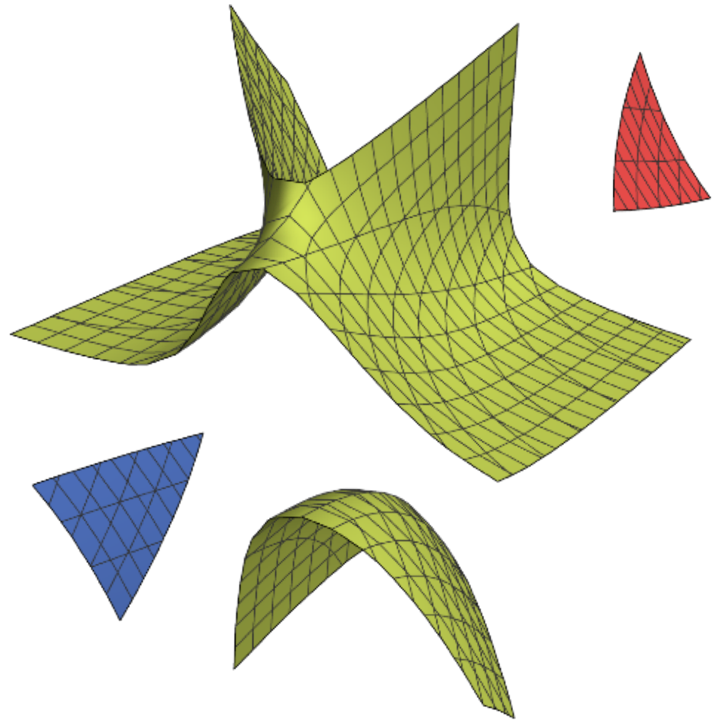
LEVERHULME TRUST

ICREA

Segre surfaces for the Painlevé equations

with Nalini Joshi & Pieter Roffelsen

ArXiv: 2405.10541



UNIVERSITY OF BIRMINGHAM

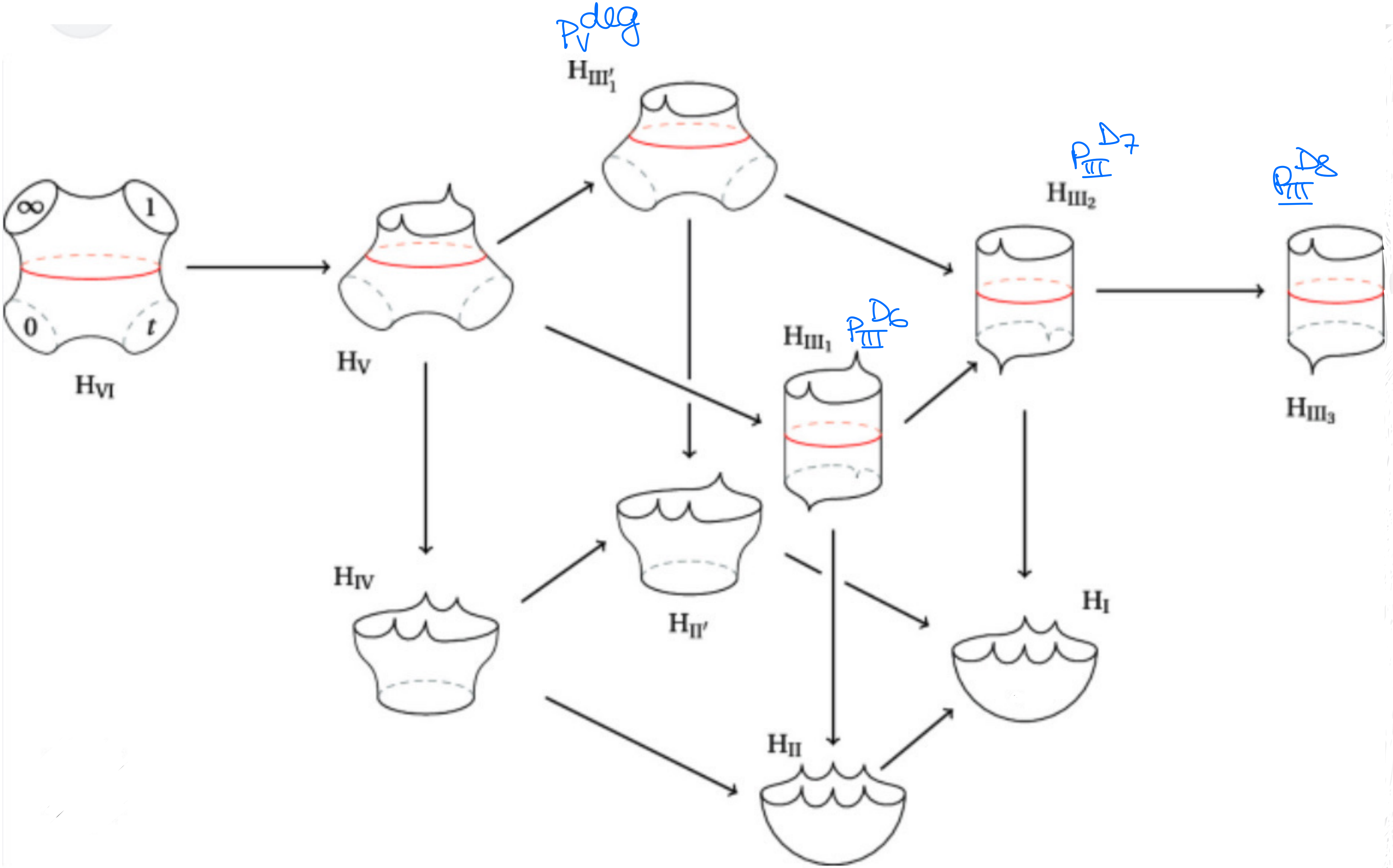
UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Monodromy manifolds of the Painlevé differential equations

P-eqs	Polynomials
PVI	$x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$
PV	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$
PV_{deg}	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_4$
PIV	$x_1x_2x_3 - x_1^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$
$PIII^{D_6}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_4$
$PIII^{D_7}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 - x_1^2 - x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2x_2 - x_3 + \omega_4$
PII^{FN}	$x_1x_2x_3 - x_1^2 + \omega_1x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

Saito van der Put

cusped character varieties



Chekhov, M.M, Rubtsov
IMRN 2016

Example PI

$x_1x_2x_3 - x_1 - x_2 - 1 = 0$

Contains 5 affine lines

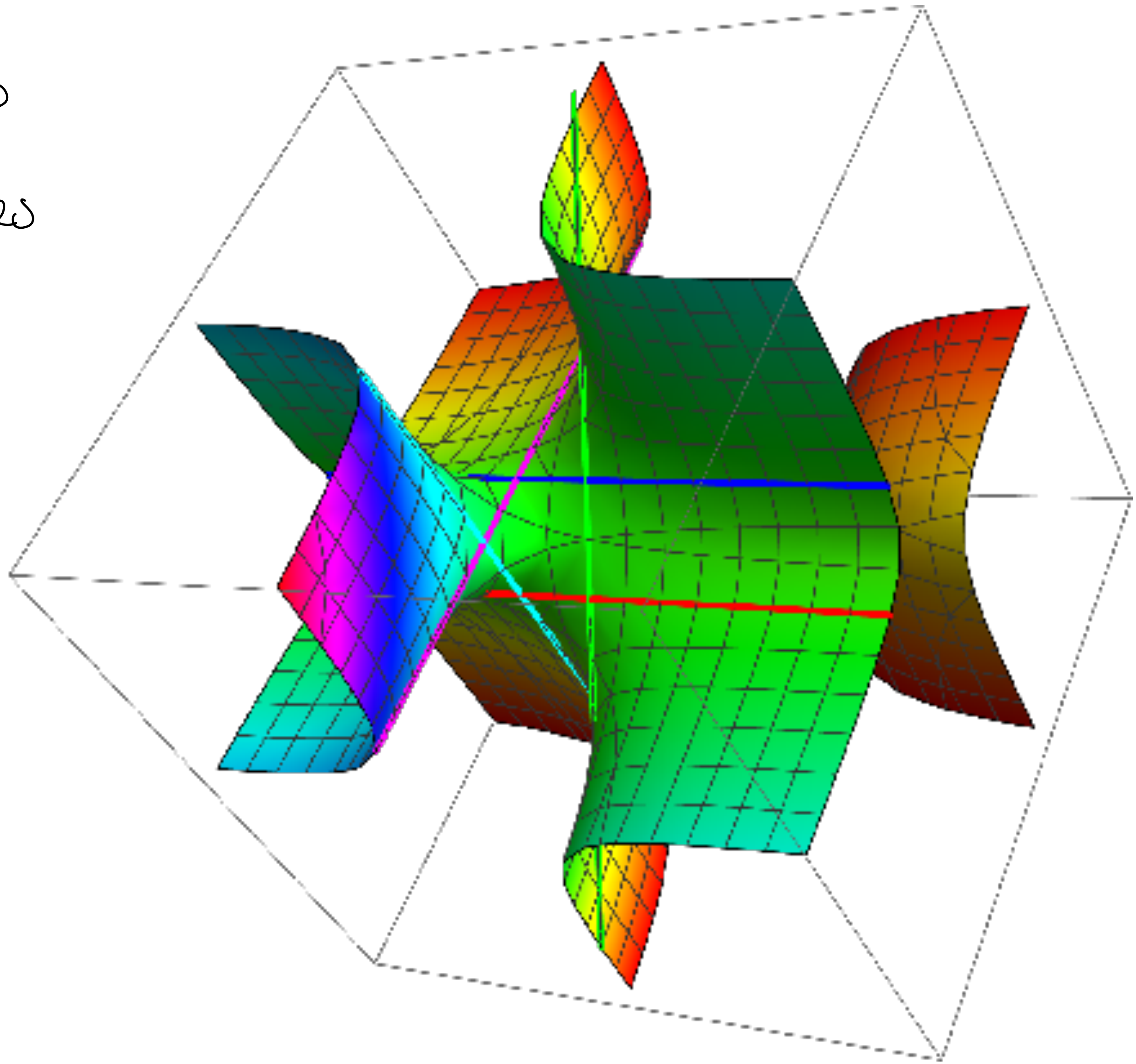
$x_1 = 0,\ x_2 = 1$

$x_1 = 1,\ x_2 = 0$

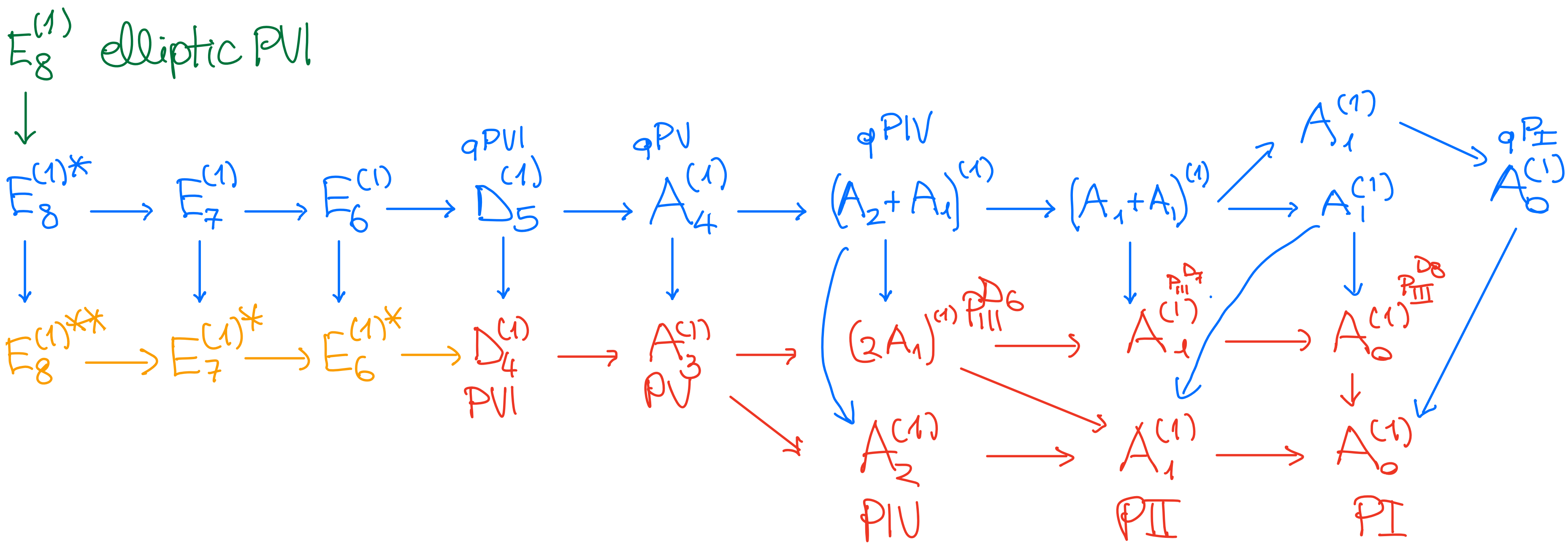
$x_3 = 0 \quad x_1 + x_2 = -1$

$x_2 = x_3 = 1$

$x_1 = x_3 = 1$



Sakai's classification by symmetry groups



qPVI [Jimbo-Sakai '96]

$$\begin{cases} f,g : q^{\mathbb{Z}} t_0 \to \mathbb{P}^1 \\ f(t)\,f(qt) = \dfrac{(g(qt)-q^{\theta_0}t)(g(qt)-q^{-\theta_0}t)}{(g(qt)-q^{\theta_\infty-1})(g(qt)-q^{-\theta_\infty})} \\ g(t)\,g(qt) = \dfrac{(f(t)-q^{\theta_t}t)(f(t)-q^{-\theta_t}t)}{q(f(t)-q^{\theta_1})(f(t)-q^{-\theta_1})} \end{cases}$$

6 parameters

$q \uparrow 1$ gives PVI

$$f(t) = f_0(t) + f_1(t)(q-1) + f_2(t)(q-1)^2 + \dots$$

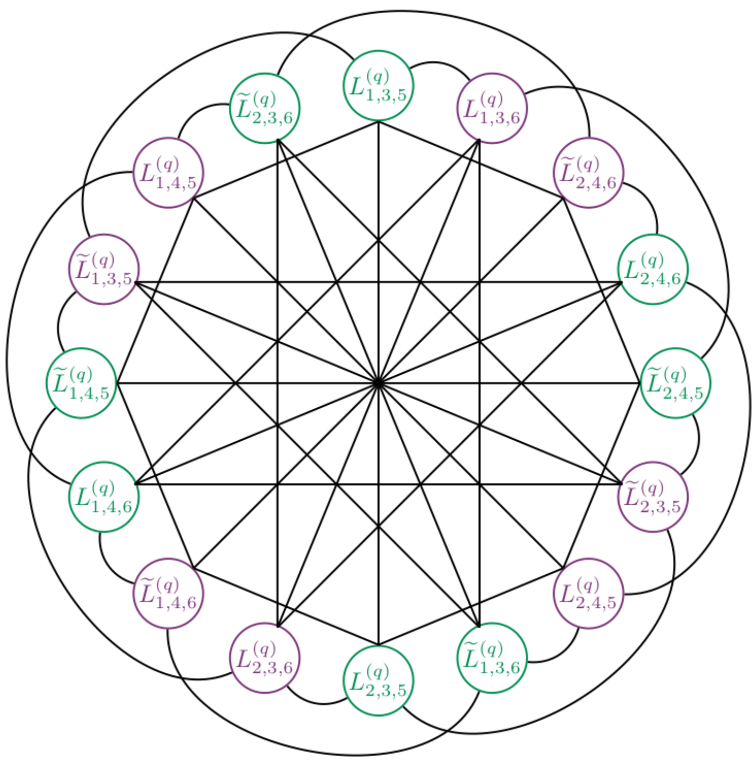
$$g(t) = g_0(t) + g_1(t)(q-1) + \dots$$

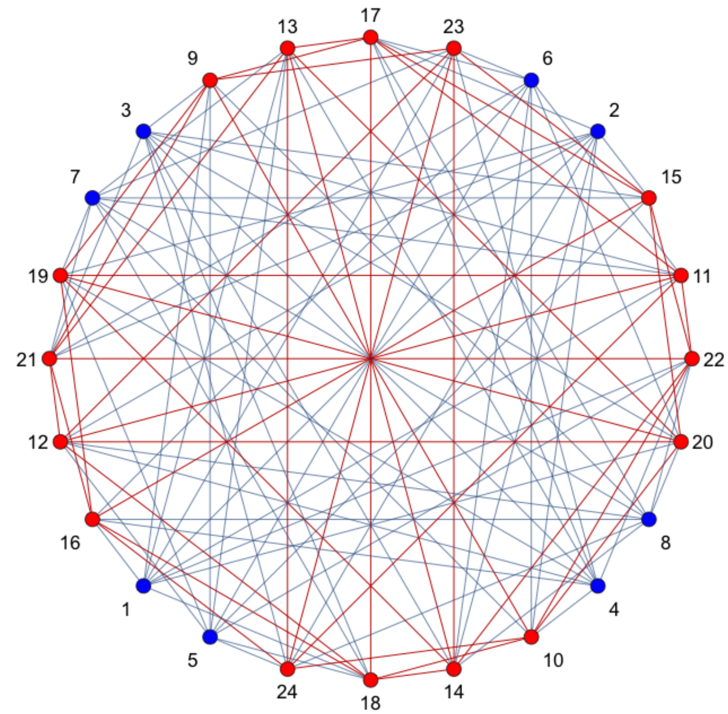
set $f_0 = u$, $g_0 = \dfrac{u-t}{u-1}$ $\Longrightarrow$ $u(t)$ satisfies PVI

$$u_{tt} = \frac{1}{2}\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right)u_t^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{u-t}\right)u_t + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{u^2}+\gamma\frac{t-1}{(u-1)^2}+\delta\frac{t(t-1)}{(u-t)^2}\right)$$

the isomorphism

Brute force





the isomorphism

Better way: use Tyurin ratios

$$
\begin{tikzcd}
\text{Tyurin ratios on } \mathcal{Z}_q \arrow[r, "q\uparrow 1", "\text{Joshi Roffelsen}"'] \arrow[d, leftrightarrow] & \text{Tyurin ratios on } \mathcal{Z}_1 \arrow[dd, leftrightarrow, bend left] \\
\text{asymptotics of } qPVI \arrow[r, "q\uparrow 1"] & \text{asymptotics of } PVI \arrow[d, leftrightarrow, "\text{Jimbo}"] \\
& x_1, x_2, x_3 \text{ in the monodromy manifold}
\end{tikzcd}
$$

$$z_1 = +\gamma^{-1}\Big(\upsilon_\infty\Big(x_2 - \upsilon_\infty^{-1}\nu_t\Big)\Big(x_3 - \upsilon_\infty^{-1}\nu_1\Big) + (\upsilon_\infty - \upsilon_\infty^{-1})\Big(x_1 - \upsilon_\infty\,\nu_0\Big)\Big),$$

$$z_2 = +\gamma^{-1}\Big(\upsilon_\infty^{-1}\Big(x_2 - \upsilon_\infty\,\nu_t\Big)\Big(x_3 - \upsilon_\infty\,\nu_1\Big) - (\upsilon_\infty - \upsilon_\infty^{-1})\Big(x_1 - \upsilon_\infty^{-1}\nu_0\Big)\Big),$$

$$z_3 = -\delta^{-1}\Big(\upsilon_0 - \upsilon_t\upsilon_1\upsilon_\infty\Big)\Big(\upsilon_0 - \frac{1}{\upsilon_t\upsilon_1\upsilon_\infty}\Big)\Big(x_2 - \frac{\upsilon_t}{\upsilon_\infty} - \frac{\upsilon_\infty}{\upsilon_t}\Big)\Big(x_3 - \frac{\upsilon_1}{\upsilon_\infty} - \frac{\upsilon_\infty}{\upsilon_1}\Big),$$

$$z_4 = -\delta^{-1}\Big(\upsilon_0 - \frac{\upsilon_t\upsilon_1}{\upsilon_\infty}\Big)\Big(\upsilon_0 - \frac{\upsilon_\infty}{\upsilon_t\upsilon_1}\Big)\Big(x_2 - \upsilon_t\upsilon_\infty - \frac{1}{\upsilon_t\upsilon_\infty}\Big)\Big(x_3 - \upsilon_1\upsilon_\infty - \frac{1}{\upsilon_1\upsilon_\infty}\Big),$$

$$z_5 = +\delta^{-1}\Big(\upsilon_0 - \frac{\upsilon_t\upsilon_\infty}{\upsilon_1}\Big)\Big(\upsilon_0 - \frac{\upsilon_1}{\upsilon_t\upsilon_\infty}\Big)\Big(x_2 - \frac{\upsilon_t}{\upsilon_\infty} - \frac{\upsilon_\infty}{\upsilon_t}\Big)\Big(x_3 - \upsilon_1\upsilon_\infty - \frac{1}{\upsilon_1\upsilon_\infty}\Big),$$

$$z_6 = +\delta^{-1}\Big(\upsilon_0 - \frac{\upsilon_1\upsilon_\infty}{\upsilon_t}\Big)\Big(\upsilon_0 - \frac{\upsilon_t}{\upsilon_1\upsilon_\infty}\Big)\Big(x_2 - \upsilon_t\upsilon_\infty - \frac{1}{\upsilon_t\upsilon_\infty}\Big)\Big(x_3 - \frac{\upsilon_1}{\upsilon_\infty} - \frac{\upsilon_\infty}{\upsilon_1}\Big),$$

$$\gamma = (\upsilon_0 - 1)(\upsilon_0^{-1} - 1)(\upsilon_\infty - \upsilon_\infty^{-1})^2,$$

$$\delta = (\upsilon_0 - 1)^2(\upsilon_t - \upsilon_t^{-1})(\upsilon_1 - \upsilon_1^{-1})(\upsilon_\infty - \upsilon_\infty^{-1})^2.$$

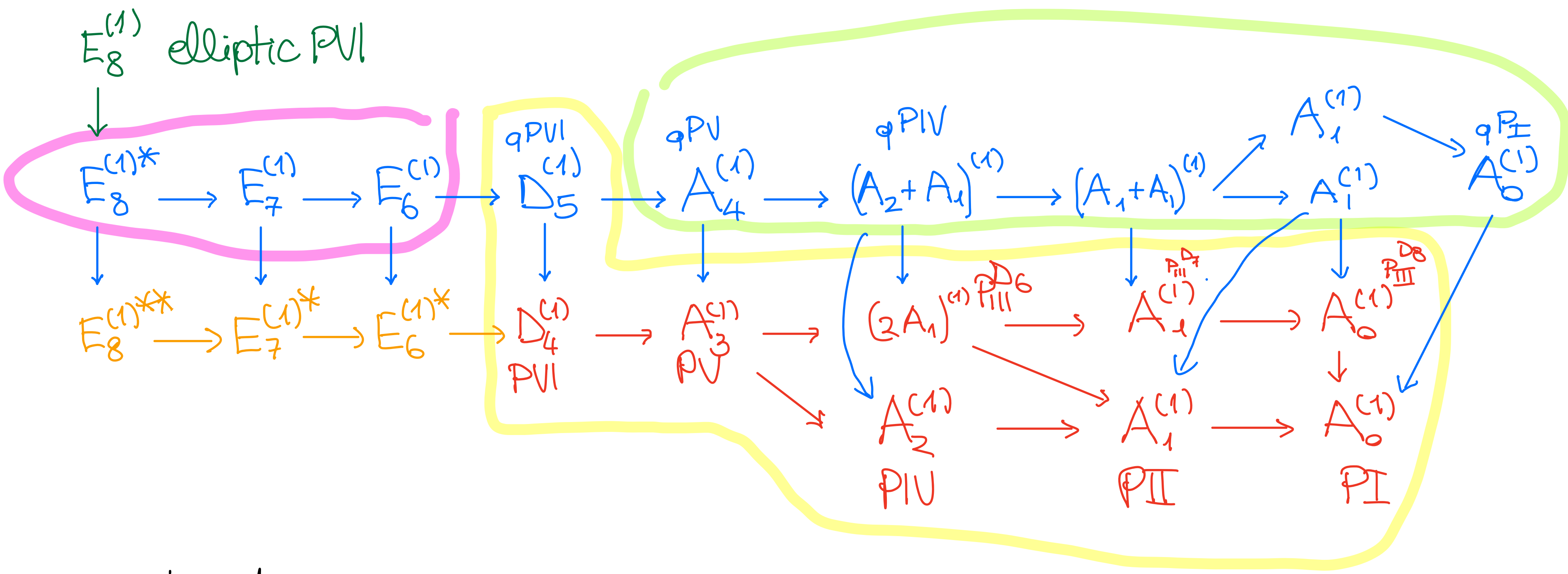
$$\upsilon_\kappa = e^{\pi i \vartheta_\kappa}$$

the other Painlevé differential equations

$$
\begin{tikzcd}
 & & P_{III} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & \mathrm{P}_{\mathrm{II}}^{\mathrm{JM}} \arrow[r] & P_I \\
 & & P_{IV} \arrow[r] \arrow[ur] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}
$$

$\mathrm{P_V}$	$z_1+z_2+z_3+z_4+z_5+z_6=0,$ $z_4+\rho_5 z_5-1=0,$ $z_3z_4-\lambda_1z_1z_2=0, \quad z_5z_6-\lambda_2z_1z_2=0.$
$\mathrm{P_V^{deg}}$	$z_1+z_3+z_4+z_5+z_6=0,$ $\rho_3z_3+z_4+\rho_5z_5+\frac{\rho_3}{\rho_5}z_6-1=0,$ $z_3z_4-z_1z_2=0, \quad z_5z_6-z_1z_2=0.$
$\mathrm{P_{IV}}$	$z_1+z_2+z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-\lambda_1z_1z_2=0, \quad z_5z_6-\lambda_2z_1z_2=0.$
$\mathrm{P_{III}^{D_6}}$	$z_1+z_2+z_3+z_4+z_5=0,$ $z_4+\rho_5z_5-1=0,$ $z_3z_4-\lambda_1z_1z_2=0, \quad z_5z_6-z_1z_2=0.$
$\mathrm{P_{III}^{D_7}}$	$z_1+z_2+z_3+z_4+z_5=0,$ $z_4+\rho_5z_5-1=0,$ $z_3z_4-z_1z_2=0, \quad z_5z_6-z_1z_2=0.$
$\mathrm{P_{II}^{JM}}$, $\mathrm{P_{II}^{FN}}$	$z_1+z_2+z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-z_1z_2=0, \quad z_5z_6-\lambda_2z_1z_2=0.$
$\mathrm{P_I}$	$z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-z_1z_2=0, \quad z_3z_4-z_5z_6=0.$

Outlook



- unified
- Expect Z_q with different parameter specializations
- Expect higher order del Pezzo

Question: mirror construction for Segre?