Perturbative connection formulas for Heun equations

Oleg Lisovyy
Institut Denis Poisson, Université de Tours, France

in collaboration with Andrii Naidiuk
based on 2208.01604 [math-ph]

Motivation and goals

- Heun accessory parameters are conjecturally related to quasiclassical limit of Virasoro conformal blocks [Zamolodchikov, '86]
- Recently, Heun connection problem has also been conjecturally solved in terms of quasiclassical conformal blocks [Bonelli, Iossa, Panea, Tanzini, '21]

We want to understand how perturbative expansions following from this solution can be computed without CFT

Outline

1. Heun equations
2. CFT heuristics and Trieste formula
3. Darboux method and Schäfke-Schmidt formula
4. Perturbative solution of the Heun connection problem

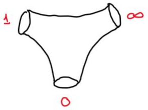
Hypergeometric equation

Consider $\psi''(z) = V(z)\,\psi(z)$, with

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\theta_\infty^2 - \theta_0^2 - \theta_1^2 + \frac{1}{4}}{z(z-1)}$$

Three regular singularities $\to$ 2nd order poles of the quadratic differential $V(z)\,dz^2$ on the Riemann sphere:

$$V(z) \sim \begin{cases} \frac{\theta_0^2 - \frac{1}{4}}{z^2} & \text{as } z \to 0, \\ \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} & \text{as } z \to 1, \\ \frac{\theta_\infty^2 - \frac{1}{4}}{z^2} & \text{as } z \to \infty. \end{cases}$$



- two 2nd order poles at $0, \infty$ correspond to $V(z) = \frac{\theta^2 - \frac{1}{4}}{z^2}$ (Euler's equation)

Frobenius solutions provide eigenbases of the operator of analytic continuation around singular points $z = 0, 1, \infty$. Their asymptotics is determined by the exponents $\theta_{0,1,\infty}$ of local monodromy, e.g.

$$\psi_{\pm}^{[0]}(z) = z^{\frac{1}{2}\mp\theta_0}(1-z)^{\frac{1}{2}-\theta_1}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2}\mp\theta_0-\theta_1-\theta_\infty,\ \frac{1}{2}\mp\theta_0-\theta_1+\theta_\infty \\ 1\mp 2\theta_0\end{array};z\right]$$

$$= z^{\frac{1}{2}\mp\theta_0}\left[1+O(z)\right] \qquad \text{as } z\to 0,$$

$$\psi_{\pm}^{[1]}(z) = (1-z)^{\frac{1}{2}\mp\theta_1}z^{\frac{1}{2}-\theta_0}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2}-\theta_0\mp\theta_1-\theta_\infty,\ \frac{1}{2}-\theta_0\mp\theta_1+\theta_\infty \\ 1\mp 2\theta_1\end{array};1-z\right]$$

$$= (1-z)^{\frac{1}{2}\mp\theta_1}\left[1+O(1-z)\right] \text{ as } z\to 1.$$

The exponents are encoded into the Riemann scheme

0	1	∞
$\frac{1}{2}-\theta_0$	$\frac{1}{2}-\theta_1$	$\frac{1}{2}-\theta_\infty$
$\frac{1}{2}+\theta_0$	$\frac{1}{2}+\theta_1$	$\frac{1}{2}+\theta_\infty$

(normal form)

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

(canonical form)

Eigenbases of Frobenius solutions are related by

$$\psi_{\epsilon}^{[0]}(z) = \sum_{\epsilon'} \mathsf{C}_{\epsilon\epsilon'} \psi_{\epsilon'}^{[1]}(z), \qquad \epsilon, \epsilon' = \pm.$$

The elements $\mathsf{C}_{\epsilon\epsilon'}$ of the connection matrix are expressed in terms of a single function

$$\mathsf{C}_{\epsilon\epsilon'} = \mathsf{C}\left(\epsilon\theta_0, \epsilon'\theta_1, \theta_\infty\right),$$

which in the hypergeometric case is given by

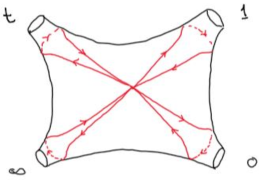
$$\boxed{\mathsf{C}\left(\theta_0, \theta_1, \theta_\infty\right) = \frac{\Gamma\left(1 - 2\theta_0\right)\Gamma\left(2\theta_1\right)}{\Gamma\left(\frac{1}{2} - \theta_0 + \theta_1 + \theta_\infty\right)\Gamma\left(\frac{1}{2} - \theta_0 + \theta_1 - \theta_\infty\right)}}$$

Heun equation

Potential:

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\theta_t^2 - \frac{1}{4}}{(z-t)^2} + \frac{\theta_\infty^2 - \theta_0^2 - \theta_1^2 - \theta_t^2 + \frac{1}{2}}{z(z-1)} + \frac{(1-t)\,\mathcal{E}}{z(z-1)(z-t)}$$

- 4 regular singular points $0,\ 1,\ \infty,\ t \Rightarrow$ 4 exponents θ_k
- 1 accessory parameter $\mathcal{E}$: not fixed by local monodromy
- we assume that $|t| > 1$

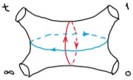


Space of monodromy data:

$$\mathcal{M} = \left\{ M_{0,1,\infty,t} \in \mathrm{SL}\left(2,\mathbb{C}\right) : M_\infty M_t M_1 M_0 = \mathbf{1}, \mathrm{Tr}\, M_k = -2\cos 2\pi\theta_k \right\} / \sim$$

- $\dim \mathcal{M} = 2$; $(\mathcal{E}, t)$ can be seen as a pair of local coordinates on $\mathcal{M}$
- another possibility is to use trace functions such as

$$\mathrm{Tr}\, M_0 M_1 = 2\cos 2\pi\sigma, \quad \mathrm{Tr}\, M_1 M_t = 2\cos 2\pi\sigma'$$



$$\begin{tikzcd}[column sep=huge]
\text{equation data: } (\mathcal{E}, t) \arrow[r, bend left, "\text{direct problem}"] & \text{monodromy data: } \sigma\text{'s} \arrow[l, bend left, "\text{inverse problem}"]
\end{tikzcd}$$

- any choice of a coordinate σ on $\mathcal{M}$ makes $\mathcal{E}$ a function of t depending on σ

"Mixed" problem:

$$\text{find } \mathcal{E}\,(t \,|\, \sigma) = \begin{array}{c} \text{reconstruct Heun equation from prescribed} \\ \text{monodromy } (\sigma) \text{ and singularity position } (t) \end{array}$$

Solved in terms of quasiclassical conformal blocks by Zamolodchikov conjecture

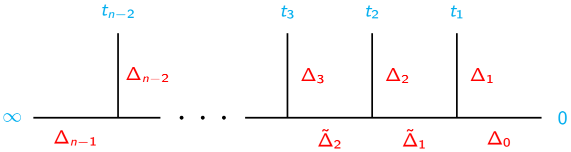
2. CFT heuristics & Trieste formula

Virasoro conformal blocks

Fix $n \geq 4$ distinct points $t_0, \ldots, t_{n-1}$ on $\mathbb{CP}^1$, using projective invariance to choose

$$t_0 = 0, \qquad t_1 = 1, \qquad t_{n-1} = \infty$$

and assuming that $|t_1| < |t_2| < \ldots < |t_{n-2}|$. Conformal block is a multivariate series assigned to a trivalent graph with n external edges, such as



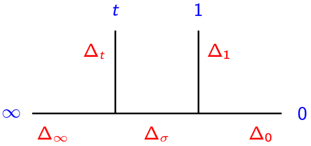
$$\mathcal{F}\left(\mathbf{t}, \boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right) = \prod_{\ell=1}^{n-3} t_\ell^{\tilde{\Delta}_\ell - \tilde{\Delta}_{\ell-1} - \Delta_\ell} \sum_{\mathbf{k} \in \mathbb{N}^{n-3}} \mathcal{F}_{\mathbf{k}}\left(\boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right) \left(\frac{t_1}{t_2}\right)^{k_1} \left(\frac{t_2}{t_3}\right)^{k_2} \ldots \left(\frac{t_{n-3}}{t_{n-2}}\right)^{k_{n-3}}$$

Remarks:

- coefs $\mathcal{F}_{\mathbf{k}}\left(\boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right)$ are fixed by the Virasoro commutation relations $\Longrightarrow$ rational functions of weights and central charge c.
- thanks to the AGT relation, there is an explicit combinatorial representation of $\mathcal{F}$ in terms of a sum over tuples of partitions.
- the series is convergent and analytic properties of $\mathcal{F}$ in each variable can be described using elementary braiding and fusion transformations.

Simplest nontrivial case: 4-point conformal block

$\mathcal{F}(t) =$



$$= t^{\Delta_\infty - \Delta_t - \Delta_\sigma}\left(1 + \sum_{k=1}^{\infty} \mathcal{F}_k t^{-k}\right)$$

- depends on 5 conformal weights Δ_k and the central charge c
- a generalization of the Gauss ${}_2F_1$ with 3 more parameters

Quasiclassical limit

Liouville parameterization:

$$c = 1 + 6Q^2, \qquad \Delta = \frac{Q^2}{4} - p^2, \qquad Q = b + b^{-1}.$$

We trade the central charge c and conformal weights Δ's for b and θ's and consider the scaling limit

$$p \to \infty, \qquad b \to 0, \qquad bp \to \theta.$$

Zamolodchikov conjecture

1. Conformal blocks have WKB type asymptotics

$$\mathcal{F}\left(\mathbf{t}; \{p_k\}\right) \sim \exp b^{-2} \mathcal{W}\left(\mathbf{t}; \{\theta_k\}\right)$$

   The series $\mathcal{W}\left(\mathbf{t}; \{\theta_k\}\right)$ is called quasiclassical conformal block.

2. The 4-point spherical quasiclassical conformal block is related to Heun accessory parameter function $\mathcal{E}\left(t \,|\, \sigma\right)$ by

$$\boxed{\mathcal{E} = t \frac{\partial \mathcal{W}}{\partial t}}$$

   where external θ_k's are Heun monodromy exponents and σ is similarly related to rescaled intermediate momentum.

Expansion of $\mathcal{W}$ at large t has the form

$$\mathcal{W}(t) = (\delta_\infty - \delta_\sigma - \delta_t)\ln t + \sum_{k=1}^{\infty} \mathcal{W}_k t^{-k},$$

where $\delta_\sigma = \frac{1}{4} - \sigma^2$, $\delta_k = \frac{1}{4} - \theta_k^2$ for $k = 0, 1, t, \infty$. First coefs are given by

$$\mathcal{W}_1 = \frac{(\delta_\sigma - \delta_0 + \delta_1)(\delta_\sigma - \delta_\infty + \delta_t)}{2\delta_\sigma},$$

$$\mathcal{W}_2 = \frac{(\delta_\sigma - \delta_0 + \delta_1)^2 (\delta_\sigma - \delta_\infty + \delta_t)^2}{8\delta_\sigma^2}\left(\frac{1}{\delta_\sigma - \delta_0 + \delta_1} + \frac{1}{\delta_\sigma - \delta_\infty + \delta_t} - \frac{1}{2\delta_\sigma}\right) + \\ + \frac{\left(\delta_\sigma^2 + 2\delta_\sigma(\delta_0 + \delta_1) - 3(\delta_0 - \delta_1)^2\right)\left(\delta_\sigma^2 + 2\delta_\sigma(\delta_\infty + \delta_t) - 3(\delta_\infty - \delta_t)^2\right)}{16\delta_\sigma^2(4\delta_\sigma + 3)}$$

The series for $t\dfrac{\partial\mathcal{W}}{\partial t}$ can be compared with the expansion of the accessory parameter function $\mathcal{E}(t\,|\,\sigma)$ order by order.

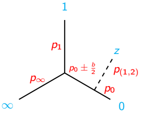
Remark: Quasiclassical conformal block $\mathcal{W}(t\,|\,\sigma)$ can be interpreted as the generating function of the canonical transformation $(\sigma, \eta) \to (\mathcal{E}, \ln t)$ on $\mathcal{M}$.

Degenerate fields

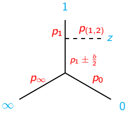
- Special fusion relations: the OPE of $\Phi_{(1,2)}(z)$ with a generic Virasoro primary with momentum p contains only two conformal families with momenta $p_{\pm} = p \pm \frac{b}{2}$.
- BPZ (Belavin-Polyakov-Zamolodhikov) constraints:

$$\mathcal{D}_{\mathrm{BPZ}}\mathcal{F}(\mathbf{t}, z) = 0$$

  - a linear PDE in position of fields
  - 2nd order in z, 1st order in positions of other fields
  - 3+1 points: hypergeometric equation in z



$= \mathcal{F}_{\pm p_0, p_1, p_\infty}(z)\,,$



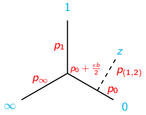
$= \mathcal{F}_{\pm p_1, p_0, p_\infty}(1-z)\,,$

where

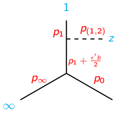
$$\mathcal{F}_{p_0, p_1, p_\infty}(z) = z^{\frac{1+b^2}{2}+bp_0}(1-z)^{\frac{1+b^2}{2}+bp_1}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+b(p_1+p_\infty+p_0)\,,\ \frac{1}{2}+b(p_1-p_\infty+p_0) \\ 1+2bp_0\end{matrix}\,;z\right]$$

Fusion transformations

Hypergeometric connection formulas for ${}_2F_1$'s can be interpreted as the fusion transformation for 3+1 point conformal blocks,



$$= \sum_{\epsilon'} \mathsf{F}_{\epsilon\epsilon'}\left(p_0, p_1, p_\infty\right)$$

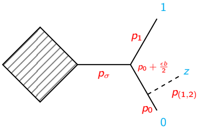


$$, \qquad \epsilon, \epsilon' = \pm$$

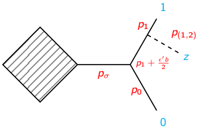
We have $\mathsf{F}_{\epsilon\epsilon'}\left(p_0, p_1, p_\infty\right) = \mathsf{F}\left(\epsilon p_0, \epsilon' p_1, p_\infty\right)$ and

$$\mathsf{F}\left(p_0, p_1, p_\infty\right) = \frac{\Gamma\left(1 - 2bp_0\right)\Gamma\left(2bp_1\right)}{\Gamma\left(\frac{1}{2} + b\left(p_1 - p_0 + p_\infty\right)\right)\Gamma\left(\frac{1}{2} + b\left(p_1 - p_0 - p_\infty\right)\right)}$$

Locality of the fusion transformations means that for more complicated conformal blocks



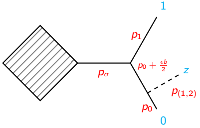
$$= \sum_{\epsilon'} \mathsf{F}_{\epsilon\epsilon'}\left(p_0, p_1, p_\sigma\right)$$



with the same fusion matrix F.

“Explanation” of Zamolodchikov conjecture

Plugging the WKB ansatz for the asymptotics of $(n+1)$-point conformal blocks



$$= \Psi_\epsilon\left(z;\mathbf{t}\right)\exp\left\{b^{-2}\mathcal{W}\left(\mathbf{t}\right)\right\}\left[1+o\left(1\right)\right] \quad \text{as } b \to 0.$$

into the BPZ constraint, the corresponding PDE becomes the generalized Heun's ODE (n Fuchsian singularities) for the amplitudes $\Psi_\pm\left(z\right)$,

$$\left[\frac{d^2}{dz^2} + \sum_{k=0}^{n-2}\frac{\delta_k}{\left(z-t_k\right)^2} + \frac{\delta_{n-1} - \sum_{k=0}^{n-2}\delta_k}{z\left(z-1\right)} + \sum_{k=2}^{n-2}\frac{\left(t_k-1\right)\mathcal{E}_k}{z\left(z-1\right)\left(z-t_k\right)}\right]\Psi_\pm\left(z\right) = 0,$$

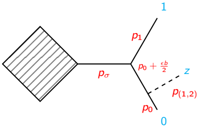
with $\delta_k = \frac{1}{4} - \theta_k^2$ and accessory parameters given by $\mathcal{E}_k = t_k\dfrac{\partial\mathcal{W}}{\partial t_k}$.

- n rescaled external momenta are related to local monodromy exponents $\frac{1}{2} \pm \theta_k$
- $n-3$ rescaled internal momenta such as $\sigma = bp_\sigma$ encode exponents of composite monodromy and parameterize accessory parameters $\mathcal{E}_2, \ldots, \mathcal{E}_{n-2}$
- we recover the usual Heun for $n = 4$

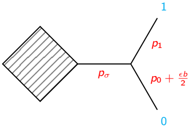
The structure of OPEs encoded in the conformal block diagrams implies that the amplitudes $\Psi_{\pm}(z)$ have $z \to 0$ expansions of the form

$$\Psi_{\pm}(z) = \mathcal{N}_{\pm} z^{\frac{1}{2} \mp \theta_0} \left[1 + \sum_{k=1}^{\infty} \Psi_{\pm,k} z^k \right],$$

and therefore give a basis of Frobenius solutions of the generalized Heun equation at $z = 0$. The normalization coefficients $\mathcal{N}_{\pm}$ are fixed by



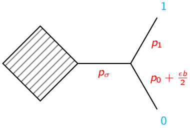
$$= z^{\frac{1+b^2}{2} - \epsilon b p_0} \Bigg[$$



$$+ O(z) \Bigg] \quad \text{as } z \to 0^+$$

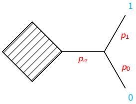
Indeed, taking the quasiclassical limit, we get

$$\mathcal{N}_\epsilon = \lim_{b \to 0}$$



$$\exp\left\{ b^{-2} \mathcal{W}(\mathbf{t}) \right\} = \mathcal{N} \exp\left\{ -\frac{\epsilon}{2} \frac{\partial \mathcal{W}}{\partial \theta_0} \right\}, \qquad \epsilon = \pm$$

where $\mathcal{N} = \lim_{b \to 0}$



$\exp\left\{ b^{-2} \mathcal{W}(\mathbf{t}) \right\} \Longrightarrow$ subleading term in the Zamolodchikov conjecture.

Trieste formula [Bonelli, Iossa, Panea, Tanzini, '21]

Denote by $\psi_{\pm}^{[0]}(z)$, $\psi_{\pm}^{[1]}(z)$ two pairs of normalized Frobenius solutions of the generalized Heun equation at $z=0$ and $z=1$:

$$\psi_{\pm}^{[0]}(z) = z^{\frac{1}{2}\mp\theta_0}\left[1+\sum_{k=1}^{\infty}\psi_{\pm,k}^{[0]}z^k\right],$$

$$\psi_{\pm}^{[1]}(z) = (1-z)^{\frac{1}{2}\mp\theta_1}\left[1+\sum_{k=1}^{\infty}\psi_{\pm,k}^{[1]}(z-1)^k\right].$$

The connection between the two bases is given by

$$\psi_{\epsilon}^{[0]}(z) = \sum_{\epsilon'} \mathsf{C}\left(\epsilon\theta_0, \epsilon'\theta_1, \sigma\right)\psi_{\epsilon'}^{[1]}(z), \qquad \epsilon, \epsilon' = \pm,$$

$$\boxed{\mathsf{C}\left(\theta_0,\theta_1,\sigma\right) = \mathsf{F}_{\mathsf{cl}}\left(\theta_0,\theta_1,\sigma\right)\exp\frac{1}{2}\left(\frac{\partial\mathcal{W}}{\partial\theta_1}-\frac{\partial\mathcal{W}}{\partial\theta_0}\right)}$$

where $\mathsf{F}_{\mathsf{cl}}\left(\theta_0,\theta_1,\theta_\infty\right) = \frac{\Gamma(1-2\theta_0)\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2}+\theta_1-\theta_0+\theta_\infty\right)\Gamma\left(\frac{1}{2}+\theta_1-\theta_0-\theta_\infty\right)}$ is the quasiclassical limit of the fusion matrix.

Practical implementation for Heun

1. Generate quasiclassical conformal block expansion

$$\mathcal{W}(t) = (\delta_\infty - \delta_\sigma - \delta_t)\ln t + \frac{(\delta_\sigma - \delta_0 + \delta_1)(\delta_\sigma - \delta_\infty + \delta_t)}{2\delta_\sigma} t^{-1} + O\left(t^{-2}\right)$$

2. Parameterize $\mathcal{E} = -\frac{1}{4} - \theta_\infty^2 + \omega^2 + \theta_t^2$ and compute the expansion of composite monodromy exponent $\sigma = \sigma(t)$ from $\mathcal{E} = t\dfrac{\partial \mathcal{W}}{\partial t}$:

$$\sigma(t) = \omega - \frac{\left(\frac{1}{4} - \omega^2 + \theta_0^2 - \theta_1^2\right)\left(\frac{1}{4} - \omega^2 + \theta_\infty^2 - \theta_t^2\right)}{4\omega\left(\frac{1}{4} - \omega^2\right)} t^{-1} + O\left(t^{-2}\right)$$

3. Plug both expansions into

$$C(\theta_0, \theta_1, \sigma) = \underbrace{\frac{\Gamma(1-2\theta_0)\,\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 + \sigma\right)\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 - \sigma\right)}}_{=\mathbf{F_{cl}}(\theta_0, \theta_1, \sigma)} \exp \frac{1}{2}\left(\frac{\partial \mathcal{W}}{\partial \theta_1} - \frac{\partial \mathcal{W}}{\partial \theta_0}\right)$$

General structure:

$$\ln \mathrm{C}\left(\theta_0,\theta_1,\sigma\right)=\ln \mathrm{F}_{\mathrm{cl}}\left(\theta_0,\theta_1,\omega\right)+\frac{1}{2}\left(\frac{\partial\mathcal{W}}{\partial\theta_1}-\frac{\partial\mathcal{W}}{\partial\theta_0}\right)$$
$$-\sum_{k=1}^{\infty}\left[\psi^{(k)}\left(\tfrac{1}{2}+\theta_1-\theta_0+\omega\right)+(-1)^k\,\psi^{(k)}\left(\tfrac{1}{2}+\theta_1-\theta_0-\omega\right)\right]\frac{(\sigma-\omega)^k}{k!}$$

Coefs of t^{-n} in $(\sigma-\omega)^k$ and $\frac{\partial\mathcal{W}}{\partial\theta_1}$, $\frac{\partial\mathcal{W}}{\partial\theta_0}$ are rational in $\theta_0,\theta_1,\theta_t,\theta_\infty$ and ω. Therefore, we have the expansion

$$\ln \mathrm{C}\left(\theta_0,\theta_1,\sigma\right)=\ln \mathrm{F}_{\mathrm{cl}}\left(\theta_0,\theta_1,\omega\right)+\sum_{k=1}^{\infty}\mathrm{f}_k t^{-k}$$

where f_k are given by linear combinations of polygamma functions, e.g.

$$\mathrm{f}_1=-\frac{\left(\frac{1}{4}-\omega^2+\theta_0^2-\theta_1^2\right)\left(\frac{1}{4}-\omega^2+\theta_\infty^2-\theta_t^2\right)}{4\omega\left(\frac{1}{4}-\omega^2\right)}\left[\psi\left(\tfrac{1}{2}+\theta_1-\theta_0+\omega\right)-\psi\left(\tfrac{1}{2}+\theta_1-\theta_0-\omega\right)\right]$$
$$-\frac{\left(\theta_0+\theta_1\right)\left(\frac{1}{4}-\omega^2+\theta_\infty^2-\theta_t^2\right)}{2\left(\frac{1}{4}-\omega^2\right)}$$

3. Darboux method and Schäfke-Schmidt formula

Motivating example. Consider the Taylor expansion of $u(z) = (1-z)^{-\theta}$ around $z = 0$:

$$u(z) = \sum_{k=0}^{\infty} u_k z^k, \qquad u_k = \frac{(\theta)_k}{k!} = \frac{\Gamma(k+\theta)}{\Gamma(\theta)\Gamma(k+1)}$$

- The ratio test ($\frac{u_{k+1}}{u_k} \overset{k\to\infty}{\longrightarrow} 1$) "detects" the position of the branch point $z = 1$
- The coefficients have the large k behavior

$$u_k = \frac{k^{\theta-1}}{\Gamma(\theta)}\left[1 + O\left(k^{-1}\right)\right] \qquad \text{as } k \to \infty.$$

It depends on the exponent θ which hints that such asymptotics can also capture the critical behavior of $u(z)$ at the branch point $z = 1$.

Darboux theorem (1878)

Let $u(z)$ be analytic in a neighborhood of $z = 0$. Suppose it has exactly one singularity $z = 1$ inside a disk $|z| = R > 1$. If $u(z)$ can be written in the form

$$u(z) = v(z) + (1 - z)^{-\theta} w(z), \qquad \theta \notin \mathbb{Z}$$

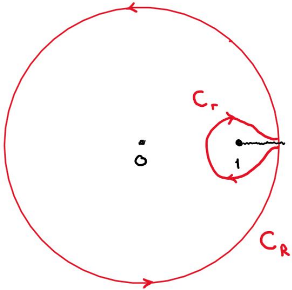
with $v(z)$, $w(z)$ analytic in a neighborhood of $z = 1$, then the coefficients of the Taylor expansion $u(z) = \sum_{k=0}^{\infty} u_k z^k$ at $z = 0$ have the asymptotics

$$u_k = \frac{w(1)}{\Gamma(\theta)} k^{\theta-1} \left[1 + O\left(k^{-1}\right)\right] \qquad \text{as } k \to \infty.$$

Proof idea:

$$u_k = \frac{1}{2\pi i} \oint_{C_R \cup C_r} z^{-k-1} u(z)\, dz$$

- the contribution of C_R is at most $O\left(R^{-k}\right)$
- plug the expression of $u(z)$ into $\oint_{C_r}$
- $\oint_{C_r} z^{-k-1} v(z)\, dz = 0$
- it suffices to estimate the asymptotics of $\oint_{C_r} z^{-k-1} (1 - z)^{-\theta} w(z)\, dz$



Application to connection problem

Consider a linear ODE $\psi''(z) = V(z)\psi(z)$ with potential

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{U(z)}{z(z-1)}$$

where $U(z)$ is holomorphic inside $|z| = R > 1$. Introduce normalized Frobenius solutions

$$\psi_{\pm}^{[0]}(z) = z^{\frac{1}{2} \mp \theta_0} \sum_{k=0}^{\infty} \psi_{\pm,k}^{[0]} z^k,$$

$$\psi_{\pm}^{[1]}(z) = (1-z)^{\frac{1}{2} \mp \theta_1} \sum_{k=0}^{\infty} \psi_{\pm,k}^{[1]} (1-z)^k,$$

with $\psi_{\pm,k}^{[0]} = \psi_{\pm,k}^{[1]} = 1$. The connection matrix relating the two bases is given by

$$\psi_{\epsilon}^{[0]}(z) = \sum_{\epsilon'} \mathsf{C}\left(\epsilon\theta_0, \epsilon'\theta_1\right) \psi_{\epsilon'}^{[1]}(z), \qquad \epsilon, \epsilon' = \pm$$

Theorem [Schäfke, Schmidt, '80]

Write the solution $\psi_+^{[0]}$ as

$$\psi_+^{[0]}(z) = z^{\frac{1}{2}-\theta_0}(1-z)^{\frac{1}{2}-\theta_1}\, u(z), \quad \text{with } u(z) = 1 + \sum_{k=1}^{\infty} u_k z^k .$$

Then

$$\boxed{\mathrm{C}(\theta_0, \theta_1) = \Gamma(2\theta_1) \lim_{k\to\infty} k^{1-2\theta_1} u_k}$$

Proof. Direct corollary of the Darboux theorem, with $v(z)$ and $w(z)$ coming from the Frobenius solutions at $z=1$.

4. Application to Heun equations

Schäfke-Schmidt theorem applies to all Heun equations with two Fuchsian singularities: the usual, confluent, and reduced confluent Heun. In the last case,

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\omega^2 - \theta_0^2 - \theta_1^2 + \frac{1}{4} + \lambda z}{z(z-1)}$$

- ω is the accessory parameter
- we look for perturbative expansion of the connection function C in λ
- for $\lambda = 0$:
  - the potential reduces to hypergeometric one with exponents

0	1	∞
$\frac{1}{2} \pm \theta_0$	$\frac{1}{2} \pm \theta_1$	$\frac{1}{2} \pm \omega$

  - the coefficients u_k are given by

$$u_k^{(\lambda=0)} = \frac{\left(\frac{1}{2} - \theta_0 + \theta_1 + \omega\right)_k \left(\frac{1}{2} - \theta_0 + \theta_1 - \omega\right)_k}{k!\,(1-2\theta_0)_k}$$

  - the C-function is the hypergeometric one

$$\mathrm{C}^{(\lambda=0)}(\theta_0,\theta_1) = \mathrm{F}_{\mathrm{cl}}(\theta_0,\theta_1,\omega) = \frac{\Gamma(1-2\theta_0)\,\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2} - \theta_0 + \theta_1 + \omega\right)\Gamma\left(\frac{1}{2} - \theta_0 + \theta_1 - \omega\right)}$$

Introducing rescaled coefficients $a_k = u_k / u_k^{(\lambda=0)}$, the Schäfke-Schmidt theorem can be reformulated as follows.

Proposition. The C-function of the reduced confluent Heun equation is given by

$$\mathrm{C}\left(\theta_0, \theta_1, \omega, t\right) = \mathrm{F}_{\mathrm{cl}}\left(\theta_0, \theta_1, \omega\right) \cdot a_\infty,$$

where $\{a_k\}$ satisfy the 3-term recurrence relation

$$a_{k+1} - a_k = -\lambda \beta_k a_{k-1}$$

subject to initial conditions $a_{-1} = 0$, $a_0 = 1$, with

$$\beta_k = -\frac{k\left(k - 2\theta_0\right)}{\left(\left(k - \frac{1}{2} - \theta_0 + \theta_1\right)^2 - \omega^2\right)\left(\left(k + \frac{1}{2} - \theta_0 + \theta_1\right)^2 - \omega^2\right)}$$

Formal solution:

$$a_\infty = \det \begin{pmatrix} 1 & -1 & & & \\ -\lambda\beta_1 & 1 & -1 & & \\ & -\lambda\beta_2 & 1 & -1 & \\ & & -\lambda\beta_3 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} = 1 - \lambda \sum_{k=1}^{\infty} \beta_k + \lambda^2 \sum_{k' \geq k+2}^{\infty} \beta_k \beta_{k'} + \ldots$$

Exponentiation gives a perturbative series involving only 1-fold sums:

$$-\ln a_\infty = \sum_{n=1}^{\infty} \frac{\mathrm{Tr}\, A^{2n}}{2n} \lambda^n, \qquad A = \begin{pmatrix} 0 & 1 & & & \\ \beta_1 & 0 & 1 & & \\ & \beta_2 & 0 & 1 & \\ & & \beta_3 & 0 & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

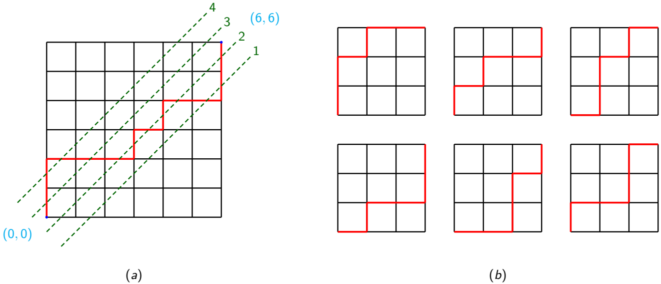
We have, for example,

$$\mathrm{Tr}\, A^2 = \sum_{k=1}^{\infty} 2\beta_k, \qquad \mathrm{Tr}\, A^4 = \sum_{k=1}^{\infty} \left(4\beta_k \beta_{k+1} + 2\beta_k^2\right),$$

$$\mathrm{Tr}\, A^6 = \sum_{k=1}^{\infty} \left(6\beta_k \beta_{k+1} \beta_{k+2} + 6\beta_k^2 \beta_{k+1} + 6\beta_k \beta_{k+1}^2 + 2\beta_k^3\right), \quad \ldots$$

NB: Since β_k is rational in k, all sums can be computed in terms of expressions rational in θ_0, θ_1, ω and polygammas $\psi^{(k)}\left(\frac{1}{2} - \theta_0 + \theta_1 \pm \omega\right) \Longrightarrow$ we recover the predictions of Trieste formula!

In general, $\operatorname{Tr} A^{2n} = \sum_{k=1}^{\infty} \sum_{\mu \vdash n}^{2^{n-1}} \mathcal{N}_\mu \cdot \beta_k^{\mu_1} \beta_{k+1}^{\mu_2} \ldots \beta_{k+\ell}^{\mu_\ell}$, where μ runs over all compositions (ordered partitions) of n and $\mathcal{N}_\mu$ are integers counting staircase walks of type μ.



(a) A staircase walk of type $(1, 3, 1, 1)$

(b) Six possible walks of type $(1, 2)$

Proposition. We have

$$\ln a_\infty = \sum_{k=1}^{\infty} \ln \left(1 - \cfrac{\lambda\beta_k}{1 - \cfrac{\lambda\beta_{k+1}}{1 - \cfrac{\lambda\beta_{k+2}}{1 - \dots}}} \right)$$

Proof. The determinant

$$D_k = \det \begin{pmatrix} 1 & -1 & & & \\ -\lambda\beta_k & 1 & -1 & & \\ & -\lambda\beta_{k+1} & 1 & -1 & \\ & & -\lambda\beta_{k+2} & 1 & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

satisfies a linear 3-term recurrence relation $D_k - D_{k+1} = -\lambda\beta_k D_{k+2}$. It can be transformed into a nonlinear 2-term Riccati equation for D_k/D_{k+1}, which is solved by the above infinite fraction. It remains to write

$$\ln a_\infty = \sum_{k=1}^{\infty} \ln \frac{D_k}{D_{k+1}}.$$

Remark. This also implies

$$\mathcal{N}_\mu = \frac{2n}{\mu_1} \prod_{\ell} \left(\frac{\mu_\ell + \mu_{\ell+1} - 1}{\mu_{\ell+1}} \right)$$

Theorem. Write the normal form of the RCHE as

$$\left[\frac{d^2}{dz^2}+\frac{\frac{1}{4}-\theta_0^2}{z^2}+\frac{\frac{1}{4}-\theta_1^2}{(z-1)^2}+\frac{\theta_0^2+\theta_1^2-\omega^2-\frac{1}{4}-\lambda z}{z(z-1)}\right]\psi(z)=0,$$

with $\theta_0,\theta_1\notin\mathbb{Z}/2$, and denote by $\psi_{\pm}^{[0]}(z)$, $\psi_{\pm}^{[1]}(z)$ its normalized Frobenius solutions at $z=0,1$. The connection between the two Frobenius bases is given by

$$\psi_{\epsilon}^{[0]}(z)=\sum_{\epsilon'=\pm}\mathrm{C}\left(\epsilon\theta_0,\epsilon'\theta_1\right)\psi_{\epsilon'}^{[1]}(z),\qquad\qquad \epsilon=\pm,$$

where $\mathrm{C}(\theta_0,\theta_1)$ admits the following representation in terms of continued fractions:

$$\boxed{\mathrm{C}(\theta_0,\theta_1)=\frac{\Gamma(1-2\theta_0)\,\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2}+\theta_1-\theta_0+\omega\right)\Gamma\left(\frac{1}{2}+\theta_1-\theta_0-\omega\right)}\exp\sum_{k=1}^{\infty}\ln\left(1-\frac{\lambda\beta_k}{1-\frac{\lambda\beta_{k+1}}{1-\ldots}}\right)}$$

with

$$\beta_k=\frac{k\,(k-2\theta_0)}{\left(\left(k+\frac{1}{2}-\theta_0+\theta_1\right)^2-\omega^2\right)\left(\left(k-\frac{1}{2}-\theta_0+\theta_1\right)^2-\omega^2\right)}.$$

Theorem. Write the normal form of the Heun equation as $\psi''(z) = V(z)\,\psi(z)$ with

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\theta_t^2 - \frac{1}{4}}{(z-t)^2} + \frac{\theta_\infty^2 - \theta_0^2 - \theta_1^2 - \theta_t^2 + \frac{1}{2}}{z(z-1)} + \frac{(1-t)\left(\omega^2 + \theta_t^2 - \theta_\infty^2 - \frac{1}{4}\right)}{z(z-1)(z-t)}$$

and assume that $|t| > 1$ and $\theta_0, \theta_1 \notin \mathbb{Z}/2$. The connection matrix relating the two normalized Frobenius bases is given by

$$\boxed{\begin{aligned} C(\theta_0,\theta_1) &= \frac{\Gamma(1-2\theta_0)\,\Gamma(2\theta_1)\,(1-\lambda)^{-\frac{1}{2}-\theta_t}}{\Gamma\left(\frac{1}{2}+\theta_1-\theta_0+\omega\right)\Gamma\left(\frac{1}{2}+\theta_1-\theta_0-\omega\right)} \times \\ &\times \exp\sum_{k=1}^{\infty} \ln\left(1 - \lambda\alpha_{k-1} - \frac{\lambda\beta_k}{1-\lambda\alpha_k - \frac{\lambda\beta_{k+1}}{1-\ldots}}\right)\end{aligned}}$$

with $\lambda = \frac{1}{t}$ and

$$\alpha_k = -\frac{\left(k+\frac{1}{2}-\theta_0-\theta_t\right)^2 - \theta_0^2 - \theta_\infty^2 + \omega^2}{\left(k+\frac{1}{2}-\theta_0+\theta_1\right)^2 - \omega^2},$$

$$\beta_k = \frac{k(k-2\theta_0)\left((k-\theta_0+\theta_1-\theta_t)^2 - \theta_\infty^2\right)}{\left(\left(k+\frac{1}{2}-\theta_0+\theta_1\right)^2-\omega^2\right)\left(\left(k-\frac{1}{2}-\theta_0+\theta_1\right)^2-\omega^2\right)}.$$

Conclusions

- Perturbative solution of the connection problem for Heun equations between two Fuchsian singularities can be systematically computed using the Schäfke-Schmidt formula
- It confirms Trieste formula expressing the connection coefficients in terms of quasiclassical Virasoro conformal blocks.
- It would be interesting to extend the method to irregular singularities and compare with CFT predictions of [Bonelli, Iossa, Panea, Tanzini, '21].
- A proof using extended symplectic structure of [Bertola, Korotkin, '19] ? connection formula for the PVI tau function ?