Hyperbolic triangle chains and finite mapping class group orbits

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algebraic solutions to isomonodromy differential equations

(Painlevé VI, Schlesinger systems)

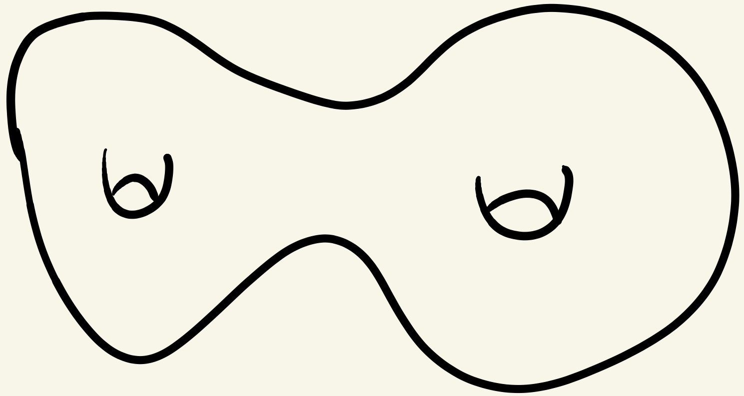
$\rightsquigarrow$

finite mapping class group orbits on $SL_2\mathbb{C}$ character varieties

(1) Set-up and history

(2) Constructing / classifying finite orbits

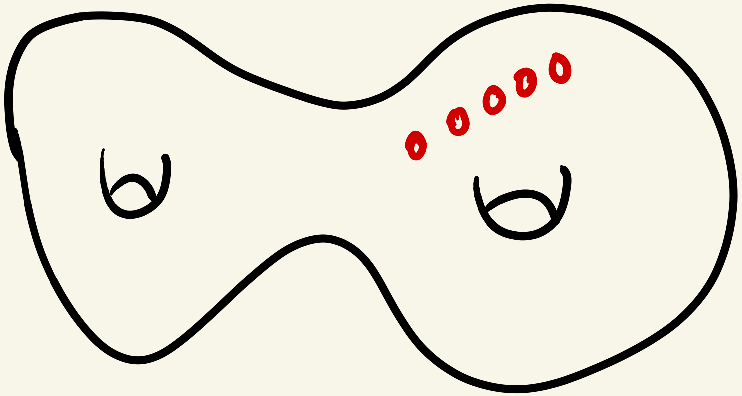
(1) Set-up and history

$S =$  : oriented surface
$g \geq 0$

mapping class group of S : $\mathrm{Mod}(S)$

$$\pi_0(\mathrm{Homeo}_+(S)) \cong \mathrm{Out}(\pi_1(S)) \cong \pi_1 \mathcal{M}_g$$

(topological) (algebraic) (geometric)

$S =$  : oriented surface

$g \geq 0$, $n \geq 0$ punctures

pure mapping class group of S : $PMod(S)$

$$\pi_0\left(Homeo^*_+(S)\right) \cong Out^*(\pi_1(S)) \cong \pi_1 \mathcal{M}_{g,n}$$

(topological) (algebraic) (geometric)

G : Lie / algebraic group

character variety of (S,G) : $\mathrm{Rep}(S,G) = \mathrm{Hom}(\pi_1 S, G) / G$

(topological quotient)

$\mathcal{T}\mathrm{Mod}(S) \hookrightarrow \mathrm{Rep}(S,G)$

(by precomposition)

<u>goal</u> : understand finite orbits of

$$\mathrm{PMod}(S) \curvearrowright \mathrm{Rep}(S, SL_2\mathbb{C})$$

<u>example 0</u> :

$\rho : \pi_1 S \longrightarrow SL_2\mathbb{C}$: finite image

$\Rightarrow [\rho] \in \mathrm{Rep}(S, SL_2\mathbb{C})$ has finite orbit

$g \geq 1, \; n \geq 0$

$$\boxed{\begin{array}{l}[\rho] \in \mathrm{Rep}(S, SL_2\mathbb{C}) \text{ has finite orbit} \\ \quad \Rightarrow^{*} \rho \text{ has finite image}\end{array}}$$

* not completely true for $g = 1$ (infinite dihedral group)

[Biswas – Gupta – Mj – Whang 22', Cousin – Heu 21']

$\underline{g=0}$

* $n=3$: $PMod(S)$ is trivial

$\Rightarrow$ every $[\rho]$ is a finite orbit

* $n=4$: $\boxed{\text{algebraic solutions to Painlevé VI} \iff \text{finite orbits in } Rep(S, SL_2\mathbb{C})}$

examples: Fuchs, Hitchin, Dubrovin, Mazzocco, Andreev, Kitaev, Boalch, ...

full list : Boalch '06 + Lisovyy–Tykhyy '14

* $n \geq 5$:

examples : Diarra, Calligaris, Mazzocco, Tykhyy, ...

Tykhyy's Conjecture

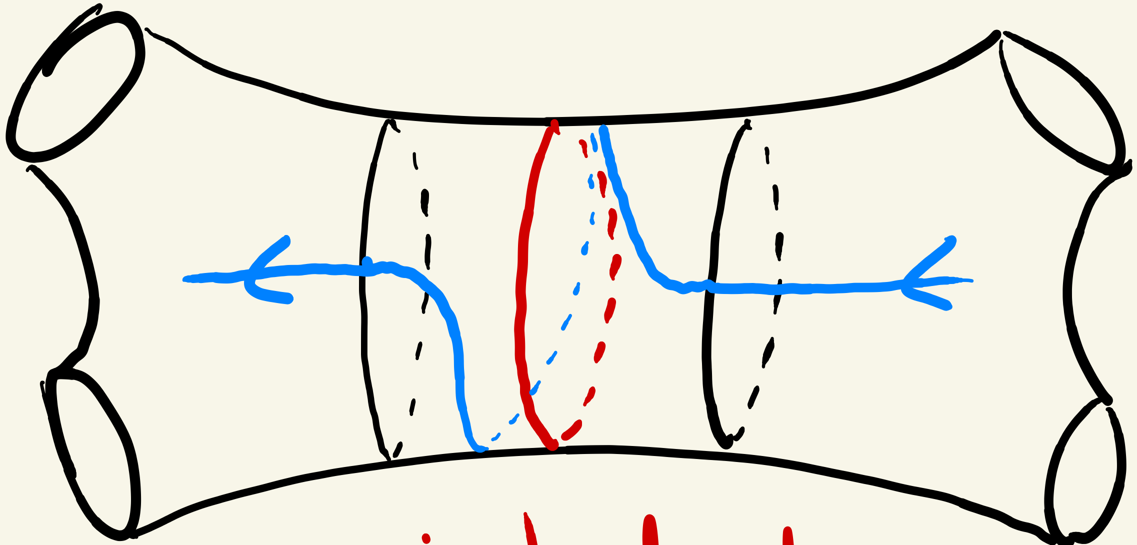
* $n = 5, 6$: list of finite orbits
* $n \geq 7$: $\nexists$ "new" finite orbits

Theorem (Bronstein-M. 24', Lam-Landesman-Litt 23')

Tykhyy's Conjecture is true

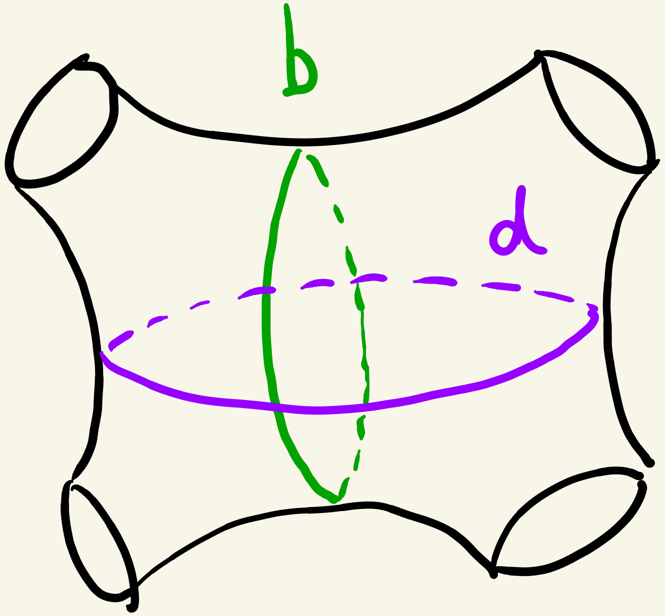
(2) Constructing/classifying finite orbits

$$\mathrm{PMod}(S) = \langle \text{Dehn twists} \rangle$$



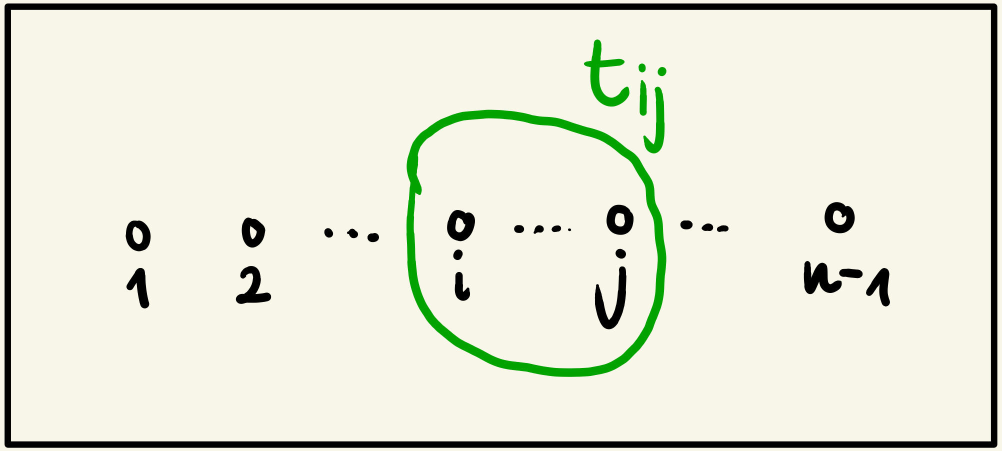
Simple closed curve a

$\underline{n=4}$



$$\mathrm{PMod}(S) = \langle \tau_b, \tau_d \rangle$$

$\underline{\text{general}}$



$$\mathrm{PMod}(S) = \langle \tau_{ij} \mid \underset{(i,j) \neq (1,n-1)}{i<j} \rangle$$

[Ghaswala - Winarski 17']

Heuristic

$[\rho] \in Rep(S, SL_2\mathbb{C})$ finite orbit "$\Rightarrow$" $\rho(a)$ is elliptic $\forall a$ simple closed curve

$Tr(\rho(a)) \in (-2,2)$

i.e. ρ is totally elliptic

"$\Rightarrow$" ρ is real

i.e. $\rho : \pi_1 S \to SU(2), SL_2\mathbb{R}$

[Procesi, Goldman-Xia]

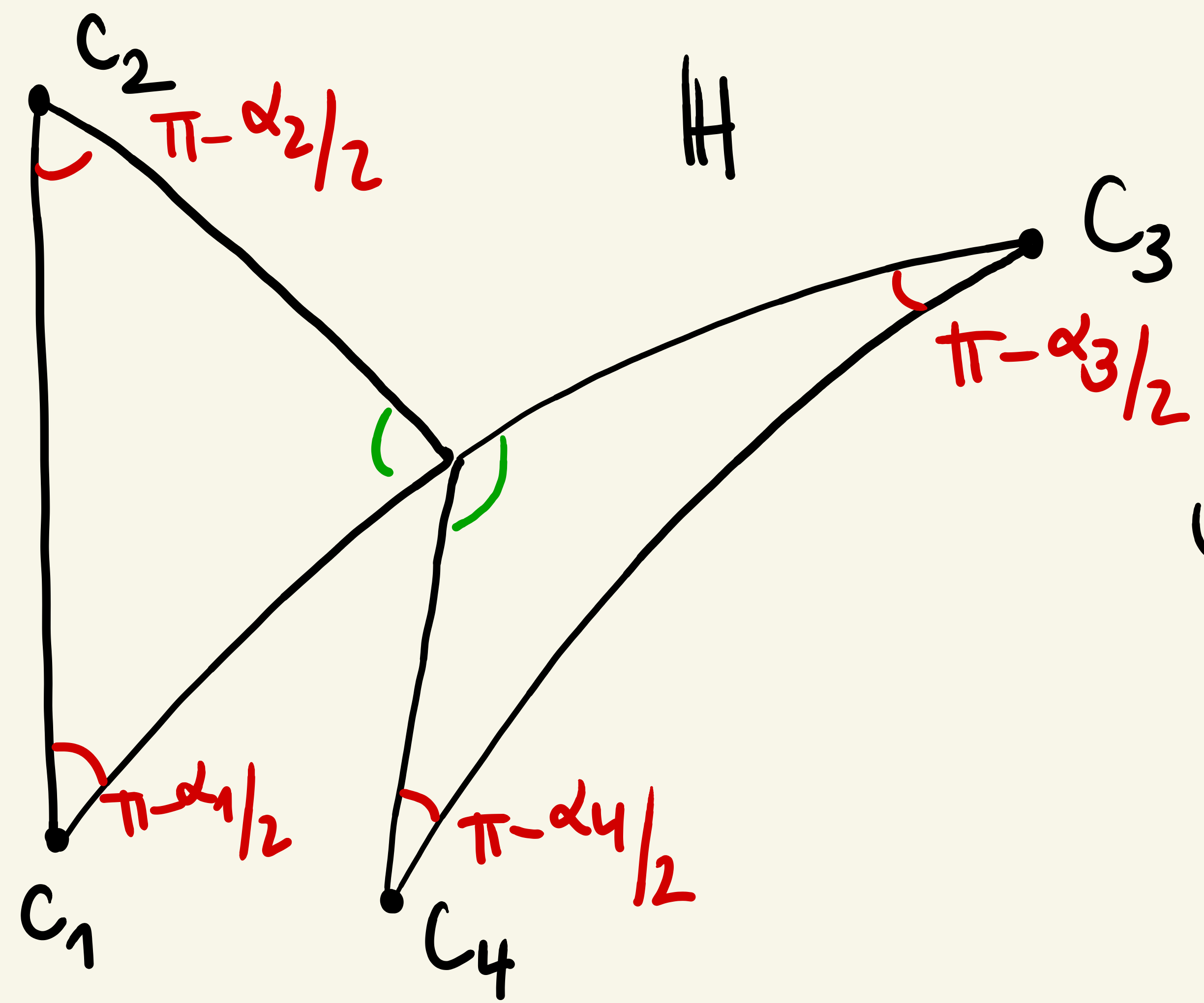
<u>moral</u>: to find finite orbits, look for totally elliptic

$$\rho : \pi_1 S \to SL_2 \mathbb{R}$$

Triangle chains (M.22')

$\alpha = (\alpha_1, \dots, \alpha_4) \in (0, 2\pi)^4$

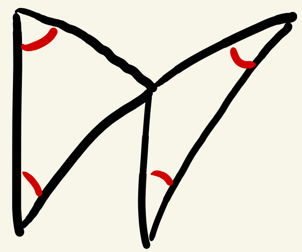
$\alpha_1 + \dots + \alpha_4 > 6\pi$

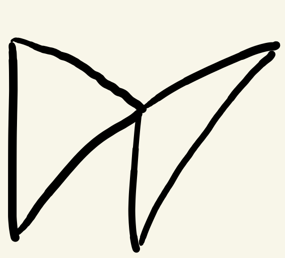
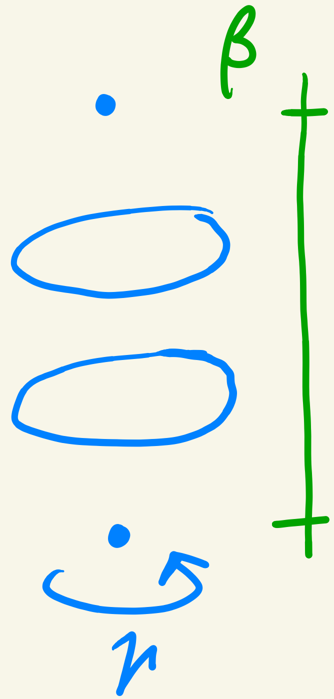


conditions:

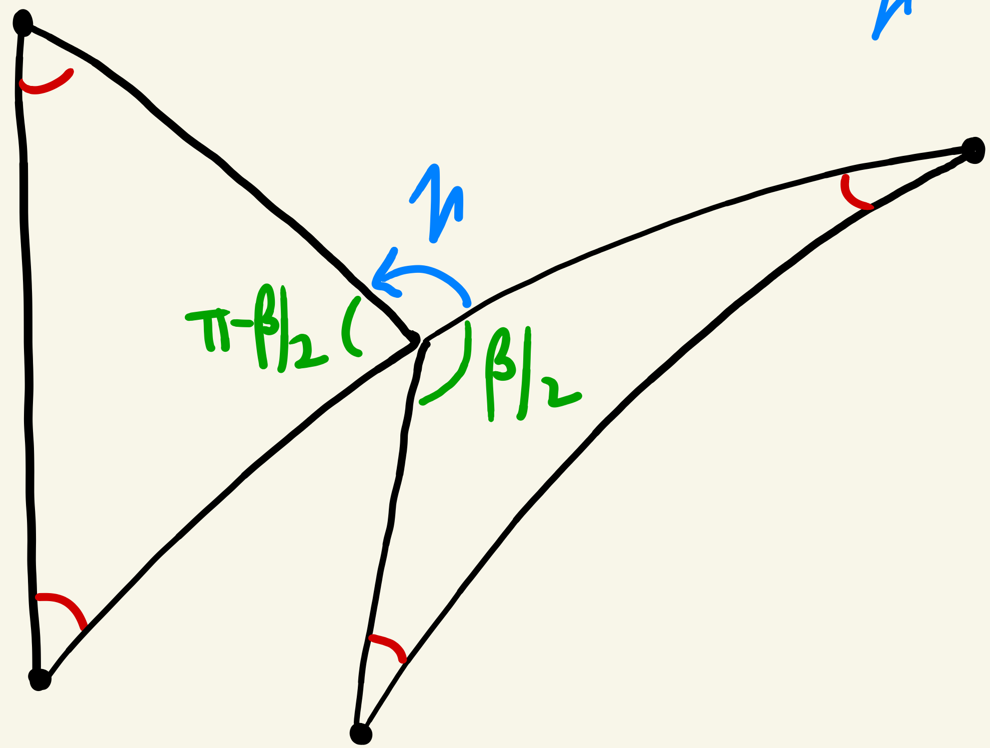
(1) $\measuredangle c_i = \pi - \alpha_i/2$

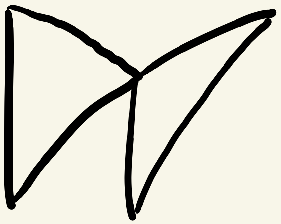
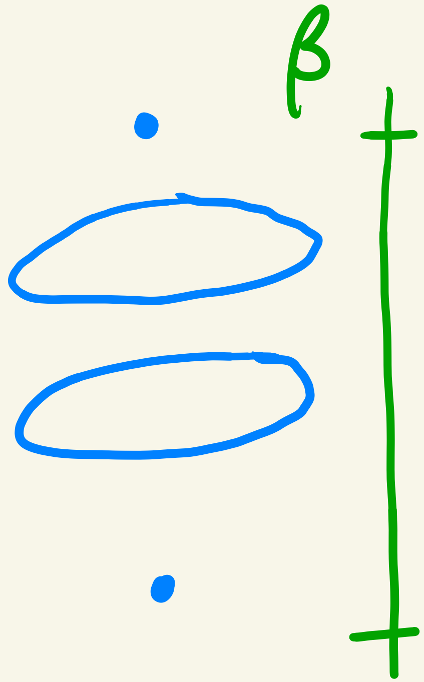
(2) $\measuredangle + \measuredangle = \pi$

$TC^{n=4}_{\alpha} := \{$  $\}_{/PSL_2\mathbb{R}} = ?$

$TC_{\alpha}^{n=4} := \Big\{$  $\Big\} \big/ _{PSL_2\mathbb{R}} \cong$ 

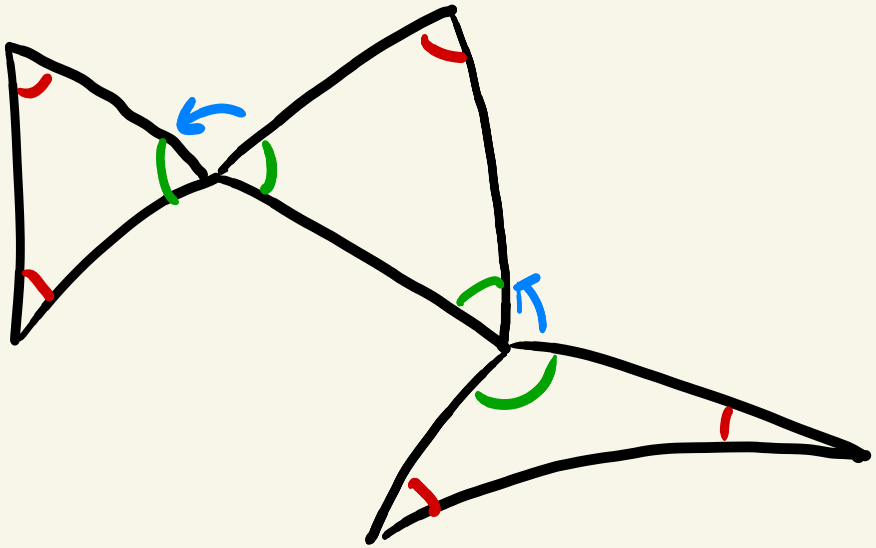
$\cong \mathbb{CP}^1$

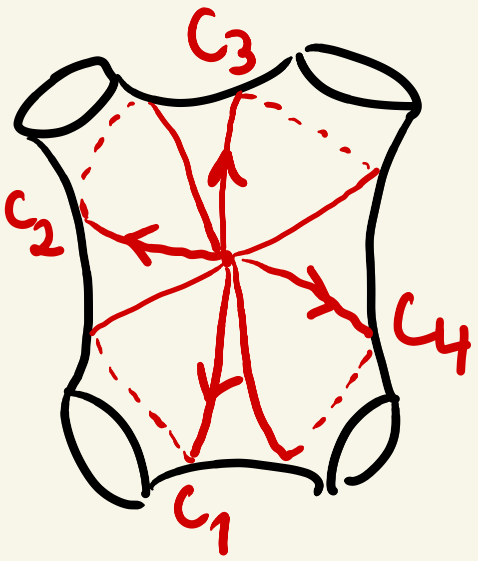


$$TC_{\alpha}^{n=4} := \left\{ \right.$$  $$\left. \right\} \Big/ PSl_2\mathbb{R} \;\cong$$ 

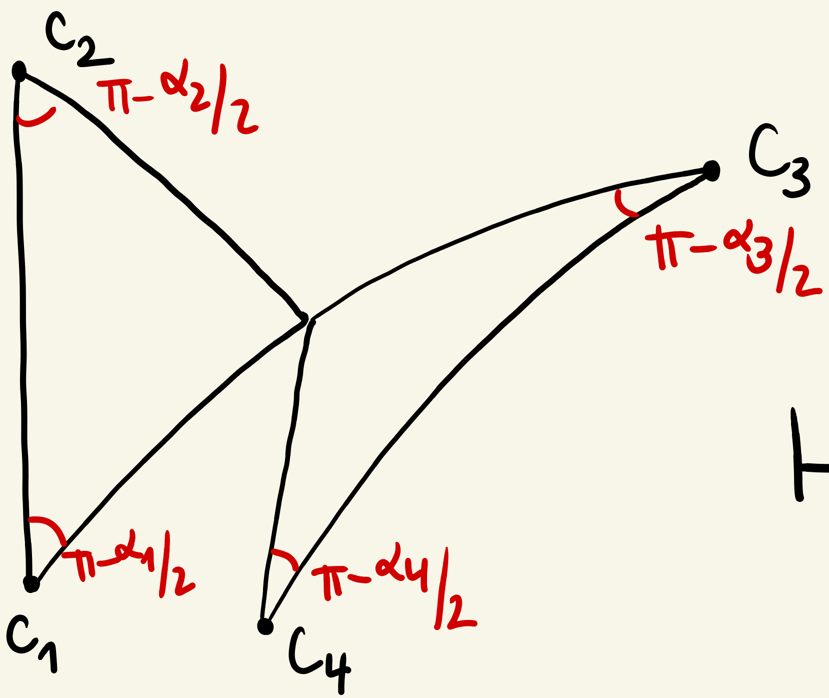
$$\cong \mathbb{CP}^1$$

<u>more generally</u> : $TC_{\alpha}^{n} \cong \mathbb{CP}^{n-3}$



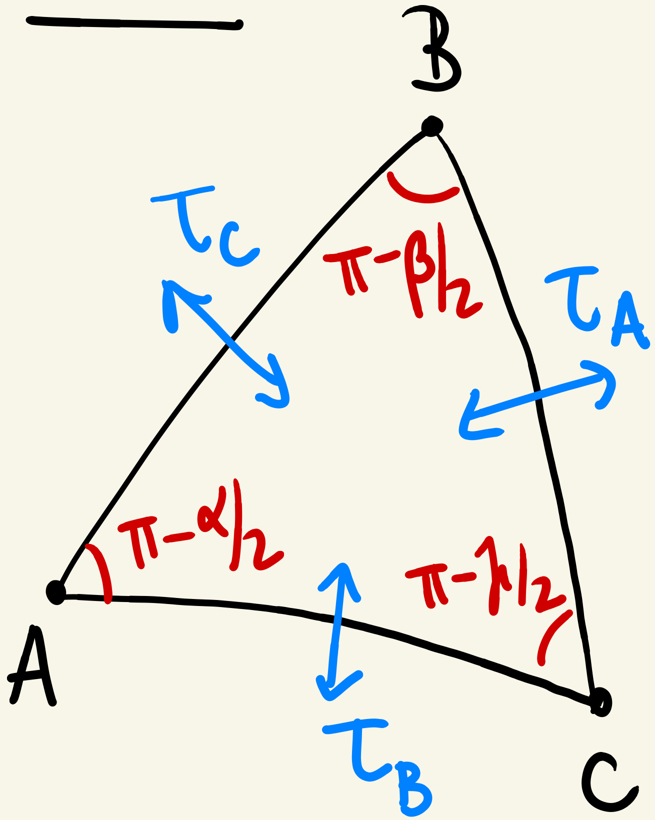
$TC_{\alpha}^{n=4} \longrightarrow \mathrm{Rep}\Big($ 

$, PSL_2\mathbb{R}\Big)$



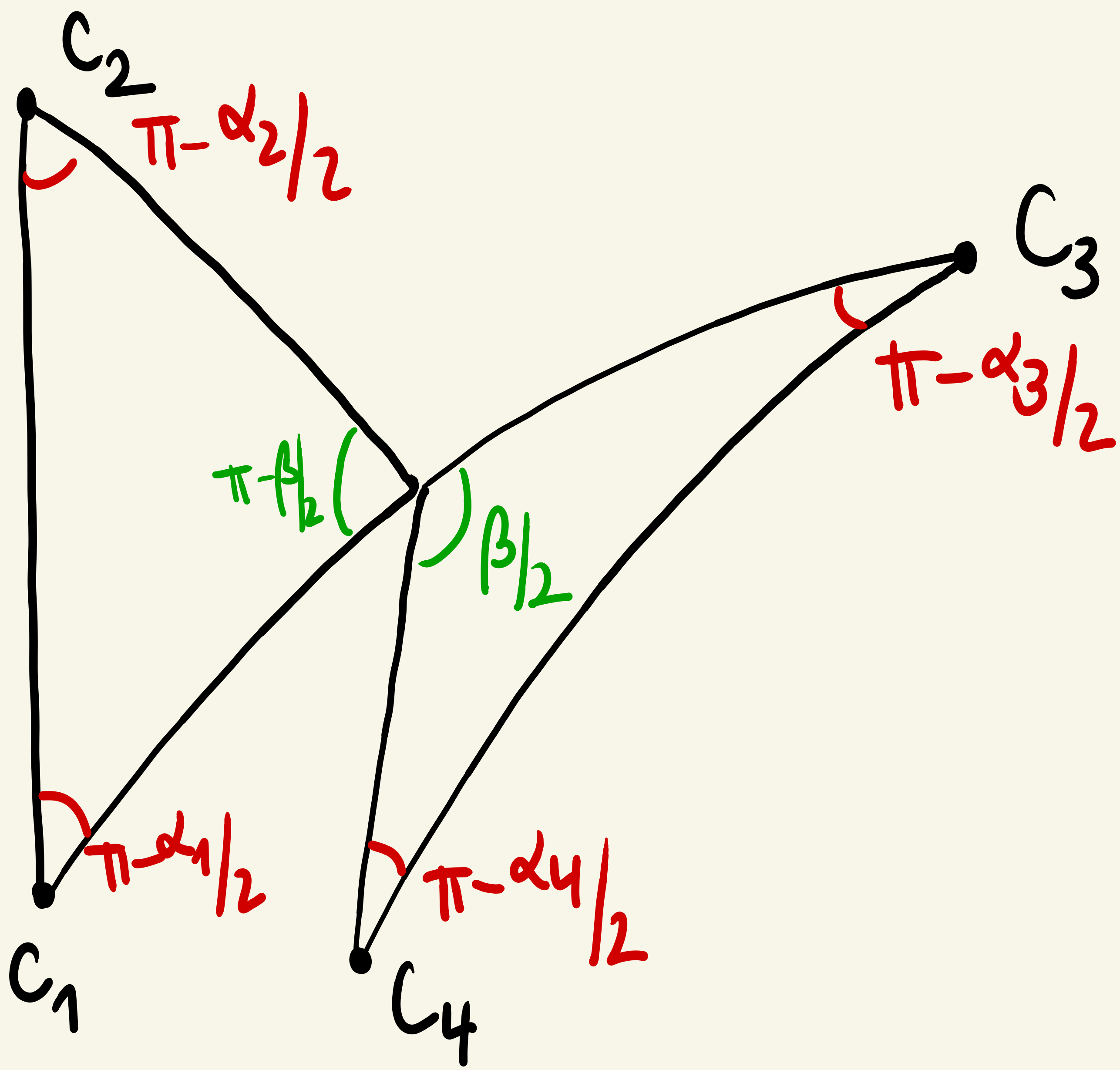
$$\langle c_1, \dots, c_4 \mid \prod c_i = 1 \rangle$$
$$\|$$
$$\mapsto \quad \rho : \pi_1 S \longrightarrow PSL_2\mathbb{R}$$
$$c_i \longmapsto \mathrm{rot}_{\alpha_i}(C_i)$$

lemma



$$\Rightarrow \underbrace{\mathrm{rot}_\alpha(A)}_{\tau_B\tau_C}\cdot\underbrace{\mathrm{rot}_\beta(B)}_{\tau_C\tau_A}\cdot\underbrace{\mathrm{rot}_\gamma(C)}_{\tau_A\tau_B}=1$$

$\square$



c_2
$\pi - \alpha_2/2$
c_3
$\pi - \alpha_3/2$
$\pi - \beta/2$
$\beta/2$
$\pi - \alpha_1/2$
$\pi - \alpha_4/2$
c_1
c_4

$g=0,\, n\geq 4$

$$\mathrm{Rep}(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}(S, PSL_2\mathbb{R})$$

$\rho(c_i)$ is elliptic of angle α_i

character variety

$g=0,\ n\geq 4$

$$\mathrm{Rep}(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}_{\alpha}(S, PSL_2\mathbb{R})$$

character variety

α-relative character variety

$g=0,\ n\geq 4$

$$\mathrm{Rep}(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}_\alpha(S, PSL_2\mathbb{R}) \supseteq \mathrm{Rep}^{DT}_\alpha$$

character variety

α- relative character variety [Deroin-Tholozan 19']

$\exists$ compact connected component

DT component

$g=0,\ n\geq 4$

$$\begin{array}{ccccc} \mathrm{Rep}(S,\mathrm{PSL}_2\mathbb{R}) & \supseteq & \mathrm{Rep}_{\alpha}(S,\mathrm{PSL}_2\mathbb{R}) & \supseteq & \mathrm{Rep}^{DT}_{\alpha} \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathrm{PMod}(S) & & \mathrm{PMod}(S) & & \mathrm{PMod}(S) \end{array}$$

$g=0, n\geq 4$

$$\mathrm{Rep}(S,\mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}_\alpha(S,\mathrm{PSL}_2\mathbb{R}) \supseteq \mathrm{Rep}^{DT}_\alpha \overset{(M.24')}{\cong} \mathcal{TC}_\alpha$$

$$\begin{array}{ccccc} \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathrm{PMod}(S) & & \mathrm{PMod}(S) & & \mathrm{PMod}(S) \end{array}$$

character variety

α-relative character variety

DT component

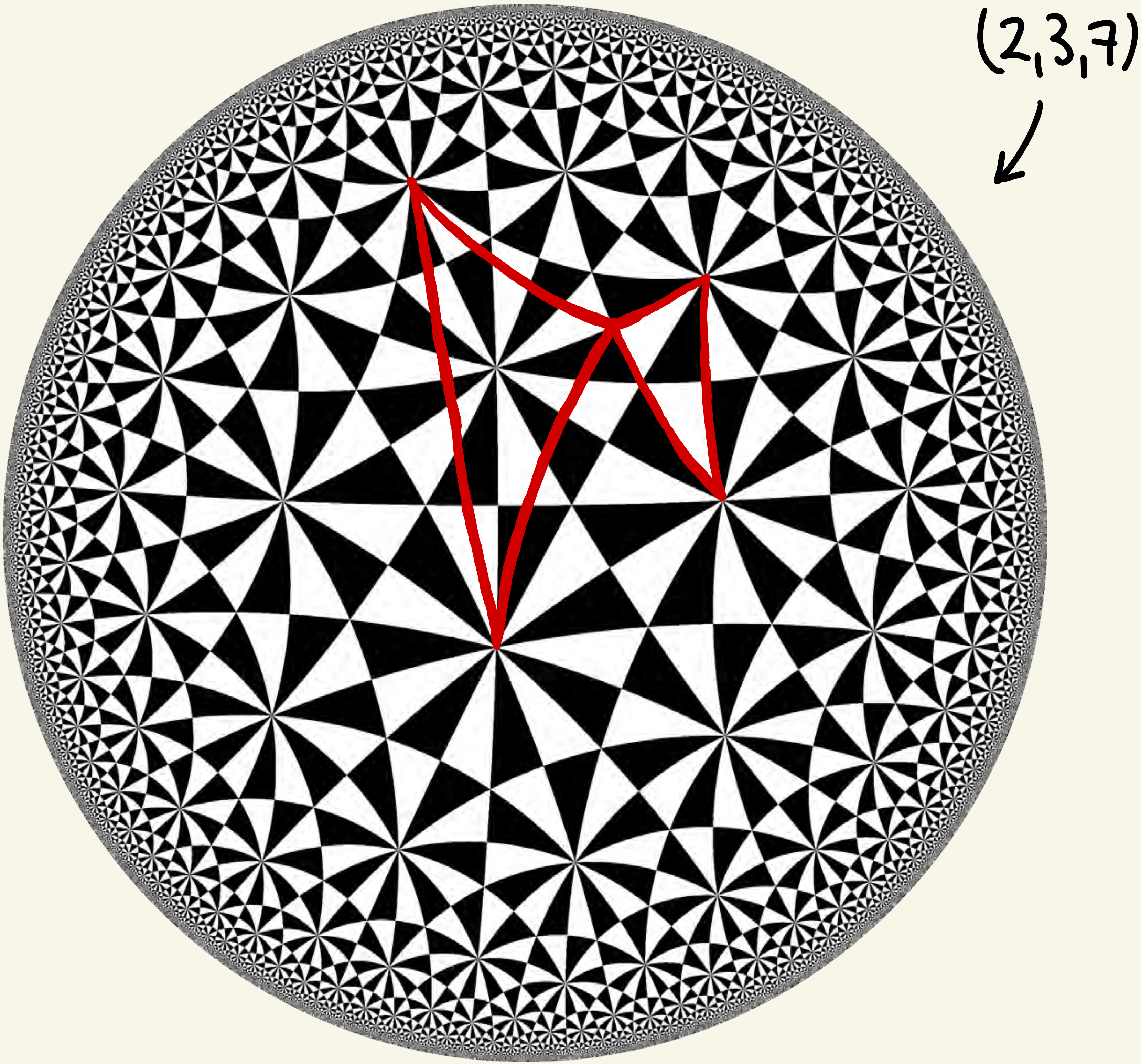
totally elliptic!

exemple 1

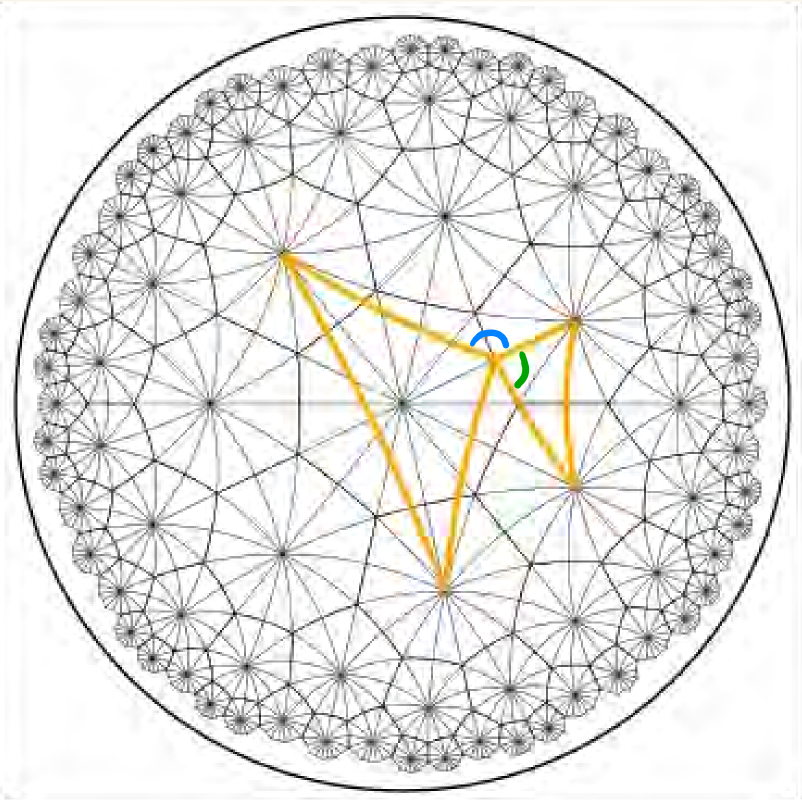
$[\rho] \in \mathrm{Rep}^{DT}_{\alpha}$ + ρ has discrete image

$\Rightarrow$ orbit of $[\rho]$ is discrete in $\mathrm{Rep}^{DT}_{\alpha}$

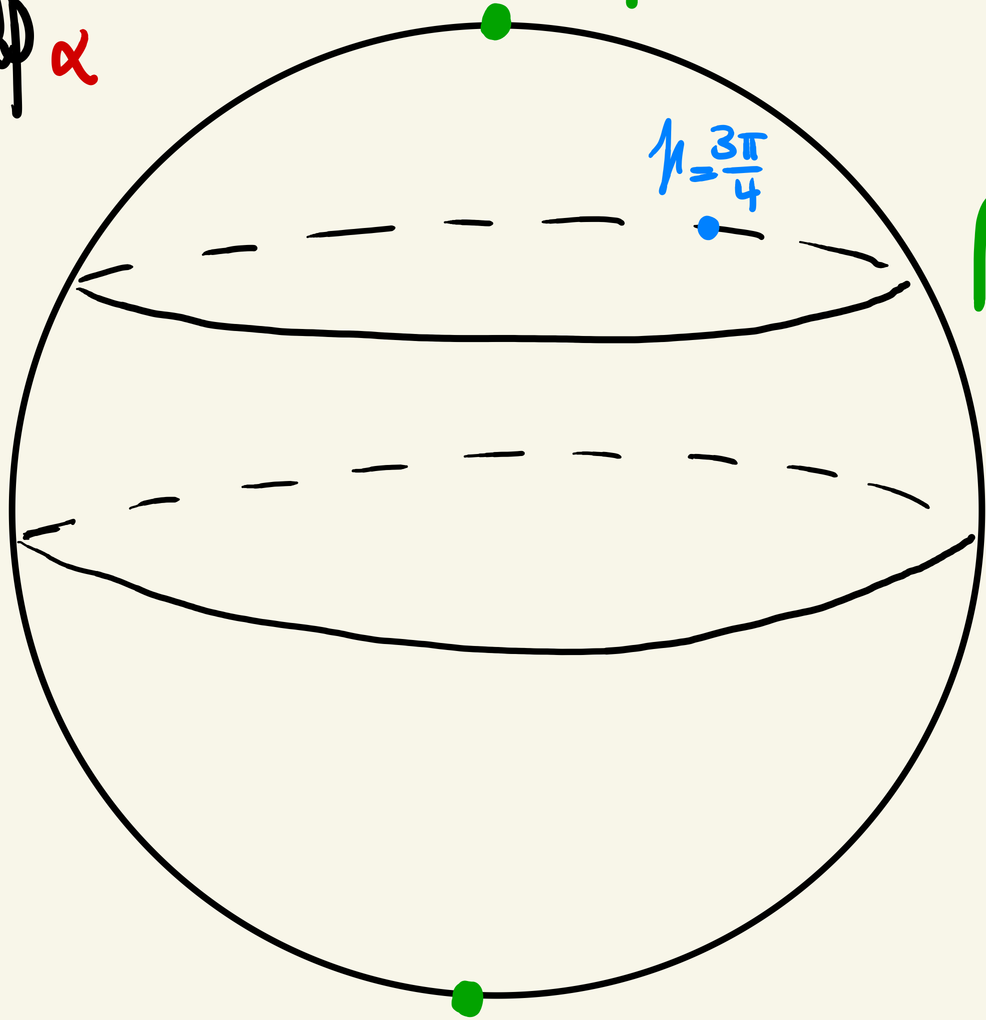
$\Rightarrow$ orbit of $[\rho]$ is finite!


(2,3,7)

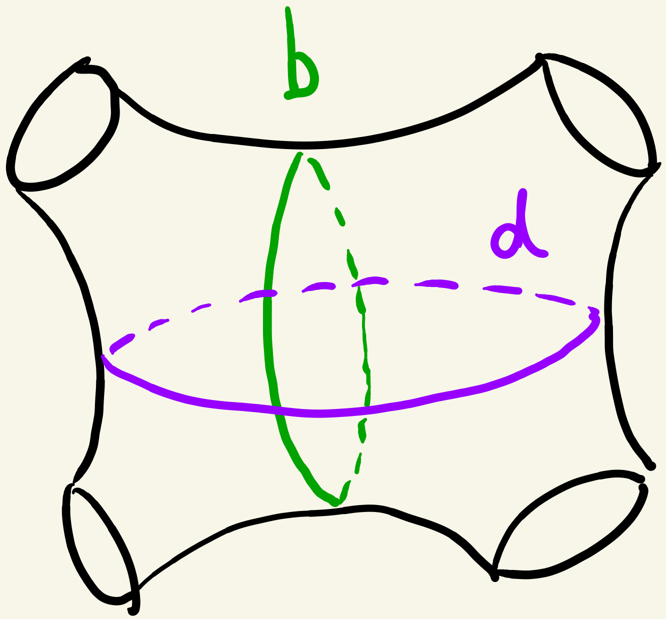
$\mathrm{Rep}_{\alpha}^{DT}$



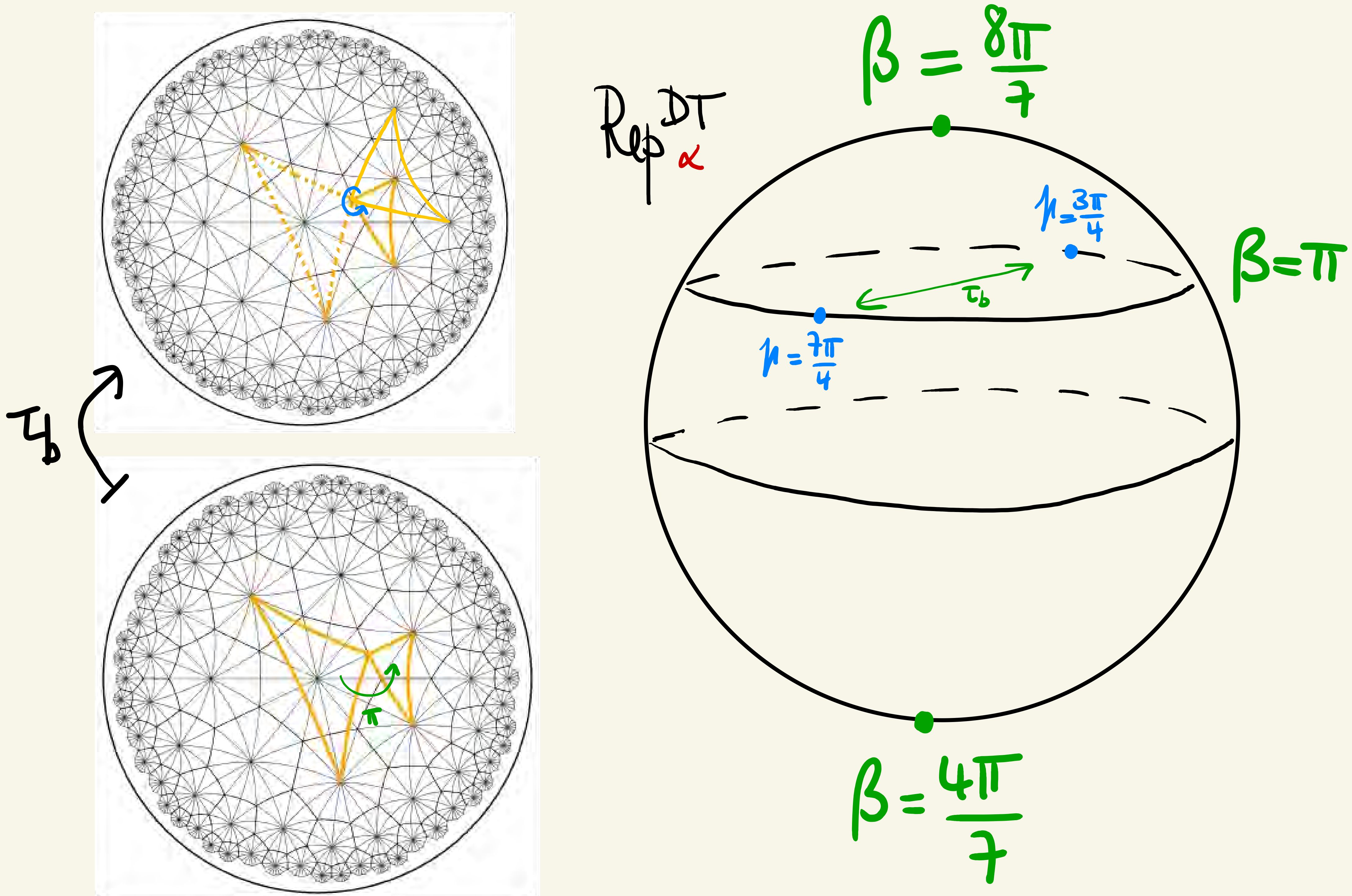
$\beta = \pi \qquad \gamma = \frac{3\pi}{4}$



$\underline{n=4}$



$$\mathrm{PMod}(S) = \langle \tau_b, \tau_d \rangle$$



τ_b
π
$\mathrm{Rep}^{DT}_{\alpha}$
$\beta = \frac{8\pi}{7}$
$\mu = \frac{3\pi}{4}$
$\beta = \pi$
τ_b
$\mu = \frac{7\pi}{4}$
$\beta = \frac{4\pi}{7}$

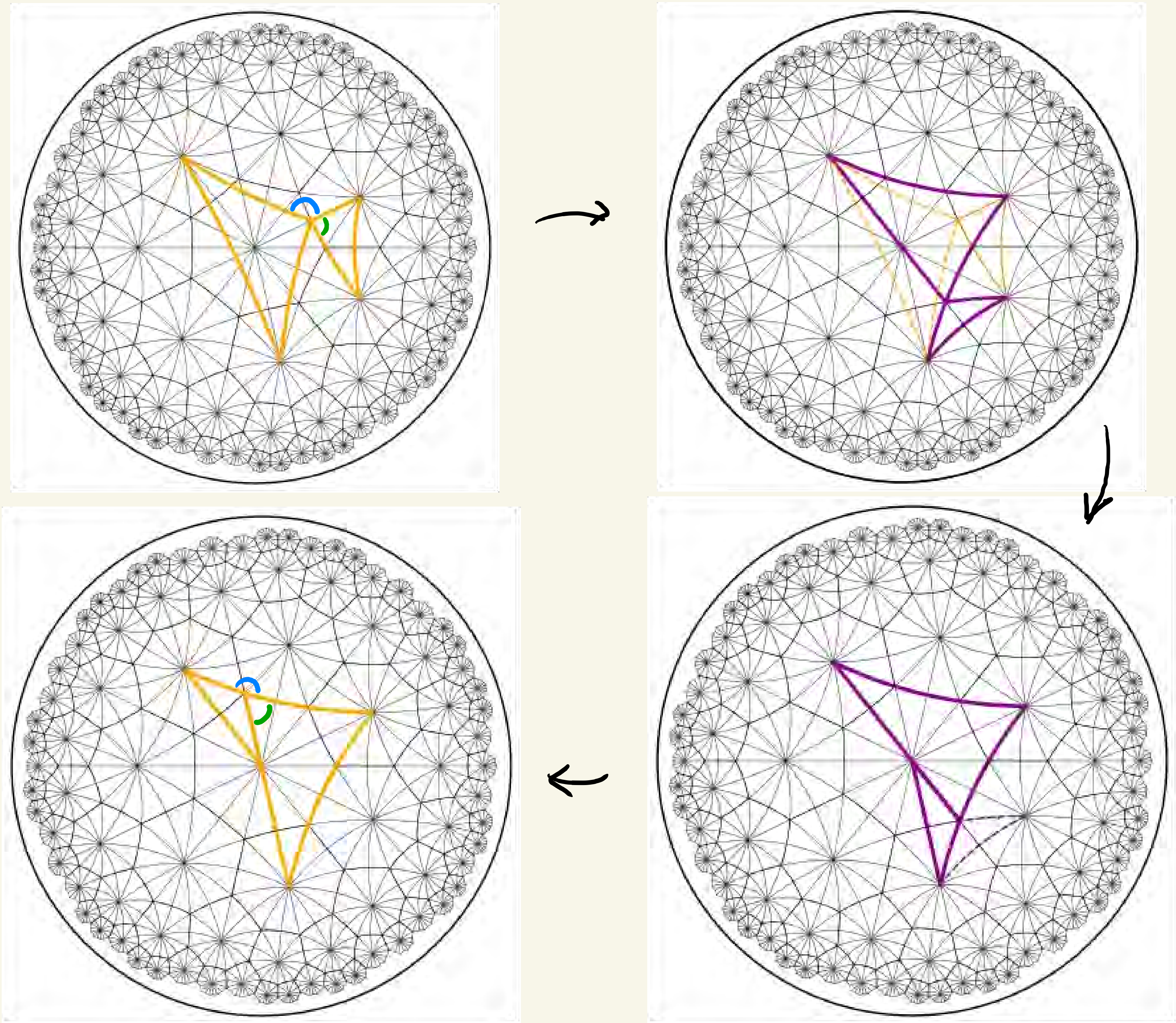
$\beta = \pi$

$\gamma = \frac{3\pi}{4}$

τ_d

$\beta = \frac{2\pi}{3}$

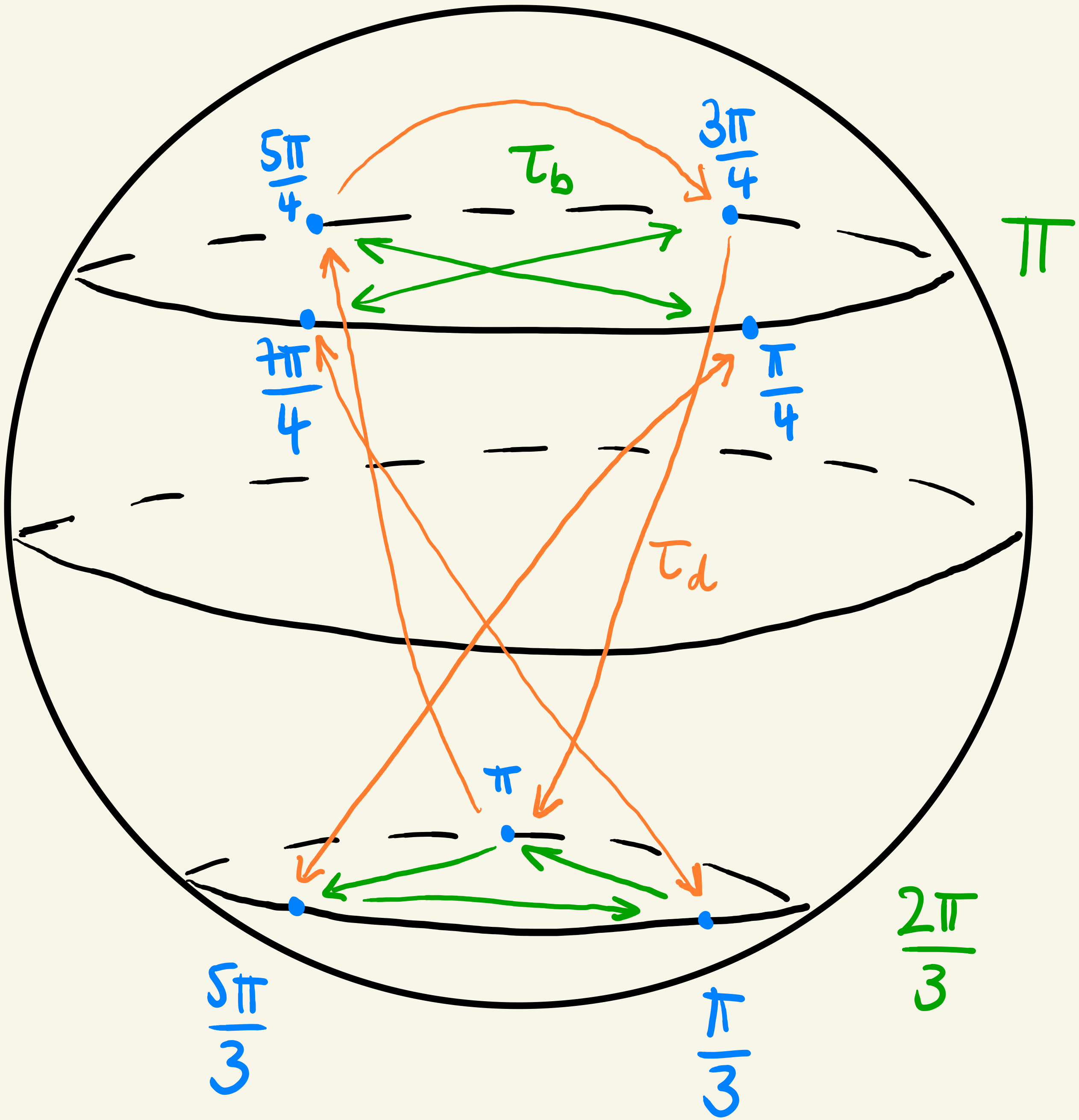
$\gamma = \pi$



length = 7

"Klein orbit"

(Boalch 05')



Lisovyy-Tykhyy's classification	Orbit length	Angle vector α	Non-peripheral trace field
Sol. II	2	$\{\theta_1, \theta_1, \theta_2, \theta_2\}, \quad \theta_1 + \theta_2 > 3\pi$	
Sol. III	3	$\{\frac{4\pi}{3}, 2\theta - 2\pi, \theta, \theta\}, \ \theta > 5\pi/3$	
Sol. IV	4	$\{\pi, \theta, \theta, \theta\}, \ \theta > 5\pi/3$	
Sol. IV*	4	$\{\theta, \theta, \theta, 3\theta - 4\pi\}, \ \theta > 5\pi/3$	
Sol. 1	5	$\left\{\frac{22\pi}{15}, \frac{8\pi}{5}, \frac{8\pi}{5}, \frac{28\pi}{15}\right\}$	$\mathbb{Q}$
Sol. 4	6	$\left\{\frac{19\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{23\pi}{12}\right\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 6	6	$\left\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 7	6	$\left\{\frac{17\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}, \frac{29\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 8	7	$\left\{\frac{10\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\right\}$	$\mathbb{Q}$
Sol. 10	8	$\left\{\frac{17\pi}{12}, \frac{7\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}\right\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 11	8	$\left\{\frac{13\pi}{10}, \frac{3\pi}{2}, \frac{19\pi}{10}, \frac{19\pi}{10}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 12	8	$\left\{\frac{3\pi}{2}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{19\pi}{10}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 13	9	$\left\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{26\pi}{15}, \frac{28\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 14	9	$\left\{\frac{14\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 15	10	$\left\{\frac{8\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}, \frac{9\pi}{5}\right\}$	$\mathbb{Q}$
Sol. 18	10	$\left\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{23\pi}{15}, \frac{9\pi}{5}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 19	10	$\left\{\frac{7\pi}{5}, \frac{29\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 20	12	$\left\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$	$\mathbb{Q}(\sqrt{2})$
Sol. 22	12	$\left\{\frac{19\pi}{15}, \frac{9\pi}{5}, \frac{9\pi}{5}, \frac{29\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 23	12	$\left\{\frac{37\pi}{30}, \frac{47\pi}{30}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 24	12	$\left\{\frac{49\pi}{30}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{59\pi}{30}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 25	12	$\left\{\frac{43\pi}{30}, \frac{49\pi}{30}, \frac{53\pi}{30}, \frac{59\pi}{30}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 26	15	$\left\{\frac{8\pi}{5}, \frac{26\pi}{15}, \frac{26\pi}{15}, \frac{26\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 27	15	$\left\{\frac{6\pi}{5}, \frac{28\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 30	16	$\left\{\frac{7\pi}{4}, \frac{7\pi}{4}, \frac{7\pi}{4}, \frac{7\pi}{4}\right\}$	$\mathbb{Q}$
Sol. 32	18	$\left\{\frac{37\pi}{21}, \frac{37\pi}{21}, \frac{37\pi}{21}, \frac{41\pi}{21}\right\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 33	18	$\left\{\frac{4\pi}{3}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\right\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 34	18	$\left\{\frac{25\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}\right\}$	$\mathbb{Q}(\cos(\pi/7))$
Sol. 37	20	$\left\{\frac{47\pi}{30}, \frac{53\pi}{30}, \frac{19\pi}{10}, \frac{19\pi}{10}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 38	20	$\left\{\frac{41\pi}{30}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{59\pi}{30}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 39	24	$\left\{\frac{3\pi}{2}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 40	30	$\left\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 41	30	$\left\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 43	40	$\left\{\frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 44	40	$\left\{\frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}\right\}$	$\mathbb{Q}(\sqrt{5})$
Sol. 45	72	$\left\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$	$\mathbb{Q}(\sqrt{5})$

TABLE 4. Lisovyy-Tykhyy's classification of finite orbits of DT representations.

Sol. II	2	$\{\theta_1, \theta_1, \theta_2, \theta_2\}, \quad \theta_1 + \theta_2 > 3\pi$
Sol. III	3	$\{\frac{4\pi}{3}, 2\theta - 2\pi, \theta, \theta\}, \ \theta > 5\pi/3$
Sol. IV	4	$\{\pi, \theta, \theta, \theta\}, \ \theta > 5\pi/3$
Sol. IV*	4	$\{\theta, \theta, \theta, 3\theta - 4\pi\}, \ \theta > 5\pi/3$
Sol. 1	5	$\left\{\frac{22\pi}{15}, \frac{8\pi}{5}, \frac{8\pi}{5}, \frac{28\pi}{15}\right\}$
Sol. 4	6	$\left\{\frac{19\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{23\pi}{12}\right\}$
Sol. 6	6	$\left\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}\right\}$
Sol. 7	6	$\left\{\frac{17\pi}{15}, \frac{5\pi}{3}, \frac{29\pi}{15}, \frac{29\pi}{15}\right\}$
Sol. 8	7	$\left\{\frac{10\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}, \frac{12\pi}{7}\right\}$

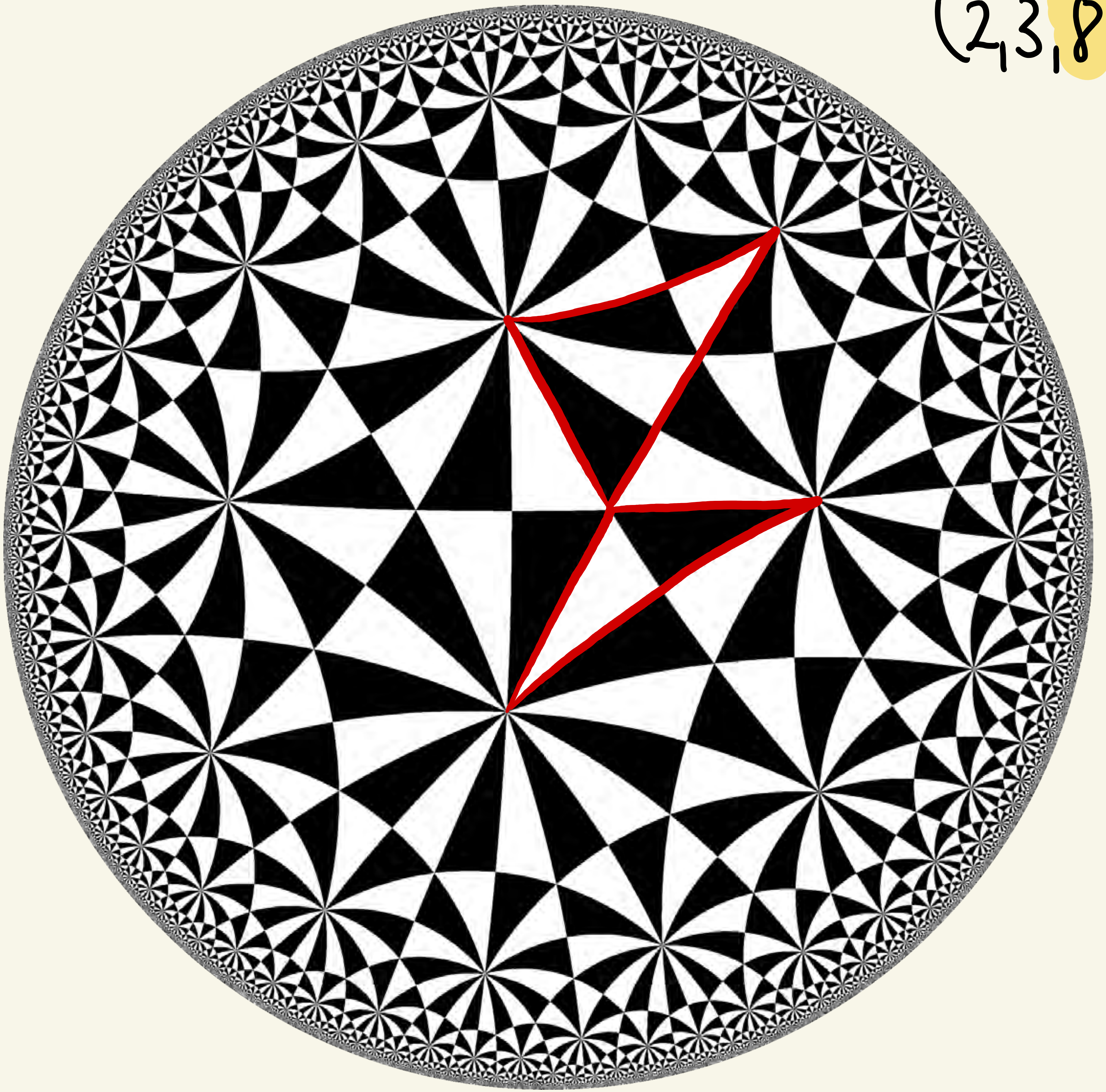
Sol. 34	18	$\left\{\frac{25\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}, \frac{41\pi}{21}\right\}$
Sol. 37	20	$\left\{\frac{47\pi}{30}, \frac{53\pi}{30}, \frac{19\pi}{10}, \frac{19\pi}{10}\right\}$
Sol. 38	20	$\left\{\frac{41\pi}{30}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{59\pi}{30}\right\}$
Sol. 39	24	$\left\{\frac{3\pi}{2}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$
Sol. 40	30	$\left\{\frac{23\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{28\pi}{15}\right\}$
Sol. 41	30	$\left\{\frac{26\pi}{15}, \frac{26\pi}{15}, \frac{29\pi}{15}, \frac{29\pi}{15}\right\}$
Sol. 43	40	$\left\{\frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}, \frac{17\pi}{10}\right\}$
Sol. 44	40	$\left\{\frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}, \frac{19\pi}{10}\right\}$
Sol. 45	72	$\left\{\frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}, \frac{11\pi}{6}\right\}$

(2,3,8)

length = 4

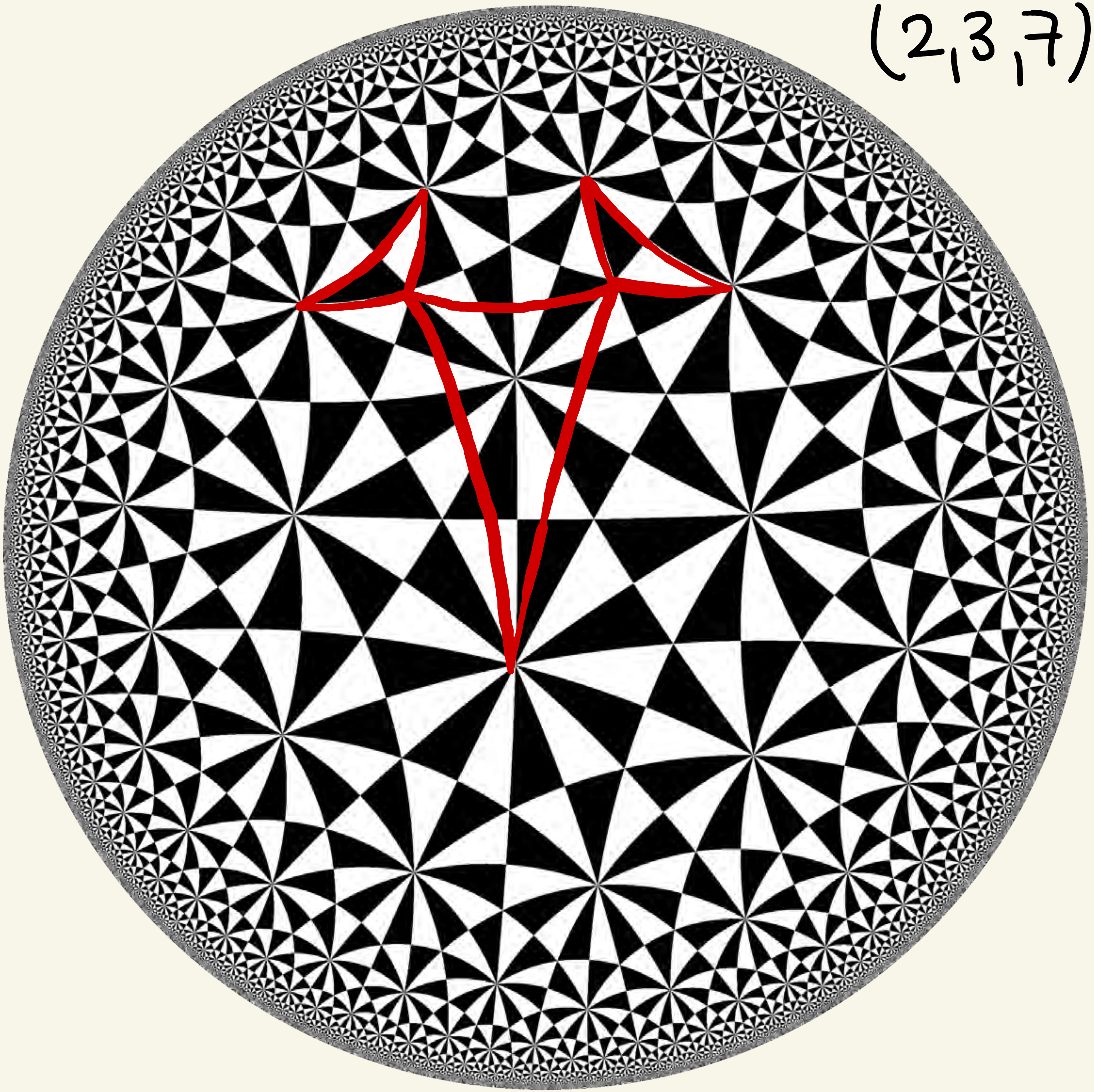
Type $\underline{\mathrm{IV}}^*$

(Dubrovin 96')



$n = 5$

length $= 105$

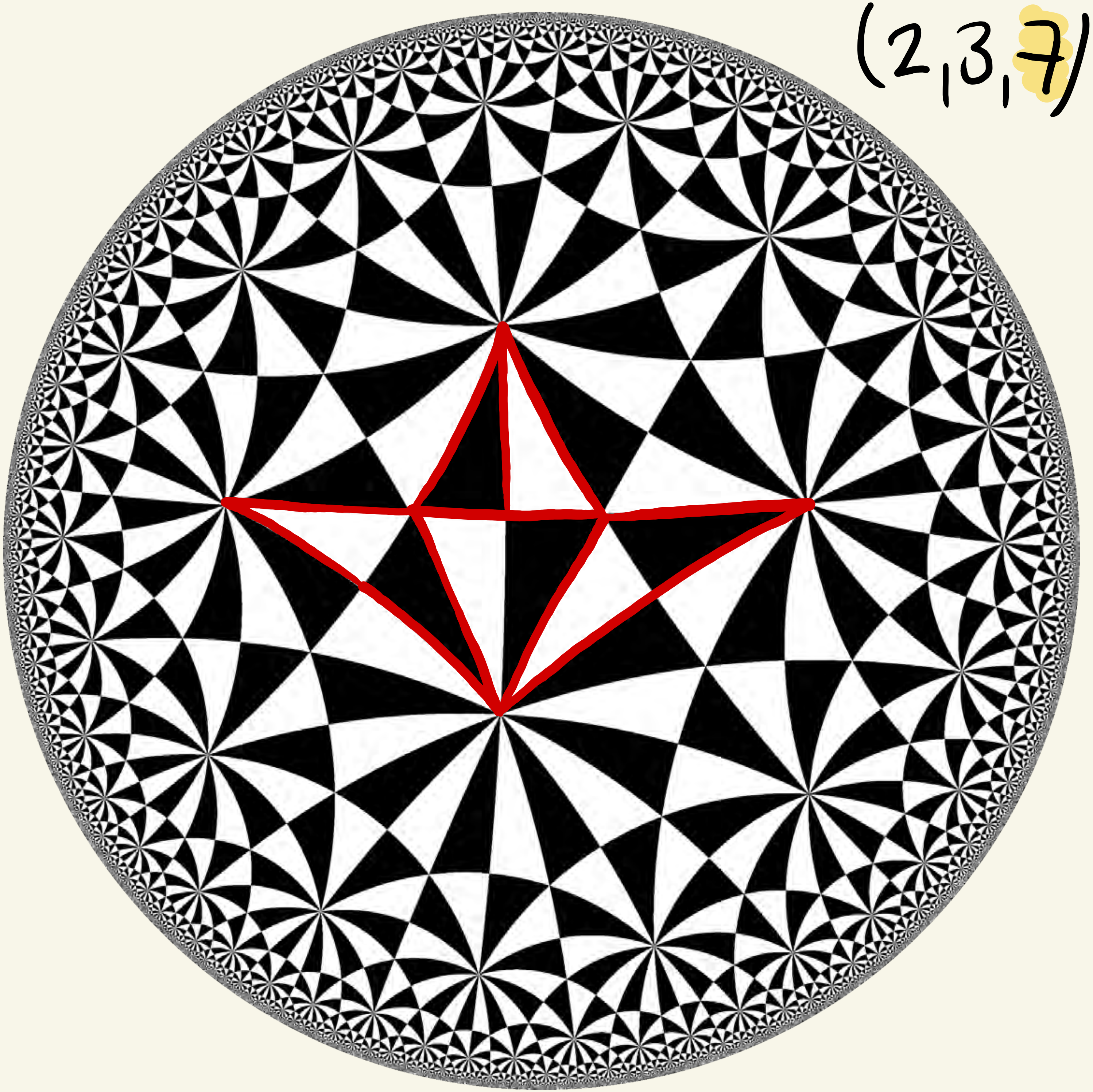


$(2,3,7)$

$(2,3,7)$

$n=6$

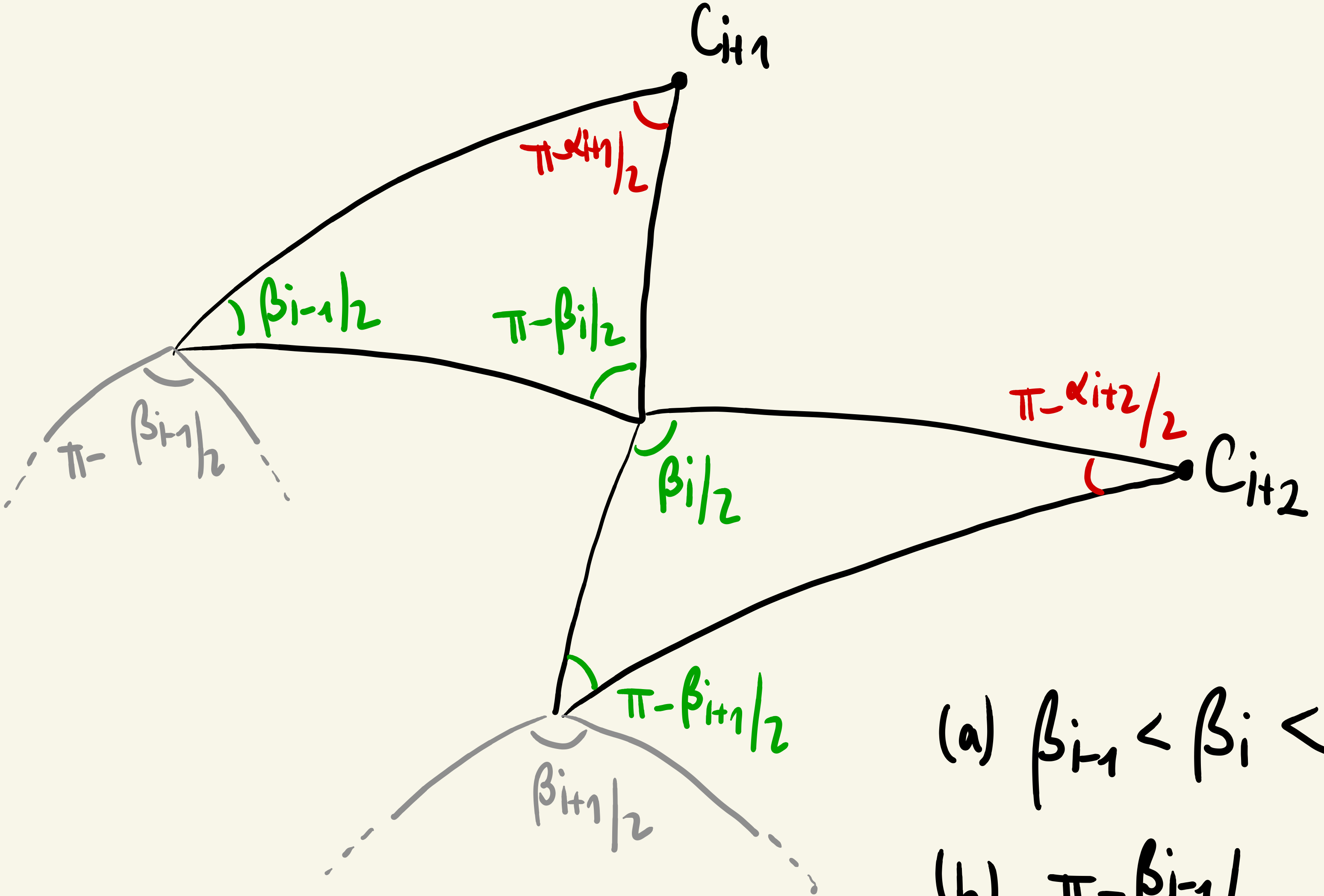
length $=40$



Classifying finite orbits arising from triangle chains

(not necessarily discrete)

* 2 triangles ($n=4$) $\rightsquigarrow$ Boalch / Lisovyy–Tykhyy
* $n-2$ triangles ($n \geq 5$)
  - each consecutive pair of triangles belongs to B/LT's list



(a) $\beta_{i-1} < \beta_i < \beta_{i+1}$

(b) $\pi - \beta_{i-1}/2$, $\beta_{i-1}/2$ are both exterior angles

$\leadsto$ a chain has at most 4 triangles, i.e. $n \leq 6$

Thank you!