

The geometry of stratification of the de Rham moduli spaces

Pengfei Huang

Nanjing University

Web-seminar on “Painlevé Equations and related topics”

Based on joint work with Zhi Hu and Runhong Zong

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Overview

I. Introduction

II. Setup

III. Main Results

I: INTRODUCTION

♣ nonabelian Hodge correspondence

Basic settings

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$$(E, \varphi : E \rightarrow E \otimes_{\mathbb{C}} \Omega_X^1 \text{ is } \mathcal{O}_X\text{-linear})$$

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- $\mathcal{M}_{\text{B}}(X, n) := \text{Hom}(\pi_1(X), \text{GL}_n(\mathbb{C})) // \text{GL}_n(\mathbb{C})$: moduli space of **local systems** of rank n X :

$$\rho : \pi_1(X, x_0) \rightarrow \text{GL}_n(\mathbb{C})$$

\rightsquigarrow affine variety, of dimension $2n^2(g-1)+2$

Nonabelian Hodge correspondence (NAHC)

Theorem 1 (Nonabelian Hodge correspondence, Simpson, 94')

We have the following identifications of moduli spaces

$$\mathcal{M}_{\text{Dol}}(X, n) \stackrel{\mathbb{R}\text{-an.}}{\cong} \mathcal{M}_{\text{dR}}(X, n) \stackrel{\mathbb{C}\text{-an.}}{\cong} \mathcal{M}_{\text{B}}(X, n).$$

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- ▶ Denote by **NAH** : $\mathcal{M}_{\text{Dol}}(X, n) \rightarrow \mathcal{M}_{\text{dR}}(X, n)$ the nonabelian Hodge map;
- ▶ This implies the moduli space of solutions to Hitchin's equations $\mathcal{M}_{\text{Sol}}(X, n)$ is **hyperKähler**!

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$$D^\lambda(fs) = fD^\lambda(s) + \lambda df \otimes s;$$

- ▶ We have a new moduli space $\mathcal{M}_{\text{Hod}}(X, n)$: the moduli space of all semistable λ -flat bundles of rank n and degree 0 over X

\rightsquigarrow quasi-projective variety, of dimension $2n^2(g-1)+3$

Twistor spaces

- We have a fibration

$$\pi : \mathcal{M}_{\text{Hod}}(X, n) \rightarrow \mathbb{A}_{\mathbb{C}}^1$$

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together with $\psi_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(\bar{X}, \bar{x})$ (where $\psi : (X, x) \rightarrow (\bar{X}, \bar{x})$) and the Riemann–Hilbert correspondence give a gluing

$$\text{TW}(X, n) := \mathcal{M}_{\text{Hod}}^s(X, n) \sqcup_{\mathbf{d}_\psi} \mathcal{M}_{\text{Hod}}^s(\bar{X}, n);$$

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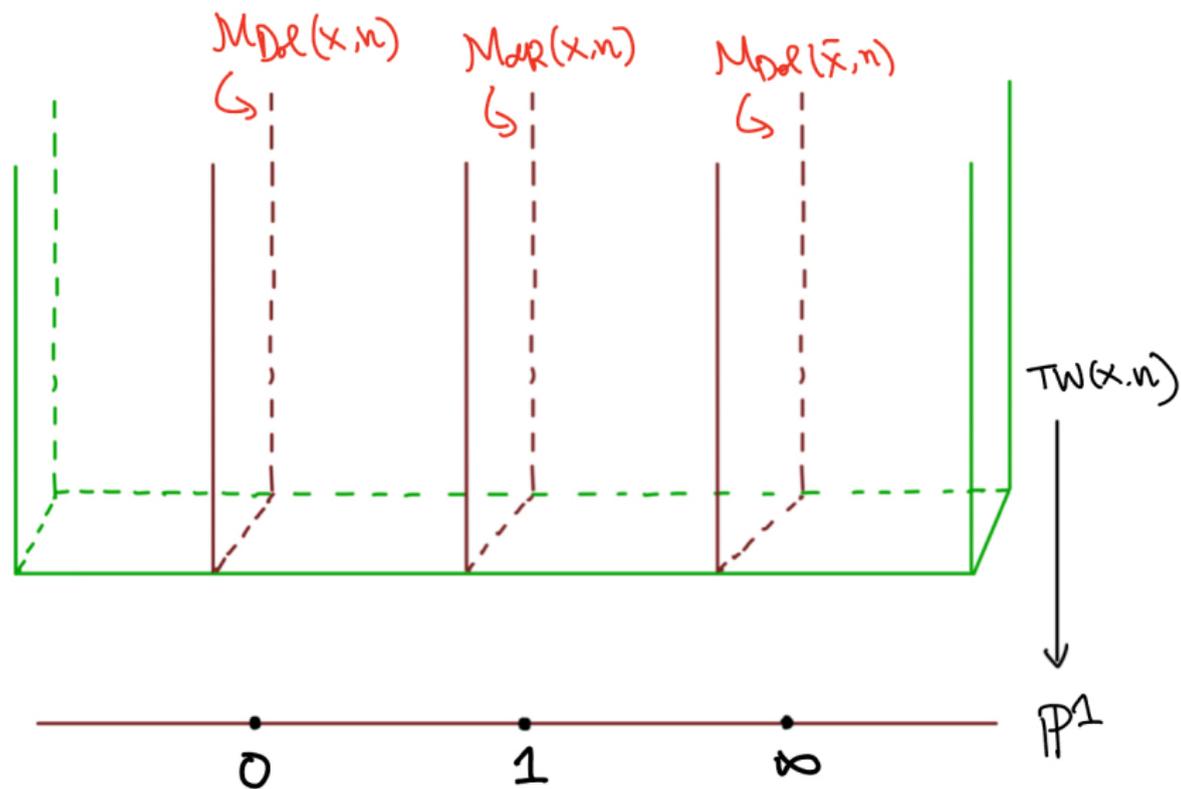
$$\text{TW}(X, n) := \mathcal{M}_{\text{Hod}}^s(X, n) \sqcup_{\text{d}_\psi} \mathcal{M}_{\text{Hod}}^s(\bar{X}, n);$$

- The resulting space satisfies

$$\text{TW}(X, n) \stackrel{\mathbb{C}\text{-an.}}{\cong} \text{TW}(\mathcal{M}_{\text{Sol}}^s(X, n)).$$

- Such $\text{TW}(X, n)$ is called the **Deligne–Hitchin twistor space**.

Picture



I: INTRODUCTION

 \mathbb{G}_m -action and stratifications

\mathbb{G}_m -action and the fixed points

- There is a natural action of \mathbb{G}_m on $\mathcal{M}_{\text{Hod}}(X, n)$ that extends the action on $\mathcal{M}_{\text{Dol}}(X, n)$, for each $t \in \mathbb{G}_m$, this action sends a λ -flat bundle to a $t\lambda$ -flat bundle:

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- **Langton theorem** (Simpson, Langer): for any $(E, D^\lambda) \in \mathcal{M}_{\text{Hod}}(X, n)$, the limit $\lim_{t \rightarrow 0} t \cdot (E, D^\lambda)$ exists in $\mathcal{M}_{\text{Hod}}(X, n)$ as a \mathbb{G}_m -fixed point;

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- Therefore,

$$\begin{aligned} \mathcal{M}_{\text{Hod}}(X, n)^{\mathbb{G}_m} &= \mathcal{M}_{\text{Dol}}(X, n)^{\mathbb{G}_m} \\ &= \left\{ (E, \varphi) = \left(\bigoplus_{i=1}^k E_i, \bigoplus_{i=1}^k \varphi_i : E_i \rightarrow E_{i+1} \otimes \Omega_X^1 \right) \right\}; \end{aligned}$$

- These fixed points, are usually called the **systems of Hodge bundles**, or **complex variations of Hodge structures** (\mathbb{C} -VHSs).

\mathbb{G}_m -action and the BB stratification

- ▶ **Remark:** In fact, the limit can be achieved via an iterated process to obtain a filtration \mathcal{F} that is **graded semistable** and satisfies the **Griffiths transversality** (w.r.t. D^λ):

$$\lim_{t \rightarrow 0} t \cdot (E, D^\lambda) = (\mathrm{Gr}_{\mathcal{F}}(E), \mathrm{Gr}_{\mathcal{F}}(D^\lambda)) \quad \text{in } \mathcal{M}_{\mathrm{Dol}}(X, n),$$

such filtration is in general not unique, but the induced graded Higgs bundle is unique in $\mathcal{M}_{\mathrm{Dol}}(X, n)$. We call it a **Simpson filtration**;

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- Now denote by $\mathrm{Fix}(X, n)$ the set of fixed points and decompose it into connected components as

$$\mathrm{Fix}(X, n) = \bigsqcup_{\alpha} \mathrm{Fix}_{\alpha},$$

so that we obtain the **Bialynichi-Birula stratification** of $\mathcal{M}_{\mathrm{Hod}}(X, n)$:

$$\mathcal{M}_{\mathrm{Hod}}(X, n) = \bigsqcup_{\alpha} G_{\alpha} := \bigsqcup_{\alpha} \left\{ (E, D^\lambda) : \lim_{t \rightarrow 0} t \cdot (E, D^\lambda) \in \mathrm{Fix}_{\alpha} \right\};$$

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- Each G_{α} is a locally closed stratum, together with a projection via taking the \mathbb{G}_m -limit

$$p_{\alpha} : G_{\alpha} \longrightarrow \mathrm{Fix}_{\alpha}, \quad (E, D^\lambda) \longmapsto \lim_{t \rightarrow 0} t \cdot (E, D^\lambda).$$

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- ▶ **Remark 1:** the **lowest stratum**, refers to the stratum flows to the connected component $\text{Fix}_L := \mathcal{U}(X, n)$, the moduli space of semistable vector bundles of rank n and degree 0:

- in $\mathcal{M}_{\text{Dol}}(X, n)$, the lowest stratum is $G_L^0 = T^*\mathcal{U}(X, n)$; (**open stratum**)
- in $\mathcal{M}_{\text{dR}}(X, n)$, the lowest stratum is $G_L^1 \rightarrow \mathcal{U}(X, n)$ twisted affine cotangent bundle of $\mathcal{U}(X, n)$; (**open stratum**)

BB stratification and oper stratification

- **Remark 2:** the **highest stratum**, refers to the stratum flows to the connected component

$$\text{Fix}_H := \left\{ \left(E = \bigoplus_{i=1}^n (L \otimes (\Omega_X^1)^{1-i}), \varphi = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{pmatrix} \right) : L \in \text{Pic}^{(n-1)(g-1)}(X) \right\},$$

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- in $\mathcal{M}_{\text{Dol}}(X, n)$, the highest stratum G_H^0 is the space of Hitchin sections; (**closed** stratum)
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- G_H^0 and G_H^1 are related by the map $\text{NAH} : \mathcal{M}_{\text{Dol}}(X, n) \rightarrow \mathcal{M}_{\text{dR}}(X, n)$, they intersect transversely:

$$\text{NAH}(G_H^0) \pitchfork G_H^1.$$

BB stratification and oper stratification

► **Remark 3:** for other fibers:

- in $\mathcal{M}_{\text{Dol}}(X, n)$: for any $u \in \text{Fix}_\alpha$ that is **stable**, the corresponding fiber

$$G_\alpha^0(u) := (p_\alpha^0)^{-1}(u) \subseteq \mathcal{M}_{\text{Dol}}(X, n)$$

is holomorphic Lagrangian; moreover, these fibers are affine; (**BAA branes**)

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- Hausel–Hitchin (2020): $G_\alpha^0(u) \subseteq \mathcal{M}_{\text{Dol}}(X, n)$ is closed if and only if u is **very stable**, that is,

$$G_\alpha^0(u) \cap \mathcal{N} = \{u\},$$

where \mathcal{N} is the nilpotent cone;

- $G_\alpha^1(u) \subseteq \mathcal{M}_{\text{dR}}(X, n)$ is closed, a strict proof for $n = 2$ was given by Dimakis–Schulz (2024).

Conformal limits and transversality

- Let $(E, \bar{\partial}_E, \varphi) \in \mathcal{M}_{\text{Dol}}(X, n)$ be a stable Higgs bundle with harmonic metric h , then for any $t \in \mathbb{R}^+$ and $\xi \in \mathbb{C}^*$,

$$D_{(t, \xi)} := \bar{\partial}_E + \partial_{h_t} + \xi^{-1} t \varphi + \xi t \varphi^{*h_t}$$

is a flat connection, where h_t is the harmonic metric corresponding to the stable $(E, \bar{\partial}_E, t\varphi)$;

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- Fixing the parameter $\hbar := t^{-1}\xi$, the limiting flat connection

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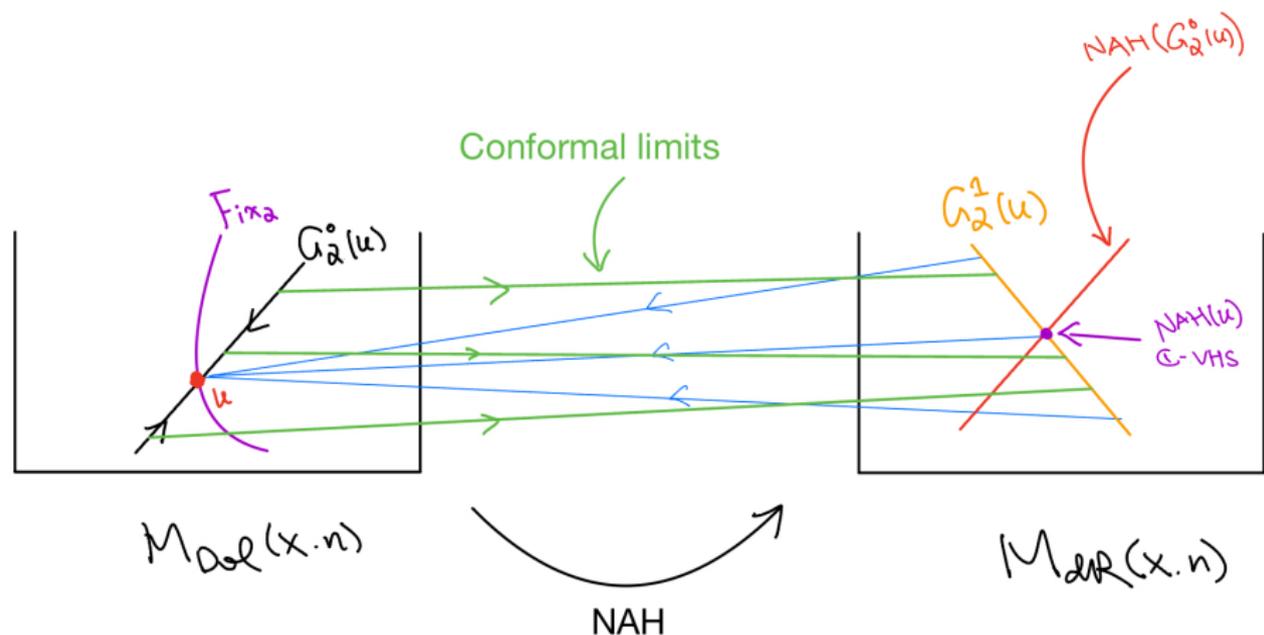
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- Collier-Wentworth** (2019): for any $u \in \text{Fix}_\alpha$ stable, the conformal limit takes $G_\alpha^0(u)$ biholomorphically to $G_\alpha^1(u)$, and moreover, under the map NAH, $G_\alpha^0(u)$ intersects with $G_\alpha^1(u)$ transversely:

$$\text{NAH}(G_\alpha^0(u)) \pitchfork G_\alpha^1(u).$$

The picture



I: INTRODUCTION

♣ Simpson's foliation conjecture

Foliation conjecture

Conjecture 2 (Foliation conjecture, Simpson, 2008)

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- ▶ For a fixed α , this holds clearly.
- ▶ This conjecture is fully open when the base X is **compact**. (“Philosophy” for topological problems: compact is difficult than noncompact)
- ▶ In the birthday conference of Esnault in 2024, Esnault and Groechenig announced a proof to this conjecture via p -adic method.

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- I do not know whether there exists a weight system so that the conjecture holds when $k \geq 6$.

II: SETUP

 some notations

Notations

- Base: X compact Riemann surface of genus g , with $\mathcal{D} = \{z_1, \dots, z_k\}$;
- (Rank, Degree) = (n, d) ;
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- **Parabolic degree:**

$$\text{pdeg}(E, \mathcal{L}, \vec{w}) := d + \sum_{i=1}^k \vec{d}^{(i)} \cdot \vec{w}^{(i)} = d + \sum_{i=1}^k \sum_{j=0}^{\ell^{(i)}} d_j^{(i)} \cdot w_j^{(i)}.$$

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- ∇^λ is called a **parabolic** (resp. **weakly parabolic**) **logarithmic λ -connection** if ∇^λ is compatible (resp. weakly compatible) with \mathcal{L} , namely

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- Under the weight system \vec{w} , we have the **\vec{w} -stability** (w.r.t. parabolic slope).

II: SETUP

 moduli spaces and the correspondence

Moduli spaces (GIT)

Theorem 3 (Boden-Yokogawa, Yokogawa, Inaba-Iwasaki-Saito, Inaba, ...)

$$\mathcal{M}_{\text{Dol}}^{\text{WP}}(d, \vec{d}, \vec{w}) \quad \text{irreducible normal quasi-projective} \quad \mathcal{M}_{\text{dR}}^{\text{WP}}(d, \vec{d}', \vec{w}') \\ \dim = n^2(2g-2+k)+1$$

$$\cup$$

$$\mathcal{M}_{\text{Dol}}^{\text{WP}}(d, \vec{d}, \vec{w}, \vec{\nu})$$

$$\dim = n^2(2g-2+k)+2-kn \quad (\text{even})$$

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$$\dim = n^2(2g-2+k)+2 - \sum_{i=1}^k \sum_{j=0}^{\ell(i)} (d_j^{(i)})^2 \quad (\text{even})$$

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- When the weight system is **generic**, the above moduli spaces are *smooth*.

Nonabelian Hodge correspondence

Theorem 4 (Simpson, Mochizuki, Szabó, ...)

Under certain relation between $(\vec{d}, \vec{w}, \vec{\nu})$ and $(\vec{d}', \vec{w}', \vec{\nu}')$, the tame nonabelian Hodge correspondence provides the following diffeomorphism

$$\mathcal{M}_{\text{Dol}}^{\text{WP}}(d, \vec{d}, \vec{w}, \vec{\nu}) \simeq \mathcal{M}_{\text{dR}}^{\text{WP}}(d, \vec{d}', \vec{w}', \vec{\nu}').$$

In particular, denote by $\mathcal{M}_{\text{Dol}}^{\text{SP}}(d, \vec{d}, \vec{w})$ the moduli space of \vec{w} -stable strongly parabolic logarithmic Higgs bundles, the above identification restricts to a diffeomorphism

$$\mathcal{M}_{\text{Dol}}^{\text{SP}}(d, \vec{d}, \vec{w}) \simeq \mathcal{M}_{\text{dR}}(d, \vec{d}, \vec{w}, \vec{w}).$$

- ▶ **Remark:** To achieve the above correspondence, we need the **parabolic degree zero** condition.

II: SETUP

♣ foliation conjecture in noncompact case

The \mathbb{G}_m -action and the fixed point loci

Proposition 5 (Simpson, Komyo, ...)

For any parabolic logarithmic flat bundle $(E, \mathcal{L}, \nabla) \in \mathcal{M}_{\text{dR}}(d, \vec{d}, \vec{w}, \vec{\nu})$, the limit

$$\Psi_{\vec{w}}(E, \mathcal{L}, \nabla) := \lim_{t \rightarrow 0} (E, \mathcal{L}, t\nabla)$$

exists in $\mathcal{M}_{\text{Dol}}^{\text{SP}}(d, \vec{d}, \vec{w})$ as a \mathbb{G}_m -fixed point. Thus defines a map

$$\Psi_{\vec{w}} : \mathcal{M}_{\text{dR}}(d, \vec{d}, \vec{w}, \vec{\nu}) \longrightarrow \text{Fix}(d, \vec{d}, \vec{w}) \subseteq \mathcal{M}_{\text{Dol}}^{\text{SP}}(d, \vec{d}, \vec{w})$$

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- Then the foliation conjecture in the parabolic case is the following

Conjecture 6 (Foliation conjecture in noncompact setting)

There exists a weight system \vec{w} so that $\Psi_{\vec{w}}$ is surjective and its fibers form a regular foliation with closed leaves on the de Rham moduli space $\mathcal{M}_{\text{dR}}(d, \vec{d}, \vec{w}, \vec{\nu})$.

III: MAIN RESULTS

 our specifies

Specifies

- Base: $X = \mathbb{P}^1$ with $\mathcal{D} = \{z_1, \dots, z_5\}$;
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- So in our setting, the moduli space $\mathcal{M}_{\text{dR}}(1, \vec{d}, \vec{w}, \vec{\nu})$ has dimension $2(5 - 3) = 4$.

The fixed point locus

- By our setting, if $(E, \mathcal{L}, \theta) \in \text{Fix}(1, \vec{d}, \vec{w})$, then E is either one of the following two forms:
 - $E_0 := \mathcal{O}(-1) \oplus \mathcal{O}(2)$;
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Proposition A (Hu-H.-Zong)

The fixed locus $\text{Fix}(1, \vec{d}, \vec{w})$ decomposes as two components

$$\text{Fix}(1, \vec{d}, \vec{w}) = \text{Fix}^0 \amalg \text{Fix}^1,$$

where

- Fix^0 (dim. 0) consists of a single point representing $(E_0, \mathcal{L}, \theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, with each $L_i = \mathcal{O}(-1)|_{z_i}$;
- Fix^1 (dim. 2) consists of points representing $(E_1, \mathcal{L}, \theta = \begin{pmatrix} 0 & \Theta \\ 0 & 0 \end{pmatrix})$, with each

$$L_i = \begin{cases} \mathcal{O}|_{z_i}, & \theta|_{z_i} \neq 0; \\ \mathcal{O}|_{z_i} \text{ or } \mathcal{O}(1)|_{z_i}, & \theta|_{z_i} = 0, \end{cases}$$

and $\Theta \in (H^0(\mathbb{P}^1, \mathcal{O}(2)) - \{0\})/\mathbb{G}_m \cong \mathbb{P}^2$.

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Theorem B (Hu-H.-Zong)

Let $\mathbb{M}_{E_1}^{\text{ind}}(\vec{d})$ be the moduli stack of indecomposable parabolic structures with dimension vector \vec{d} on E_1 , then there exists a non-separated scheme \mathfrak{F} such that

- (1) $\mathbb{M}_{E_1}^{\text{ind}}(\vec{d})$ is a \mathbb{G}_m -gerbe over \mathfrak{F} ;
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Realizing Fix^1 as a non-separated scheme

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- The stacks $\mathbb{M}_{E_1}^{\text{ind}}(\vec{d})$, $\mathbb{M}^{\text{ind}}(d, \vec{d})$ (stack of rank 2 indecomposable parabolic bundles of type (d, \vec{d})) and the corresponding coarse moduli spaces are also considered by Loray–Saito, and Donagi–Pantev. (“gluing del Pezzo surfaces of degree 4, with \mathbb{P}^2 s”)

Constructing the non-separated scheme \mathfrak{F}

- By **Proposition A**, the parabolic structure at the point in \mathcal{D} has 2 choices if it is the zero of the Higgs field θ ;
- The zeros of the Higgs field θ (two points in \mathbb{P}^1) define a double cover of \mathbb{P}^2 , with 15 special points, and 5 special lines lying in \mathbb{P}^2 ;
- Among the 15 points, 5 are determined by the condition when θ has zeros of order 2 at z_i ;
- Blowing up \mathbb{P}^2 at these 5 points produce a del Pezzo surfaces of degree 4, denoted as \mathfrak{B} ;
- \mathfrak{F} is obtained by gluing \mathfrak{B} and a copy $\underline{\mathfrak{B}}$ of \mathfrak{B} , so that each special line in \mathfrak{B} is glued to the corresponding exceptional curve in $\underline{\mathfrak{B}}$, and vice versa.

Surjectivity of $\Psi_{\vec{w}}$ and analyzing $\Psi_{\vec{w}}^{-1}(\text{Fix}^1)$

Theorem C (Hu-H.-Zong)

Let $\Psi_{\vec{w}} : \mathcal{M}_{\text{dR}}(1, \vec{d}, \vec{w}, \vec{\nu}) \rightarrow \text{Fix}(1, \vec{d}, \vec{w})$ be the \mathbb{G}_m -limit map, then

- (1) $\Psi_{\vec{w}}$ is surjective;
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- For any $\mathbb{E} = (E, \mathcal{L}, \nabla) \in \text{Fix}(1, \vec{d}, \vec{w})$, by describing the fiber $\Psi_{\vec{w}}^{-1}(\mathbb{E})$ to show $\Psi_{\vec{w}}$ is surjective. Divide into two cases: $\mathbb{E} \in \text{Fix}^1$, or $\mathbb{E} \in \text{Fix}^0$;

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- Thus we can calculate the dimension of $\Psi_{\vec{w}}^{-1}(\mathbb{E})$ for each case, it always equals to 2.

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 - So we can find a family $(E_t, \nabla_t, \varphi_t)$ in $\Psi_{\vec{w}}^{-1}(\text{Fix}^1)$ with limit

$$\lim_{t \rightarrow 0} (E_t, \nabla_t, \varphi_t) = (E, \nabla, \varphi).$$

Thank You So Much!