On the generic existence of Stokes graph / WKB spec net

Classical story

C : compact Riemann surface

Q : mero quadratic diff $\in \Gamma(C, K^{\otimes 2}(D))$

$\quad \parallel$ local

$\quad f(z)\, dz^{\otimes 2}$

Def $\theta \in S^1$

A θ-trajectory of Q (θ-preStokes curve)

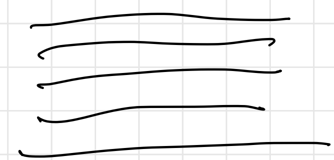
$:\Longleftrightarrow$ A real curve satisfying

$$\operatorname{Im}\left(e^{i\theta}\sqrt{f}\, dz\right) = 0$$

local

e.g. ① $Q = dz^{\otimes 2}$ on $\mathbb{C}$

$\operatorname{Im}(dz) = 0$



② $Q = z\, dz^{\otimes 2}$ on $\mathbb{C}$

Def Q is strongly GMN

:$\Leftrightarrow$
1. $\forall$zero is simple
2. $\exists$zero
3. $\forall$pole is order ≥ 2
4. $\exists$pole

$\Longrightarrow$ Strebel, Bridgeland–Smith

$\exists V \subset S^1$ open dense subset s.t.

$\forall \theta \in V$ $\nexists$ closed

$\nexists$ recurrent trajectory

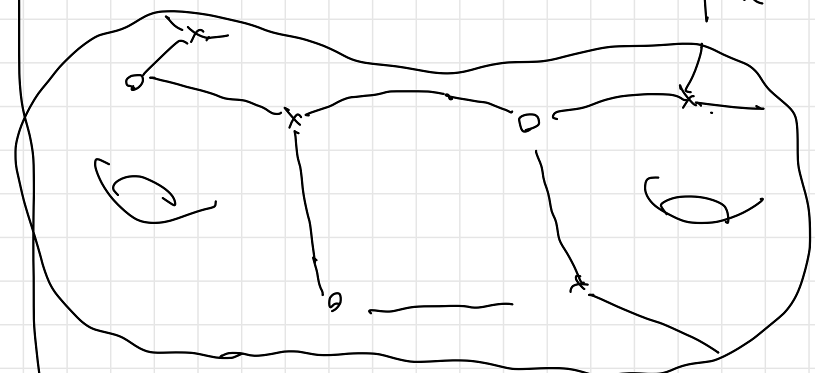
$\nexists$ saddle

For $\theta \in V$,

draw θ-preStokes curves from the zeros of Q

$\times$: zero

$\circ$: pole

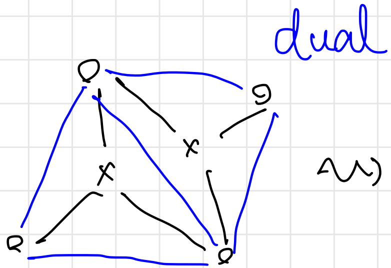


Stokes graph

on

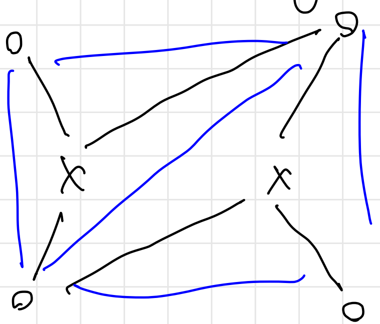
Spectral network of Q

①

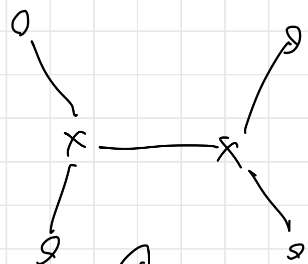


$\rightsquigarrow$ Ideal triangulation of C

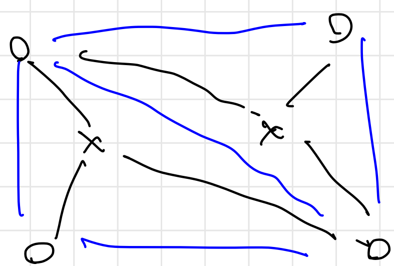
Rotating θ



$\longleftrightarrow$



$\updownarrow$



flip

$\rightsquigarrow$ Fock-Goncharov coordinate on Character Variety

② $\left((\hbar\partial)^2 - Q\right)\psi = 0$

$\rightsquigarrow$ formal WKB sol

$\rightsquigarrow$ $\exists$ resummation on the complement of Stokes graph (if $\nexists$ saddles)

③ QFT ....

⇝ Higher order/rank/deg ver. ?

Higher Teichmüller,

Higher order WKB, QFT, ...

Previous works

Honda – Kawai – Takei, Gaiotto – Moore – Neitzke (physics)

Today we consider along their line, give a mathematical foundation.

⇓

Contents

1. Def, proofs

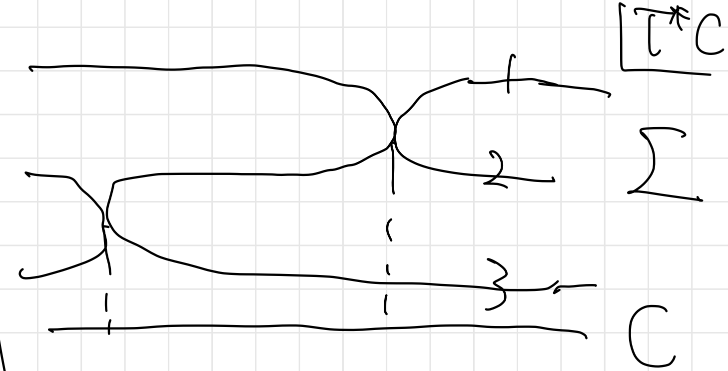
2. Applications.

1. <u>Setup</u>: C : cpt R.S.

$$\Phi = (\Phi_k, \dots, \Phi_1)$$
$$\| $$
$$\bigoplus_{i=1}^{k} \Phi_i \in \bigoplus_{i=1}^{k} \Gamma(C, K^{\otimes i}(D))$$

$$\rightsquigarrow \Sigma := \{\xi^n - \sum \xi^{n-i}\Phi_i = 0\} \subset T^*C$$

ramified covering with n sheets $\Big\downarrow$ C



$$\lambda = \xi\, dz$$
(canonical 1-form)
Liouville

<u>Def</u> For a pair of sheets (ij)

θ-preStokes curve of type (ij)

$\Longleftrightarrow$ A curve satisfying

$$\mathrm{Im}(\lambda|_i - \lambda|_j) = 0$$

(Orientation put by $\mathrm{Re}(\lambda|_i - \lambda|_j) > 0$)

e.g. $\Phi = \Phi_2 = Q$

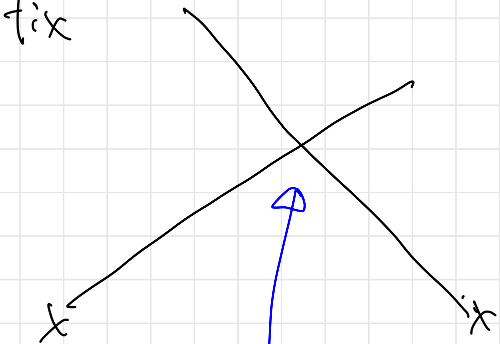
$$\Sigma = \{\xi^2 - Q = 0\}$$

$$\lambda|_{\text{sheet}} = \pm\sqrt{Q}$$

$$\lambda|_1 - \lambda|_2 = \pm 2\sqrt{Q}$$

zero $\rightsquigarrow$ branching value

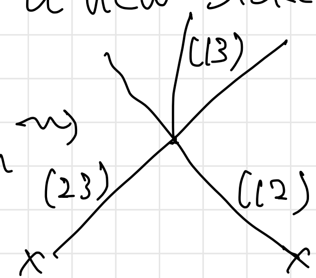
θ : fix



New phenomenon

"Intersection"

WKB/QFT demands us that Put a new Stokes line!



$\exists$ collision $\rightsquigarrow$

Collision ⇝ generation ⇝ ......

Roughly speaking,

the final resulting object is

<u>WKB spectral network</u>.

<u>Q</u>

① Want to exclude pathological (e.g. recurrent) network.

② Want to make a meaningful picture.
(If ∃ ∞ly many lines, the picture is not nice.)

③ Is the process well-def?

④ <u>Want</u> ∄ non-existence of "Saddles"
<u>prescriptions</u>

①

<u>Def</u> Φ is strongly GMN

$$:\Longleftrightarrow \left(\sum_C \times \sum, \ (\lambda_i - \lambda_j)^2 \right)$$

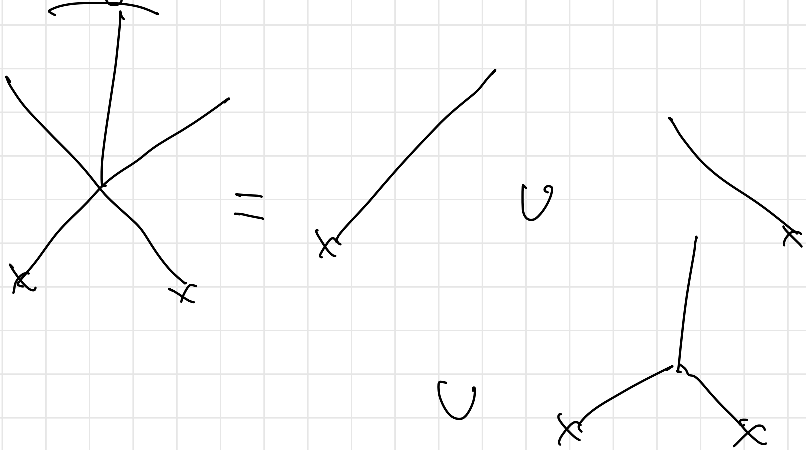
is strongly GMN

(+ more stronger pole condition)

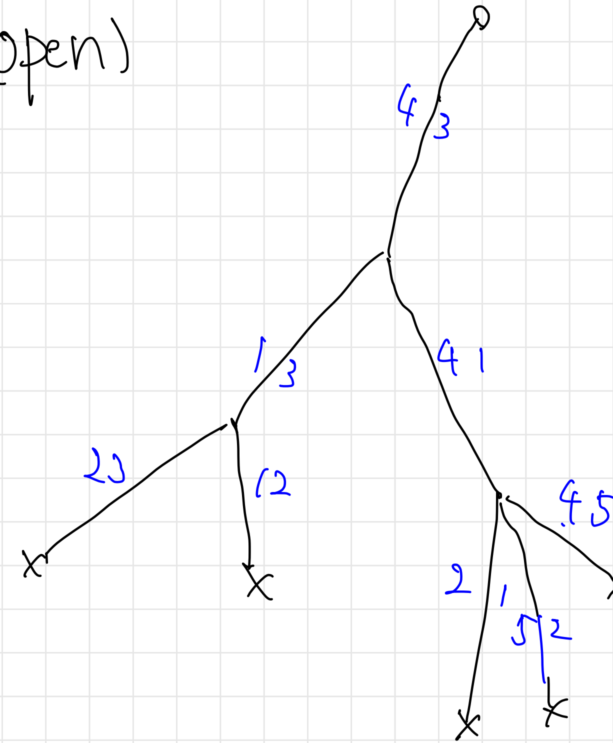
Trajectories here are proj'd down to trajectories on C of Φ.

② Idea: A spectral network is a union of trees.

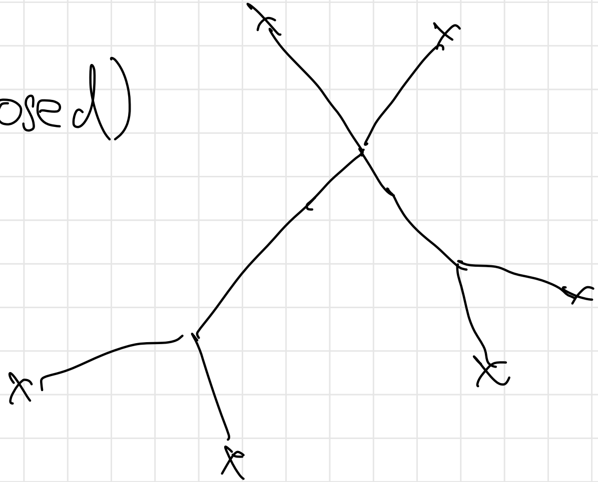
e.g.



Def (Stokes tree, phys name BPS-state) (open)



(closed)



2. $\theta \in S^1$

$\mathcal{S}_\theta$: the set of States trees

3. θ is unobstructed

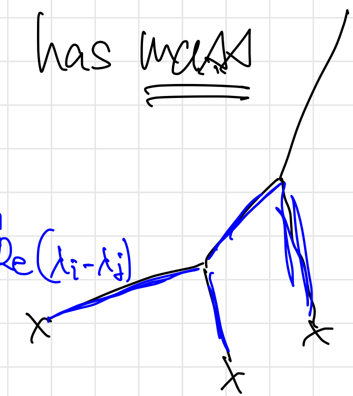
$:\Longleftrightarrow \mathcal{S}_\theta \not\ni$ closed

So-called Spec net $:= \bigcup_{S \in \mathcal{S}_\theta} S$

$\exists$ increasing $\mathbb{R}$-filtration on $\mathcal{S}_\theta$

Each tree S has mass

$m(S) := \sum \int \mathrm{Re}(\lambda_i - \lambda_j)$



Lemma (k) ("Gromov compactness")

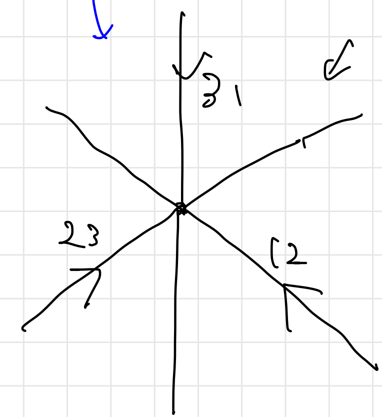
$\forall L \in \mathbb{R}_{>0}$,

$\{ S \in \mathcal{S}_\theta \mid m(S) < L \}$ is a finite set.

③ Generically $\nexists$ closed free

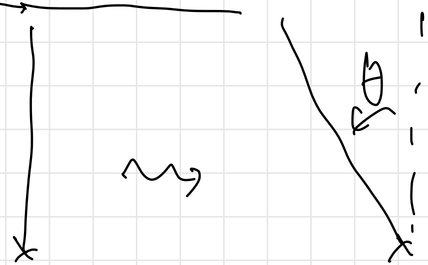
④ This is a generalization of saddle.

This stops our procedure.

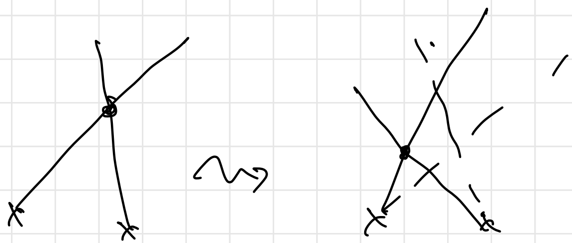


One can collaspe this by perturbing θ.

Move θ to the left

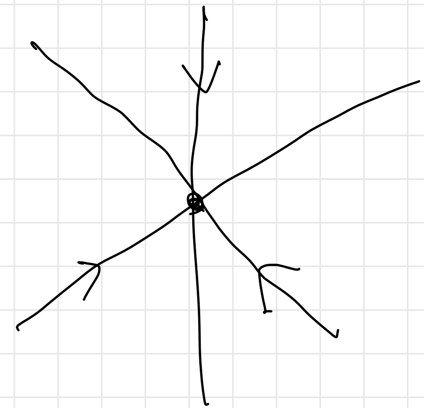


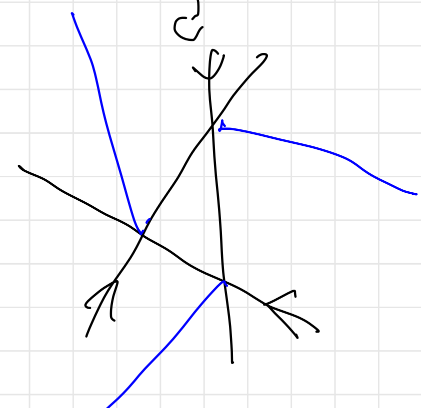
left



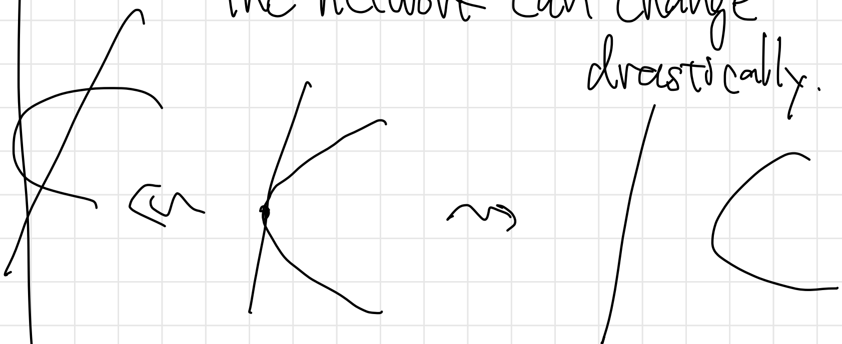
also moves to the left

new preStokes also move to the left.





Caution! By perturbing θ, the network can change drastically.



↑ e.g.

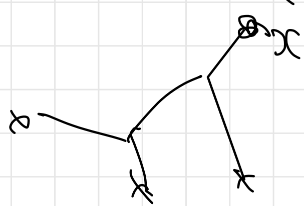
We have to implement the above argument in "stable angles."

Thm(K) A generic θ is unobstructed.

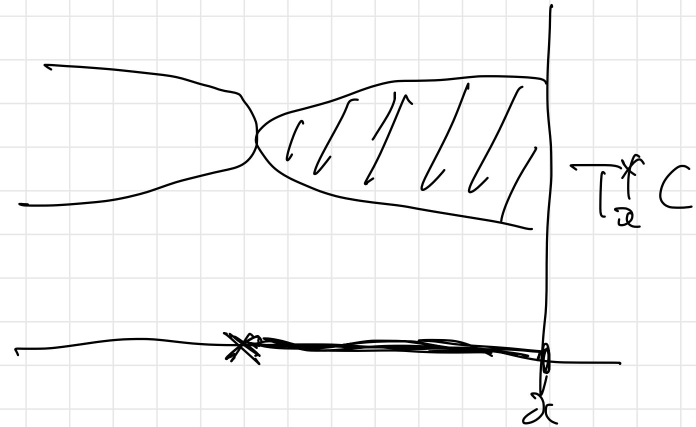
App 1 (Lag intersection Floer theory)

Expectation (Folklore)

open Stokes tree ended at $x \in C$ $\overset{1:1}{\longleftrightarrow}$ holo disk bdd by Σ & $T^*_x C$



close ——— $\longleftrightarrow$ ——— by Σ



proof Exact case: Nho.

Non-exact case:
WIP with Iwaki-Ohta.

Cor

For generic θ,

$e^{i\theta} \cdot \Sigma$ is unobstructed in the sense of FOOO

$\Updownarrow$

Fukaya A_∞-alg of Σ is uncurved.

Solomon-Verbitsky

up to technicality

App ? "Fukaya-Sheaf Corresp"

Conj

$\mathrm{Fuk}(T^*M) \hookrightarrow Sh^{\mathbb{R}^\delta}_{>0}(M \times \mathbb{R}_t)$

equiv Tamarkin cat

App 3 (WKB analysis)

One can make a precise conj. for exact WKB.

Pervious work

($k \geq 0$) 2-order $\hbar$-diff eq — exact WKB — Spec net with wall-crossing factor $\longleftrightarrow$ SQ

$\hbar$-RH

$\hbar$-diff eq $\xrightarrow{\exists \text{ functor}}$ equiv Tamarkin

(rational Lag by K-Petr-Shende)

$Fuk(T^*C) \ni \Sigma \longmapsto$ "SQ" candidate

One can construct it directly by using spec net. (with wall-crossing factor.)

Thank you!