Description of generalized isomonodromic deformations of rank 2 linear differential equations using apparent singularities

Web-seminar on Painlevé Equations and related topics

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1. Linear equations and their isomonodromic deformations
2. Classification of Algebraic solution of irregular Garnier system
3. Explicit description of irregular Garnier system
4. Our algebraic solution of irregular Garnier system

Introduction

(This talk is based on arXiv/2003.08045 (PRIMS), 2205.14979 (FE))

- I am interested in the algebraic geometric aspect of **(generalized) isomonodromic deformations.**
  - A (generalized) isomonodromic deformation is the deformation of a linear differential equation (for example, Fuchsian system) which preserves its (generalized) monodromy.
  - **Painlevé equations** are derived by the (generalized) isomonodromic deformations.
  - By (generalized) isomonodromic deformations, we have generalization of Painlevé equations (for example, **Garnier system**, **irregular Garnier system**.)
- **Motivation**: To supplement the classification of algebraic solutions of irregular Garnier systems due to Diarra–Loray (Compositio Math. 2020)

  (There is a lack in their list of algebraic solutions.)
- First we recall linear differential equations.

  ($\mathbb{P}^1$ is the **complex** projective line.)

Linear differential equation

We consider the following linear differential equation on $\mathbb{P}^1$:

$$\frac{d\boldsymbol{y}}{dx} = A(x)\boldsymbol{y} \quad \left(\text{where } A(x) = \sum_{i=1}^{\nu}\sum_{k=1}^{m_i} \frac{A_{-k}^{(i)}}{(x-t_i)^k} - \sum_{k=2}^{m_\infty} A_{-k}^{(\infty)} x^{k-2}\right)$$

where $A_{-k}^{(i)}$ are constant $r \times r$ matrices. (It appeared in Jimbo–Miwa–Ueno).

- Laurent expansion of $A(x)$ at $x = t_i$:

$$A(x) = \frac{A_{-m_i}^{(i)}}{(x-t_i)^{m_i}} + \frac{A_{-m_i+1}^{(i)}}{(x-t_i)^{m_i-1}} + \cdots + \frac{A_{-1}^{(i)}}{x-t_i} + \text{ [holo. part]}$$

- **Assumpstion**: The eigenvalues of the leading coeff. $A_{-m_i}^{(i)}$ are distinct from each other. By this assumption, we may diagonalize it as follows:

$$\frac{\mathrm{diag}(\nu_1^{(i)}, \nu_2^{(i)}, \ldots, \nu_r^{(i)})}{(x-t_i)^{m_i}} + \text{ [holo. part (diagonal)]},$$

  where $\nu_j^{(i)}$ are polynomials in $(x - t_i)$ with $\deg(\nu_j^{(i)}) \leq m_i - 1$.

Parameters of linear differential equation

- We will consider the generalized isomonodromic deformation of $\frac{d\boldsymbol{y}}{dx} = A(x)\boldsymbol{y}$. So we have to discuss about parameters of $A(x)$.

$$A(x) = \sum_{i=1}^{\nu}\sum_{k=1}^{m_i} \frac{A_{-k}^{(i)}}{(x-t_i)^k} - \sum_{k=2}^{m_\infty} A_{-k}^{(\infty)} x^{k-2}$$

$$\longmapsto \frac{\mathrm{diag}(\nu_1^{(i)}, \nu_2^{(i)}, \dots, \nu_r^{(i)})}{(x-t_i)^{m_i}} dx \ + \ \text{[holo. part] at } x = t_i$$

where $\nu_j^{(i)}$ are polynomials in $(x-t_i)$ with $\deg(\nu_j^{(i)}) \leq m_i - 1$.

- We set $n := \sum_{i=1}^{\nu} m_i + m_\infty$.
- $A(x)$ has the following parameters:

(i-a) Positions of singular points $t_1, t_2, \dots, t_\nu, \infty$.

(i-b) The coefficients of $\nu_j^{(i)}$ except the coefficient of $(x-t_i)^{m_i-1}$.

(ii) The coefficient of $(x-t_i)^{m_i-1}$ (residue part).

(iii) Remaining parameters (that is called "**accessory parameters**").

The dimension of the space of parameters

Now we count the dimension of the space of parameters.

(i) $\nu + 1$: number of singularities.

$r(n - \nu - 1)$: number of coefficients of $\nu_j^{(i)}$ except the residue parts.

By considering Möbius transformation, the number of parameters of this kind is

$$(\nu + 1) + r(n - \nu - 1) - 3.$$

(ii) $r(\nu + 1)$: number of the coefficient of $(x - t_i)^{m_i - 1}$.

By considering the Fuchs relation, the number of parameters of this kind is

$$r(\nu + 1) - 1.$$

(iii) There are $(n - 1)$-$r \times r$ matrices in $A(x)$,

By considering gauge transformations, the number of accessory parameters is

$$(n - 1)r^2 - (r^2 - 1) - \{r(n - \nu - 1) + r(\nu + 1) - 1\}$$
$$= -2r^2 + nr(r - 1) + 2.$$

Role of each parameter

When we consider the generalized isomonodromic deformation of $\frac{d\boldsymbol{y}}{dx} = A(x)\boldsymbol{y}$,

$$A(x) = \sum_{i=1}^{\nu} \sum_{k=1}^{m_i} \frac{A_{-k}^{(i)}}{(x - t_i)^k} - \sum_{k=2}^{m_\infty} A_{-k}^{(\infty)} x^{k-2},$$

the roles of parameters of the linear equations is as follows:

- **Independent variable of the deformation**:
  - position of singularities $t_1, t_2, \ldots, t_\nu, \infty$
  - the coefficients of $\nu_j^{(i)}$ except the residue parts.
- **Dependent variable of the deformation**: accessory parameters
- **Fixed parameters**:
  - the coefficient of $(x - t_i)^{m_i - 1}$ (residue part).

Our setting (irregular Garnier system)

- We assume that $\frac{d\boldsymbol{y}}{dx} = A(x)\boldsymbol{y}$ is rank 2. That is, $A(x)$ is 2×2 matrix.
- Moreover, we assume that the principal part of each singularity of ∇ is diagonalizable as follows:

$$\begin{pmatrix} \nu^{(i)}_{-m_i} & 0 \\ 0 & -\nu^{(i)}_{-m_i} \end{pmatrix} \frac{dx}{(x-t_i)^{m_i}} + \cdots + \begin{pmatrix} \nu^{(i)}_{-2} & 0 \\ 0 & -\nu^{(i)}_{-2} \end{pmatrix} \frac{dx}{(x-t_i)^2} + \begin{pmatrix} \nu^{(i)}_{+,-1} & 0 \\ 0 & \nu^{(i)}_{-,-1} \end{pmatrix} \frac{dx}{x-t_i}$$

- Parameters of a family of linear differential equations:
  - **dependent variable**: accessory parameters ($\sharp = 2n-6$),
  - **independent variable**: positions of singularities, and $\nu^{(i)}_{-j}$ $(j = 2, 3, \ldots, m_i)$ ($\sharp = n-3$)

The system of (nonlinear) differential equations given by the isomonodromic deformations of this linear equation is called an **irregular Garnier system**.

How to give an explicit formula of the system

(1) First, we have to introduce **explicit** accessory parameters

- Now we use **apparent singularities** as explicit accessory parameters.
  - Okamoto, Iwasaki, Dubrovin–Mazzocco, Saito–Szabo, Diarra–Loray, Marchal–Orantin–Alameddine, Marchal–Alameddine,.. et al.

(2) Second, we have to give an **explicit** family of the linear differential equations parametrized by apparent singularities.

- it was done already, for example, by Diarra–Loray.

(3) Third, we have to calculate isomonodromic deformations concretely by using the explicit family of the linear differential equations.

- We use the **isomonodromy 2-form**, roughly speaking, this is the pull-back of the Goldman symplectic form on the character variety under the Riemann-Hilbert map (regular case).
- Explicit calculation of the isomonodromy 2-form by using apparent singularities was done by K (arXiv:2003.08045, PRIMS)

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Introduction

- In general, solutions of irregular Garnier system are highly transcendental.
- But, special irregular Garnier systems have special solution which is **algebraic**.
- Idea of construction of an algebraic solution is as follows:
  - If we have an **isomonodromic** and **algebraic** family of linear differential equations, then we have an **algebraic** solution of an irregular Garnier system by calculating the apparent singularities.
  - How to construct isomonodromic and algebraic family of linear differential equations ?
  - For example, pull-back method.

Pull-back method

Pull-back method

(1) Prepare a linear equation on $\mathbb{P}^1$ and a family of branch coverings $\mathbb{P}^1 \to \mathbb{P}^1$.

(2) By taking the pull-back of the linear equation under the branch coverings, we have a family of linear equations on $\mathbb{P}^1$, which parametrized by the parameters of the family of branch coverings.

(3) By construction of the family of linear equations, this family is isomonodromic and algebraic.

- **When do irregular Garnier systems have an algebraic solution constructed by the pull-back method?**
- Diarra–Loray have given an answer.
- **If an algebraic solution is not classical, then this algebraic solution is given by the pull-back method.**
- classical solution: solutions comes from the deformation of a rank 2 differential system with diagonal or dihedral differential Galois group.

Classification by Diarra–Loray

Theorem due to Diarra–Loray (2020)

Up to isomorphisms, there are exactly 3 non classical algebraic solutions, for irregular Garnier systems of rank $N > 1$. The list of corresponding formal data is as follows:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 1 & 1 \\ 1/3 & 0 & 1 \end{array}, \quad \begin{array}{cc} 0 & \infty \\ \hline 1 & 2 \\ 0 & 1 \end{array}, \quad \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}, \quad \begin{array}{l} \text{Positions of singularities} \\ \hline \text{Poincaré rank } (m_i - 1) \\ \text{exponent} \end{array}$$

- Here exponent means the difference of "eigenvalues" of the **residue part**.
- The algebraic solutions of first and second cases are already given.
  - The first and second irregular Garnier systems are 2-variable.
- **On the other hand, the algebraic solution of third case had not been given, because the corresponding explicit irregular Garnier system had not been known.**
  - (This is 3-variable Garnier system. It is more complicated than 2-variable cases.)

Our target

- Our target is the irregular Gariner system corresponding to $\begin{array}{ccc} 0 & 1 & \infty \\ \hline 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}$.
- This irregular Gariner system is derived by the isomonodromic deformation of connections

$$\nabla\colon \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \Omega^1_{\mathbb{P}^1}(2[0] + 2[1] + 2[\infty])$$

  - We consider connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ instead of connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ (rank 2 linear ODE).
  - (We may transform connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ into on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ by a birational bunde transformation.)
  - Assume that $\nabla|_{2[0]}, \nabla|_{2[1]}, \nabla|_{2[\infty]}$ are diagonalized as

$$\begin{pmatrix} 2\,t_1 & 0 \\ 0 & -2\,t_1 \end{pmatrix} \frac{dx}{x^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix} \frac{dx}{x}$$

$$\begin{pmatrix} 2\,t_2 & 0 \\ 0 & -2\,t_2 \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix} \frac{dx}{x-1} \qquad (t_i \text{ become independent variables})$$

$$\begin{pmatrix} 2\,t_3 & 0 \\ 0 & -2\,t_3 \end{pmatrix} \frac{dw}{w^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix} \frac{dw}{w},$$

Introduction

We want to derive the 3-variable Garnier system corresponding to $\begin{array}{ccc} 0 & 1 & \infty \\ \hline 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}$.

Strategy

(1) We use **apparent singularities** as explicit accessory parameters.

(2) We use an **explicit** family of connections due to Diarra–Loray (arXiv:1907.07678, Period. Math. Hungar.).

(3) We consider the **isomonodromy 2-form** and calculate this 2-form by using apparent singularities (arXiv:2003.08045, PRIMS)

First we recall a definition of apparent singularities.

Definition of apparent singularities

- $D = \sum_{i=1}^{\nu} n_i[x_i] + n_\infty[\infty] \quad (n := \deg(D))$
- $\nabla\colon \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \Omega^1_{\mathbb{P}^1}(D)$
  - Assume that $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ is not ∇-invariant.
- The composition φ_∇

$$\mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\nabla} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \Omega^1_{\mathbb{P}^1}(D)$$
$$\longrightarrow (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)/\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \Omega^1_{\mathbb{P}^1}(D) \cong \mathcal{O}_{\mathbb{P}^1}(n-2)$$

  is an $\mathcal{O}_{\mathbb{P}^1}$-morphism.

Definition

We call zeros of φ_∇ **apparent singularities** of ∇.

- Assume that the apparent singularities consist of distinct points $q_1, \ldots, q_{n-3}$.
- (# of accessory parameters is $2(n-3)$)

Definition of p_i

- $\phi_\nabla := \mathrm{id} \oplus \varphi_\nabla \colon \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \dashrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$
- $\nabla \longmapsto (\phi_\nabla)_*(\nabla)$
  - $(\phi_\nabla)_*(\nabla)$ is a connection on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$, which has simple poles at $q_1, \dots q_{n-3}$ with residual eigenvalues 0 and -1 at each q_j.
- By automorphisms of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$, we may normalize $(\phi_\nabla)_*(\nabla)$ as

$$\begin{pmatrix} 0 & \frac{1}{P(x)} \\ * & * \end{pmatrix} dx, \quad P(x) := \prod_{i=1}^{\nu} (x - x_i)^{n_i}$$

Definition

We define p_j so that $\begin{pmatrix} 1 \\ p_j \end{pmatrix}$ is in the 0-eigenspace of the residue matrix of normalized $(\phi_\nabla)_*(\nabla)$ at q_j

Diarra–Loray's normal form (1)

Now we recall a family of connections due to Diarra–Loray (arXiv:1907.07678, Period. Math. Hungar.)

- **Accessory parameters**: $\{(q_1, p_1), \ldots, (q_{n-3}, p_{n-3})\} \in \mathrm{Sym}^{n-3}(\mathbb{C}^2)$
- Family of connections parametrized by the accessory parameters:

$$d + \Omega \colon \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2) \to (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)) \otimes \Omega^2_{\mathbb{P}^1}(D + q_1 + \cdots + q_{n-3})$$

$$\Omega = \begin{pmatrix} 0 & \frac{1}{P(x)} \\ c_2(x) & d_2(x) \end{pmatrix} dx$$

  - $P(x) := \prod_{i=1}^{\nu} (x - x_i)^{n_i}$
  - $c_2(x) := \sum_{i=1}^{\nu} \frac{C_i(x)}{(x-x_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{p_j}{x-q_j} + \tilde{C}(x) + x^{n-3} C_\infty(x)$
  - $d_2(x) := \sum_{i=1}^{\nu} \frac{D_i(x)}{(x-x_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{-1}{x-q_j} + D_\infty(x)$

poly. in x	$C_i, D_i (i = 1, \ldots, \nu)$	C_∞	D_∞	$\tilde{C}$
degree	$\leq n_i - 1$	$\leq n_\infty - 1$	$\leq n_\infty - 2$	$\leq n - 4$

Diarra–Loray's normal form (2)

- We have a correspondence by ϕ_∇:

  Connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$ with apparent singularities at $q_1, \ldots, q_{n-3}$
  $\longleftrightarrow$ Connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$

- If we fix local data at $x_1, x_2, \ldots, x_\nu, \infty$,
  then C_i, D_i $(i = 1, 2, \ldots, \nu)$, C_∞, D_∞ are determined.
  - In our situation, $D = 2[0] + 2[1] + 2[\infty]$ $(x_1 = 0, x_2 = 1$ $(\nu = 2))$

$$\begin{cases} C_0(x) = 4\,t_1^2(1-2\,x) \\ C_1(x) = 4\,t_2^2(2\,x-1) \\ C_\infty(x) = 4\,t_3^2(x-2) \end{cases} \qquad \begin{cases} D_0(x) = -\dfrac{x}{3} \\ D_1(x) = -\dfrac{x-1}{3} \\ D_\infty(x) = 0 \end{cases}$$

Diarra–Loray's normal form (3)

- By the condition that $q_1, \ldots, q_{n-3}$ are apparent, $\tilde{C}$ is determined.
- In our situation, we have

$$\tilde{C} = \tilde{C}_{q_1}(x-q_2)(x-q_3) + \tilde{C}_{q_2}(x-q_1)(x-q_3) + \tilde{C}_{q_3}(x-q_2)(x-q_1),$$

where

$$\tilde{C}_{q_j} = \frac{1}{Q'(q_j)}\left(\frac{p_j^2}{q_j^2(q_j-1)^2} + \frac{q_j p_j + 12\, t_1^2(2\, q_j - 1)}{3\, q_j^2} + \frac{(q_j-1)p_j - 12\, t_2^2(2\, q_j - 1)}{3\,(q_j-1)^2} + \sum_{k\in\{1,2,3\}\setminus\{j\}} \frac{p_j - p_k}{q_j - q_k} - 4\, t_3^2 q_j^3(q_j-2)\right),$$

and $Q'(x) = (x-q_1)(x-q_2) + (x-q_2)(x-q_3) + (x-q_3)(x-q_1)$

- So in our situation, Ω is parametrized by $q_1, q_2, q_3, p_1, p_2, p_3; t_1, t_2, t_3$.

Isomonodromy 2-form

- Let Ω be the Diarra–Loray's normal form:

  $d+\Omega\colon \mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n-2)\to(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n-2))\otimes\Omega^2_{\mathbb{P}^1}(D+q_1+\cdots+q_{n-3})$

  parametrized by $q_1,q_2,q_3,p_1,p_2,p_3;t_1,t_2,t_3$ (in our situation $(n=6)$).
- We take (formal) solutions of $d+\Omega=0$:
  - ψ_i : formal fundamental matrix solution of $d+\Omega=0$ at $x=x_i$ $(x_1=0,x_2=1,x_3=\infty)$
  - ψ_{q_j} : fundamental matrix solution of $d+\Omega=0$ at $x=q_j$ (q_j is apparent)

Isomonodromy 2-form (Krichever 2002)

ω is the 2-form on the parameter space of $q_1,q_2,q_3,p_1,p_2,p_3;t_1,t_2,t_3$ defined by

$$\omega(\delta_1,\delta_2):=\frac{1}{2}\sum_{i=1}^{3}\operatorname{res}_{x=x_i}\operatorname{Tr}(\delta(\Omega)\wedge\delta(\psi_i)\psi_i^{-1}) + \frac{1}{2}\sum_{j=1}^{3}\operatorname{res}_{x=q_j}\operatorname{Tr}(\delta(\Omega)\wedge\delta(\psi_{q_j})\psi_{q_j}^{-1})$$

Explicit formula of Isomonodromy 2-form (1)

We have explicit description of Ω and we may give explicit description of solutions ψ_i and ψ_{q_j}. So we can calculate ω explicitly.

- Change of variable: From p_j to η_j:

$$\begin{aligned}\eta_j &:= \frac{p_j}{q_j^2(q_j-1)^2} - \frac{D_0(q_j)}{q_j^2} - \frac{D_1(q_j)}{(q_j-1)^2} - D_\infty(q_j)\\ &= \frac{p_j}{q_j^2(q_j-1)^2} + \frac{1}{3\,q_j} + \frac{1}{3(q_j-1)}\end{aligned}$$

- We consider the diagonalizations until the constant terms:

$$\begin{pmatrix} 2\,t_1 & 0 \\ 0 & -2\,t_1 \end{pmatrix}\frac{dx}{x^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix}\frac{dx}{x} + \begin{pmatrix} \theta_1^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) & 0 \\ 0 & \theta_1^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \end{pmatrix} dx + O(x)$$

$$\begin{pmatrix} 2\,t_2 & 0 \\ 0 & -2\,t_2 \end{pmatrix}\frac{dx}{(x-1)^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix}\frac{dx}{x-1} + \begin{pmatrix} \theta_2^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) & 0 \\ 0 & \theta_2^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \end{pmatrix} dx + O(x-1)$$

$$\begin{pmatrix} 2\,t_3 & 0 \\ 0 & -2\,t_3 \end{pmatrix}\frac{dw}{w^2} + \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{pmatrix}\frac{dw}{w} + \begin{pmatrix} \theta_3^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) & 0 \\ 0 & \theta_3^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \end{pmatrix} dw + O(w).$$

Explicit formula of Isomonodromy 2-form (2)

- By explicit calculation of ω, we have the following equality

$$\omega = \sum_{j=1}^{3} d\eta_j \wedge dq_j - d\theta_1^+ \wedge d(2t_1) - d\theta_2^+ \wedge d(2t_2) - d\theta_3^+ \wedge d(2t_3)$$
$$- d\theta_1^- \wedge d(-2t_1) - d\theta_2^- \wedge d(-2t_2) - d\theta_3^- \wedge d(-2t_3)$$
$$+ \sum_{i_1<i_2} f_{i_1,i_2}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta})\, dt_{i_1} \wedge dt_{i_2}.$$

- We set

$$\begin{cases} H_{t_1}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) := 2\,\theta_1^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) - 2\,\theta_1^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}), \\ H_{t_2}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) := 2\,\theta_2^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) - 2\,\theta_2^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}), \\ H_{t_3}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) := 2\,\theta_3^-(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) - 2\,\theta_3^+(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}). \end{cases}$$

- Then

$$\omega = \sum_{j=1}^{3} d\eta_j \wedge dq_j + dH_{t_1} \wedge dt_1 + dH_{t_2} \wedge dt_2 + dH_{t_3} \wedge dt_3$$
$$+ \sum_{i_1<i_2} f_{i_1,i_2}\, dt_{i_1} \wedge dt_{i_2}.$$

Hamiltonian (1)

- Since ω is isomonodromy 2-form (that is, the interior product with IMD vanishes), we obtain the corresponding irregular Garnier system is

$$\begin{cases} \dfrac{\partial q_j}{\partial t_i} = \dfrac{\partial H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})}{\partial \eta_j} \\ \dfrac{\partial \eta_j}{\partial t_i} = -\dfrac{\partial H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})}{\partial q_j} \end{cases} \qquad (i = 1, 2, 3,\ j = 1, 2, 3).$$

- Explicit forms of the hamiltonians $H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})$ are in the next page.

Hamiltonian (2)

$Q(x) := (x-q_1)(x-q_2)(x-q_3)$, $\sigma_1 := q_1+q_2+q_3$, $\sigma_2 := q_1q_2+q_2q_3+q_3q_1$, $\sigma_3 = q_1q_2q_3$.

$$\begin{cases} H_{t_1}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \\ = -\dfrac{q_1q_2q_3}{t_1}\displaystyle\sum_{j=1}^{3}\left(\frac{q_j\,(q_j-1)^2}{Q'(q_j)}{\eta_j}^2 - \frac{\left(5\,{q_j}^2-9\,q_j+4\right)}{3\,Q'(q_j)}\eta_j\right) + \sum_{j=1}^{3}\frac{4\,t_1}{{q_j}^2} + \frac{144\,{t_1}^2+144\,{t_2}^2-13}{36\,t_1} \\ \quad + \dfrac{4\,{t_3}^2\sigma_3\,(\sigma_1-2)}{t_1} - \dfrac{4\,t_1\,(2\,\sigma_2-\sigma_1)}{q_1q_2q_3} \\ \quad - \dfrac{4\,{t_2}^2\left({\sigma_1}^2-2\,\sigma_1\sigma_2+3\,\sigma_1\sigma_3+{\sigma_2}^2-2\,\sigma_2\sigma_3-2\,\sigma_1+2\,\sigma_2-4\,\sigma_3+1\right)}{t_1\,(q_1-1)^2\,(q_2-1)^2\,(q_3-1)^2} \\ H_{t_2}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \\ = -\dfrac{(q_1-1)\,(q_2-1)\,(q_3-1)}{t_2}\displaystyle\sum_{j=1}^{3}\left(\frac{{q_j}^2\,(q_j-1)}{Q'(q_j)}{\eta_j}^2 - \frac{q_j\,(5\,q_j-1)}{3\,Q'(q_j)}\eta_j\right) + \sum_{j=1}^{3}\frac{4\,t_2}{(q_j-1)^2} \\ \quad + \dfrac{144\,{t_1}^2+144\,{t_2}^2-13}{36\,t_2} + \dfrac{4\,{t_3}^2(\sigma_1-\sigma_2+\sigma_3-1)\,(\sigma_1-1)}{t_2} + \dfrac{4\,t_2\,(2\,\sigma_2-3\,\sigma_1+3)}{(q_1-1)\,(q_2-1)\,(q_3-1)} \\ \quad - \dfrac{4\,{t_1}^2\left(\sigma_1\sigma_2-\sigma_1\sigma_3-{\sigma_2}^2+2\,\sigma_2\sigma_3-\sigma_2+\sigma_3\right)}{t_2{q_1}^2{q_2}^2{q_3}^2} \\ H_{t_3}(\boldsymbol{t},\boldsymbol{q},\boldsymbol{\eta}) \\ = -\dfrac{1}{t_3}\displaystyle\sum_{j=1}^{3}\left(\frac{{q_j}^2\,(q_j-1)^2}{Q'(q_j)}{\eta_j}^2 - \frac{q_j\left(2\,{q_j}^2-3\,q_j+1\right)}{3\,Q'(q_j)}\eta_j\right) + 4\,t_3({\sigma_1}^2-2\,\sigma_1-\sigma_2+1) - \frac{1}{36\,t_3} \\ \quad - \dfrac{4\,{t_1}^2\,(2\,\sigma_3-\sigma_2)}{t_3{q_1}^2{q_2}^2{q_3}^2} + \dfrac{4\,{t_2}^2\,(2\,\sigma_3-\sigma_2+1)}{t_3\,(q_1-1)^2\,(q_2-1)^2\,(q_3-1)^2} \end{cases}$$

Introduction

- Now we have the 3-variable Garnier system corresponding to $\begin{array}{ccc} 0 & 1 & \infty \\ \hline 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}$.
- We want to give an algebraic solution of this equation.
- By Diarra–Loray, we already have the corresponding isomonodromic family of connections by Pull-back method.
- First we recall the construction of this isomonodromic family .

Pull-back method (recall)

(1) Prepare a linear equation on $\mathbb{P}^1$ and a family of branch coverings $\mathbb{P}^1 \to \mathbb{P}^1$.

(2) By taking the pull-back of the linear equation under the branch coverings, we have a family of linear equations on $\mathbb{P}^1$, which parametrized by the parameters of the family of branch coverings.

(3) By construction of the family of linear equations, this family is isomonodromic and algebraic.

Construction of an isomonodromic family

- We consider the following linear equation on $\mathbb{P}^1_z$:

$$\frac{d^2u}{dz^2} + \frac{2}{3\,z}\frac{du}{dz} - \frac{1}{z}u = 0.$$

  - it has a regular singular point at $z = 0$
  - it has a ramified irregular singular point at $z = \infty$ (Poincaré rank $1/2$)

- We consider the branched covering $\phi_{\boldsymbol{s}} \colon \mathbb{P}^1_x \to \mathbb{P}^1_z$ defined by

$$x \longmapsto \phi_{\boldsymbol{s}}(x) = \frac{\left(s_3 x\,(x-1) + s_2 x + s_1\,(1-x)\right)^3}{x^2\,(x-1)^2}.$$

- This branched covering is parametrized by $(s_1, s_2, s_3) \in \mathbb{C}^3$.
- We take the pull-back of the linear equation above. Then we have the linear equation on $\mathbb{P}^1_x$ associated to the connection

$$d + \begin{pmatrix} 0 & \frac{1}{x^2(x-1)^2} \\ \frac{4\,s_3(s_3x^2-(s_1-s_2+s_3)x+s_1)Q(x;\boldsymbol{s})^2}{x^2(x-1)^2} & -\frac{1}{3\,x} - \frac{1}{3(x-1)} - \sum_{i=1}^{2}\frac{Q'(x;\boldsymbol{s})}{Q(x;\boldsymbol{s})} \end{pmatrix} dx,$$

where we put $Q(x;\boldsymbol{s}) := x^3 + \dfrac{s_1 - s_2 - 3\,s_3}{2\,s_3}x^2 + \dfrac{-3\,s_1 - s_2 + s_3}{2\,s_3}x + \dfrac{s_1}{s_3}$.

Algebraic solution

- The apparent singularities q_1, q_2, q_3 are the zeros of $Q(x;\boldsymbol{s})$,
- $\eta_j = \frac{1}{3\,q_j} + \frac{1}{3\,(q_j-1)}$.
- By comparing the principal parts, we have $t_i^2 = s_i^3$.

Main theorem

Functions $q_j(t_1, t_2, t_3), \eta_j(t_1, t_2, t_3)$ $(j = 1, 2, 3)$ are defined by

$$\begin{cases} q_j^3 + \dfrac{s_1 - s_2 - 3\,s_3}{2\,s_3} q_j^2 + \dfrac{-3\,s_1 - s_2 + s_3}{2\,s_3} q_j + \dfrac{s_1}{s_3} = 0 & j = 1, 2, 3 \\ \eta_j = \dfrac{1}{3\,q_j} + \dfrac{1}{3\,(q_j - 1)} & j = 1, 2, 3 \\ t_i^2 = s_i^3 & i = 1, 2, 3. \end{cases}$$

implicitly. Then these functions satisfy the following system:

$$\begin{cases} \dfrac{\partial q_j}{\partial t_i} = \dfrac{\partial H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})}{\partial \eta_j} \\ \dfrac{\partial \eta_j}{\partial t_i} = -\dfrac{\partial H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})}{\partial q_j} \end{cases} \qquad (i = 1, 2, 3,\ j = 1, 2, 3)$$

τ function

- $H_{t_i}(\boldsymbol{t})$: the substitution of our algebraic solution to the Hamiltonians $H_{t_i}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\eta})$.
  - Our Hamilton system is nonautonomous. So this is not conserved quantity.
- We will calculate the τ-**function** with respect to our algebraic solution.
- We consider the following 1-form:

$$\varpi = H_{t_1}(\boldsymbol{t})dt_1 + H_{t_2}(\boldsymbol{t})dt_2 + H_{t_3}(\boldsymbol{t})dt_3$$

- Now, this 1-from is exact. If we set

$$F(s_1, s_2, s_3) := \frac{{s_1}^3 + {s_2}^3 + {s_3}^3}{2} + \frac{\ln(s_1 s_2 s_3)}{24} - \frac{9\left((s_2 - s_3)^2 s_1 + (s_1 - s_3)^2 s_2 + (s_2 - s_1)^2 s_3\right)}{2} - 18\, s_1 s_2 s_3,$$

  then $\varpi = dF(s_1, s_2, s_3) \quad (t_i^2 = s_i^3)$.
- So the τ-function with respect to our algebraic solution is

$$\tau = c \cdot e^{F(s_1, s_2, s_3)},$$

  where c is a constant.