Deligne-Simpson Problem via Spectral Correspondence

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Joint work with Sukjoo Lee

Deligne-Simpson problem

- (Multiplicative) Deligne-Simpson problem (DSP):
  given conjugacy classes C_i in $GL_r(\mathbb{C})$ for $i = 1, \ldots, n$,
  when does there exist irreducible solutions to the equation

$$M_1 \ldots M_n = Id_r$$

  with $M_i \in C_i$. Here "irreducible" means M_i have no common invariant subspace.
- Recall that $\pi_1(\mathbb{C}P^1 \setminus \{p_1, \ldots, p_n\}) = \langle \gamma_1, \ldots, \gamma_n | \gamma_1 \ldots \gamma_n = e \rangle$.
- Equivalently, when does there exist irreducible local systems on $\mathbb{CP}^1 \setminus \{p_1, \ldots, p_n\}$ with prescribed conjugacy classes of the monodromies?
- Equivalently, when is the corresponding character variety non-empty?

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- Known results (increasing generality)
  - '90s Simpson: nonabelian Hodge correspondence.
  - '90s Katz: middle convolution for perverse sheaves.
  - '00s Kostov: reduction and deformation method.
  - '00s Crawley-Boevey-Shaw: representations of certain multiplicative preprojective algebra (sufficient condition).
  - A recent result announced by Cheng Shu completes the necessary condition conjectured by Crawley-Boevey-Shaw.

Variants of DSP

- DSP for higher genus: replace $\mathbb{CP}^1 \setminus \{p_1, \ldots, p_n\}$ by an n-punctured genus $g \geq 1$ Riemann surface.
- Hausel-Letellier-Rodriguez-Villegas: DSP for higher genus is always solvable for generic eigenvalues.
- Other versions of DSP: additive, irregular, other structure groups ...

Simpson's criterion on DSP

Denote

- $d_i = \dim(C_i)$
- $K = \sum_{i=1}^{n} d_i - 2(r^2 - 1)$ this is the expected dimension of the moduli space (character variety)
- $r_i = \min_{\lambda \in \mathbb{C}}\{\text{rank}(Y - \lambda I)\}$ with $Y \in C_i$.
  Note that $r - r_i =$ maximum number of Jordan blocks corresponding to the same eigenvalues.

Theorem (Simpson) For generic eigenvalues and when one of the conjugacy classes C_i has distinct eigenvalues, then DSP is solvable if and only if

1. $K \geq 0$
2. $r_1 + \cdots + \widehat{r_i} + \cdots + r_n \geq n$ for $i = 1, \ldots, n$.

NAHC on punctured curves

- Non-abelian Hodge correspondence (NAHC) due to Hitchin, Donaldson, Corlette, Simpson:

$$\{\text{stable Higgs bundles of rank } r \text{ degree } 0 \text{ on } C\}$$
$$\updownarrow$$
$$\{\text{irreducible local systems } \pi_1(C) \to GL_r(\mathbb{C})\}$$

- Generalization of NAHC to punctured curve due to Simpson:

$$\{\text{stable parabolic Higgs bundles of rank } r \text{ parabolic degree } 0 \text{ on } C\}$$
$$\updownarrow$$
$$\{\text{stable filtered local systems on } C \setminus \{p_1, \ldots, p_n\} \text{ of degree } 0\}$$

- When the filtration is trivial, stable filtered local systems are just irreducible local systems.
- Simpson constructs parabolic Higgs bundles as systems of Hodge bundles and then convert to local systems via NAHC.
- Our idea: use spectral correspondence to produce parabolic Higgs bundles

Higgs bundles

- Let C be a smooth algebraic curve/Riemann surface.
- Denote by K_C the canonical bundle of C.
- A Higgs bundle (E, ϕ) on C is a pair consisting of a vector bundle E on C and a Higgs field $\phi \in H^0(C, End(E) \otimes K_C)$.
- Locally, the Higgs field ϕ can be thought of as a matrix with entries in holomorphic 1-forms.
- (Spectral curve) Each Higgs bundle determines a finite cover $\widetilde{C} \subset Tot(K_C)$ over C, parametrizing the "eigenvalues" of the Higgs field ϕ.

Classical spectral correspondence

The classical spectral correspondence (due to Hitchin, Beauville-Narasimhan-Ramanan, Simpson) says that

stable Higgs bundles on C	$\xleftrightarrow{\text{spectral}}$	stable pure dimension one sheaves supported on spectral curves $\widetilde{C} \subset Tot(K_C)$

- When the spectral curve $\widetilde{C}$ is smooth, the pure dimension one sheaves are just line bundles on $\widetilde{C}$, parametrizing the "eigenspaces".
- Let $\overline{Tot(K_C)} = \mathbb{P}(K_C \oplus \mathcal{O})$ be the compactification of $Tot(K_C)$.
- At the level of moduli spaces: $\mathcal{H}^{hol} \cong \mathcal{X}$
  - $\mathcal{H}^{hol}$ is the moduli of stable Higgs bundles of rank r on C.
  - $\mathcal{X}$ is the moduli of stable pure dimension one sheaves F on $\overline{Tot(K_C)}$ with curve class

$$\Sigma := rC_0, \quad \text{where } C_0 \text{ is the zero section}$$

compactly supported on $Tot(K_C)$.

Hitchin map

- (Hitchin base) $B^{hol} = \bigoplus_{i=1}^{r} H^0(C, K_C^{\otimes i})$.
- There is the well-known Hitchin map $\mathcal{H}^{hol} \to B^{hol}$ defined by

$$(E, \phi) \mapsto (tr(\phi), tr(\wedge^2 \phi), \ldots, \det(\phi))$$

- We also have the Fitting support map $\mathcal{X} \to |rC_0|^\circ$ which sends a pure dimension one sheaf to its Fitting support.
- Here $|rC_0|^\circ \subset |rC_0|$ is the (open) affine subspace of effective divisors lying in $Tot(K_C)$.
- They fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{hol} & \xrightarrow{\sim} & \mathcal{X} \\ \downarrow & & \downarrow \\ B^{hol} & \xrightarrow{\sim} & |rC_0|^\circ \end{array}$$

Parabolic Higgs bundles

- Fix $p \in C$, a partition of r $\underline{m} = m_1 \geq \cdots \geq m_\ell$ with $\sum m_j = r$.
- A parabolic Higgs bundle $(E, \phi, E_p^\bullet, \underline{\alpha})$ consists of
  - vector bundle E on C
  - (meromorphic) Higgs field $\phi \in H^0(C, End(E) \otimes K_C(p))$
  - quasi-parabolic structure (partial flag) $E_p^\bullet : 0 = E_p^\ell \subset \cdots \subset E_p^1 \subset E_p^0 = E|_p$ such that $res_p(\phi)$ preserves the partial flag and $\dim(E_p^{j-1}/E_p^j) = m_j$
  - parabolic weights $1 > \alpha_\ell > \ldots > \alpha_1 \geq 0$
- A parabolic Higgs bundle is called $\underline{\xi}$-parabolic if $res_p(\phi)|_{E_p^{j-1}/E_p^j} = \xi_j \cdot Id_{E_p^{j-1}/E_p^j}$ where $\underline{\xi} = (\xi_1, \ldots, \xi_\ell) \in \mathbb{C}^\ell$.
- In particular, $(\underline{\xi}, \underline{m})$ are the eigenvalues of $res_p(\phi)$ and their multiplicities.
- When $\underline{\xi} = 0$, they are usually called strongly parabolic.

Parabolic spectral correspondence

- A result of Diaconescu-Donagi-Pantev (DDP) (also Kontsevich-Soibelman and Szabo) shows that for generic $\underline{\xi}$

$$\mathcal{H}(\underline{m})_{\underline{\xi}} \cong \mathcal{M}(\underline{m})_{\underline{\xi}}$$

  where

  - $\underline{\xi}$ is generic if the ξ_j's are mutually distinct.
  - $\mathcal{H}(\underline{m})_{\underline{\xi}}$ is the moduli of $\underline{\xi}$-parabolic Higgs bundles with flag type $\underline{m}$.
  - $\mathcal{M}(\underline{m})_{\underline{\xi}}$ is the moduli of stable pure dimension one sheaves on a surface $Z_{\underline{\xi}}$ with a given curve class

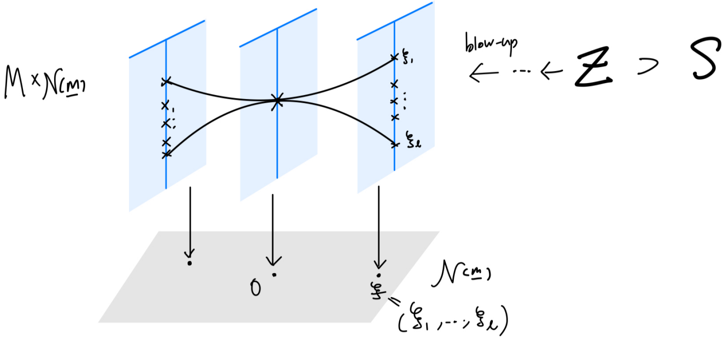
$$\Sigma(\underline{m})_{\underline{\xi}} \tag{1}$$

    compactly supported on an open subset $S_{\underline{\xi}} \subset Z_{\underline{\xi}}$.

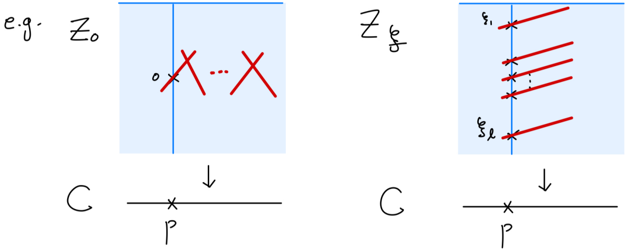
- In fact, they did it for irregular Higgs bundles.
- Q: what are $Z_{\underline{\xi}}$ and $\Sigma(\underline{m})_{\underline{\xi}}$?

Family of surfaces

- $\mathcal{N}(\underline{m}) = \{\underline{\xi} = (\xi_1, \ldots, \xi_\ell) | \sum m_j \xi_j = 0\}$.
- $M = \mathbb{P}(K_C(p) \oplus \mathcal{O})$.



Curve classes



- $S_{\underline{\xi}}$ is the complement of the (strict transform of) fiber over p and the infinity divisor in $Z_{\underline{\xi}}$. ("remove the blue lines")
- $f : Z_{\underline{\xi}} = M_\ell \to \cdots \to M_1 \to M \to C$ and $p^j_\ell : Z_{\underline{\xi}} \to M_j$.
- $C_0 \subset M$ the zero section and $E_j \subset M_j$ the exceptional divisor.
- $$\Sigma(\underline{m})_{\underline{\xi}} := rf^* C_0 - \sum_{i=1}^{\ell} m_i \Xi_i, \quad \text{where } \Xi_i = (p^j_\ell)^* E_j$$

Relative spectral correspondence

- In joint work with Sukjoo Lee, we extend the construction of DDP to all $\underline{\xi} \in \mathcal{N}(\underline{m})$: there exists a closed embedding

$$\begin{array}{ccc} \boldsymbol{Q} : \boldsymbol{\mathcal{H}}(\underline{m}) & \hookrightarrow & \boldsymbol{\mathcal{M}}(\underline{m}) \\ & \searrow \quad \swarrow & \\ & \mathcal{N}(\underline{m}) & \end{array}$$

  where
  - $\boldsymbol{\mathcal{H}}(\underline{m})$ is the relative moduli of $\underline{\xi}$-parabolic Higgs bundles over $\mathcal{N}(\underline{m})$.
  - $\boldsymbol{\mathcal{M}}(\underline{m})$ is the relative moduli space of stable pure dimension one sheaves on $\boldsymbol{Z}/\mathcal{N}(\underline{m})$ with the relative curve class $\boldsymbol{\Sigma}(\underline{m})/\mathcal{N}(\underline{m})$ and compactly supported on $\boldsymbol{S}/\mathcal{N}(\underline{m})$.
- In particular, the parabolic structures (flags) are encoded in the sheaves.
- $\boldsymbol{Q}$ is an isomorphism over (1) generic $\underline{\xi} \in \mathcal{N}$ and (2) locus of integral curves.
- For $\underline{\xi} = 0$ (the strongly parabolic case), this agrees with a result of Su-Wang-Wen.

Parabolic Hitchin maps

- Everything we said extends naturally from $D = p$ to $D = p_1 + \cdots + p_n$.
- Notation shift: we will write $\vec{m}$ and $\vec{\xi}$ when $D = p_1 + \cdots + p_n$.
- Parabolic Hitchin map: $\mathcal{H}(\vec{m})_{\underline{\xi}} \to A := \bigoplus_{\mu=1}^{r} H^0(C, K_C(D)^{\otimes \mu})$.
- Never surjective.
- For $\underline{\xi} = 0$, Baraglia-Kamgarpour and Su-Wang-Wen show that the image lies in

$$A(\vec{m})_0 = \bigoplus_{\mu=1}^{r} H^0\left(C, L(\vec{m})_\mu\right) \subset A.$$

where $L(\vec{m})_\mu$ is a subsheaf of $(K_C(D))^{\otimes \mu}$:

$$L(\vec{m})_\mu := (K_C(D))^{\otimes \mu} \otimes \mathcal{O}_C \left(- \sum_{i=1}^{n} (\gamma_{P^i}(\mu)\, p_i) \right) \tag{2}$$

and $\gamma_{P^i} : \{1, \ldots, r\} \to \{1, \ldots, r\}$ is a function determined by the partition (flag type) $P^i = (\underline{m}_i)$ at p_i.

Parabolic Hitchin map vs Fitting support map

- How about the image for $\vec{\xi} \neq 0$?
- Let $\boldsymbol{A}(\vec{m}) = A \times \mathcal{N}(\vec{m})$.
- Let $B(\vec{m})_{\vec{\xi}} \subset |\Sigma(\vec{m})_{\vec{\xi}}|$ affine subspace of effective divisors lying in $S_{\vec{\xi}}$.
- Call $B(\vec{m})_{\vec{\xi}}$ a parabolic Hitchin base.
- Vary $B(\vec{m})_{\vec{\xi}}$ over $\mathcal{N}(\vec{m}) \rightsquigarrow$ family of parabolic Hitchin bases $\boldsymbol{B}(\vec{m})$.
- Then we have the Fitting support map $Fitt : \boldsymbol{\mathcal{M}}(\vec{m}) \to \boldsymbol{B}(\vec{m})$ which sends a pure dimension one sheaf to its Fitting support.
- The Hitchin map and Fitting support map fit into the diagram

$$\begin{array}{ccc} \boldsymbol{\mathcal{H}}(\vec{m}) & \xhookrightarrow{\boldsymbol{Q}} & \boldsymbol{\mathcal{M}}(\vec{m}) \\ \downarrow & & \downarrow \\ \boldsymbol{A}(\vec{m}) & \xhookleftarrow[\iota]{} & \boldsymbol{B}(\vec{m}) \end{array}$$

  where ι is a natural inclusion.
- Q: is $B(\vec{m})_{\vec{\xi}}$ non-empty?
- Q: does $\boldsymbol{B}(\vec{m})$ forms a bundle over $\mathcal{N}(\vec{m})$?

Balasubramanian-Distler-Donagi (BDD)

- In the context of 6D superconformal field theory, Balasubramanian-Distler-Donagi consider the relative meromorphic Hitchin systems over the moduli space of stable pointed curves $\overline{\mathcal{M}_{g,n}}$.
- According to the classification of $4D$ $N = 2$ theories, there is a OK/bad dichotomy.
- Mathematically, the dichotomy is translated to a condition in terms of the Hitchin bases.
- The collection of line bundles $L(\vec{m})_\mu$, for $\mu = 2, \ldots, r$ satisfies the *OK condition* if

$$H^1\left(C, L(\vec{m})_\mu\right) = 0. \quad (3)$$

- They show that the OK condition implies that their family of Hitchin bases (in their setup) over $\overline{\mathcal{M}_{g,n}}$ forms a vector bundle.
- Conjecture (BDD): the OK condition implies the sufficiency of the Deligne-Simpson problem.

Theorem (L-Lee) If the OK condition holds, then

1. $B(\vec{m})_{\vec{\xi}}$ is non-empty for all $\vec{\xi} \in \mathcal{N}(\vec{m})$;
2. $\boldsymbol{B}(\vec{m})$ forms an affine bundle over $\mathcal{N}(\vec{m})$.

Sketch of proof for part (1):

- Recall that $\Sigma(\vec{m})_{\vec{\xi}} := rf^*C_0 - \sum_{i,j} m_{i,j}\Xi_{i,j}$ where $\Xi_{i,j}$ are pullback of exceptional divisors.
- (Constraint problem I) An element in $B(\vec{m})_{\vec{\xi}} \subset |\Sigma(\vec{m})_{\vec{\xi}}|$ can be characterized as a spectral curve in the ruled surface M passing through the points $\vec{\xi}$ (and its blow-ups) with some prescribed multiplicities determined by $\vec{m}$.
- (Constraint problem II) A spectral curve in M is defined by $y^r + s_1 y^{r-1} + \cdots + s_r = 0$, so the problem is further translated into the existence of sections s_μ of line bundles whose local derivatives satisfy certain system of linear equations.

Solving the constraint problem

- Let L be a line bundle and $L' = L\left(-\sum_{i=1}^{n} t(p_i)p_i\right)$ where $t(p_i) \geq 0$.
- There is the long exact sequence

$$\ldots \to H^0(C, L) \xrightarrow{q} H^0(C, L|_{t(p_1)p_1+\cdots+t(p_n)p_n}) \to H^1(C, L') \to \ldots \quad (4)$$

- If $H^1(C, L') = 0$, then the map q is surjective i.e. there exists a section in L whose local derivatives around p_i can be any given value in $\mathbb{C}$ up to order $t(p_i) - 1$.
- The OK condition $H^1(C, L(\vec{m})_\mu) = 0$ provides exactly such surjectivity to obtain a solution to constraint problem II.

Back to Deligne-Simpson problem

- Want to use the non-emptiness of the parabolic Hitchin base $B(\vec{m})_{\vec{\xi}}$ to deduce DSP.
- Under NAHC, we need to produce a parabolic Higgs bundles that is
  1. stable of parabolic degree 0
  2. its residues at the p_i's have the prescribed conjugacy classes.
- For generic enough $\vec{\xi}$, a curve in $B(\vec{m})_{\vec{\xi}}$ is integral.
  If we pick a line bundle over it, the corresponding Higgs bundle on C will be automatically stable.

Conjugacy classes of residues

- For simplicity, suppose $D = p$ and $\underline{\xi} = 0$.
- Fix $\underline{m} = (m_1, ..., m_\ell)$.
- Let $X \in B(\underline{m})_{\underline{\xi}}$ be an integral curve.
- Let L be a line bundle on X.
- Let (E, Φ) be the induced Higgs bundle on C.
- Lemma The conjugacy class (Jordan normal form) of $res_p(\Phi)$ with eigenvalue 0 is given by the conjugate partition of $\underline{m}$.

Jordan normal form (elementary linear algebra)

- V a vector space of dimension r.
- $B : V \to V$ a nilpotent operator.
- The Jordan normal form is determined by the sequence of subspaces:

$$0 \subset W_1 \subset \cdots \subset W_\ell = V, \quad \text{where } W_j := \ker(B^j)$$

  for some ℓ.
- Let $b_j = \dim(W_j/W_{j-1})$ for $j = 1, \ldots, \ell$.
- Let u_j the number of Jordan blocks of size j.
- Then u_j can be computed as follows:

$$\begin{aligned} u_1 + u_2 + \cdots + u_\ell &= b_1 \\ u_2 + \cdots + u_\ell &= b_2 \\ &\vdots \\ u_\ell &= b_\ell \end{aligned}$$

- Then the Jordan normal form of B in the form of partition is given by the conjugate partition of $(b_1, \ldots, b_\ell)$.

Intersection pattern

On the other hand, if we write $\Xi_i = E_i + \cdots + E_\ell$ and $\Sigma = \Sigma(\underline{m})_{\underline{\xi}}$, then their intersections satisfy

$$\begin{aligned} E_1 \cdot \Sigma + E_2 \cdot \Sigma + \cdots + E_\ell \cdot \Sigma = \Xi_1 \cdot \Sigma &= m_1 \\ E_2 \cdot \Sigma + \cdots + E_\ell \cdot \Sigma = \Xi_2 \cdot \Sigma &= m_2 \\ &\vdots \\ E_\ell \cdot \Sigma = \Xi_\ell \cdot \Sigma &= m_\ell \end{aligned}$$

The similar pattern suggests that the Jordan normal form is encoded in the intersection numbers.

DSP via spectral correspondence

We can now prove the BDD's conjecture on DSP.
Strategy:

1. Choose suitable $\vec{m}, \vec{\xi}$ to match the desired conjugacy classes in DSP.
2. The OK condition implies that $B(\vec{m})_{\vec{\xi}}$ is non-empty.
3. Pick an integral curve X in $B(\vec{m})_{\vec{\xi}}$. Then

$$\begin{array}{c}\text{Line bundles on } X \\ \big\downarrow \text{spectral correspondence} \\ \text{stable parabolic Higgs bundles of rank } r \text{ parabolic degree 0 on } C \text{ with} \\ \text{prescribed conjugacy classes of the residues} \\ \big\downarrow \text{NACH} \\ \text{stable filtered local systems on } C \setminus \{p_1, \ldots, p_n\} \text{ of degree 0 with prescribed} \\ \text{conjugacy classes of the monodromy} \\ \big\downarrow \text{when } C = \mathbb{CP}^1 \\ \text{Solutions to DSP}\end{array}$$

Final result

- Identify each conjugacy class C_i with a collection of partitions $\{P^{\lambda_{i,1}}, \ldots, P^{\lambda_{i,e(i)}}\}$, labeled by the eigenvalues $\lambda_{i,j}$ of C_i.
- Then we define the following partition of r

$$P^i := \widehat{P}^{\lambda_{i,1}} \cup \cdots \cup \widehat{P}^{\lambda_{i,e(i)}}$$

where $\widehat{P}^{\lambda_{i,j}}$ is the conjugate partition of $P^{\lambda_{i,j}}$.

Theorem (L-Lee) Let $n \geq 3$. Let $C_1, \ldots, C_n \subset GL_r(\mathbb{C})$ be a collection of conjugacy classes whose collection of eigenvalues is multiplicatively generic. Suppose that the following conditions hold:

1. $\prod_{i=1}^n \det(C_i) = 1$.
2. $\sum_{i=1}^n \gamma_{P^i}(\mu) < (n-2)\mu + 2$ for $\mu = 2, \ldots, r$.

Then the DSP is solvable for the tuple of conjugacy classes $(C_1, \ldots, C_n)$.

Remarks

- Condition (2) in the theorem above is a reformulation of the OK condition, hence confirming the conjecture of BDD from the viewpoint of parabolic Hitchin bases $B(\vec{m})_{\vec{\xi}}$.
- Also, our approach extends naturally to curves of genus > 0 and provide another proof of DSP for higher genus.
- Since the spectral correspondence developed by Kontsevich-Soibelman, Szabo, Diaconescu-Donagi-Pantev applies more generally to irregular Higgs bundles, we expect our approach to produce solutions to the irregular DSP as well.

The End

Thank you!