Galois descent for generalized monodromy data

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Painlevé Equations and related topics
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Monodromy

Examples: Differential equations on the complex affine line $X = \mathbb{A}^1(\mathbb{C})$ (coordinate z):

- $z\frac{d}{dz}f(z) + f(z) = 0$

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- $z\frac{d}{dz}f(z) + \gamma f(z) = 0 \quad \rightsquigarrow f(z) = c \cdot z^{-\gamma}$
  sheaf of solutions: local system on $X \setminus \{0\}$ with stalk $\mathbb{C}$ and monodromy $e^{-2\pi i\gamma}$

Riemann–Hilbert correspondences

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Correspondence between differential equations and topological data
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Theorem (Deligne '70)

X complex manifold, $D \subset X$ divisor

$$\left\{ \begin{array}{c} \text{meromorphic connections on } X \\ \text{with regular poles at } D \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{local systems} \\ \text{on } X \setminus D \end{array} \right\}$$

Motivation

Two facts:

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Our case of interest: **Hypergeometric differential equations**

Fedorov '18: Real structures of hypergeometric equations with regular singularities
Barco–Hien–H.–Sevenheck '23: Allow irregular singularities and consider more general subfields of $\mathbb{C}$

Plan of the talk

1. Riemann–Hilbert correspondence for irregular singularities (d'après D'Agnolo–Kashiwara)
2. Galois descent
3. Results for hypergeometric systems

D-modules: An algebraic theory of differential equations

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local sections: finite sums of the form

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Observation

Let $P \in \mathcal{D}_X$, then consider the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X/(P)$. One has

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \{f \in \mathcal{O}_X ; Pf = 0\}$$

Regular Riemann–Hilbert correspondence

$$\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \xrightarrow{\mathcal{S}ol_X = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(-,\mathcal{O}_X)} \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$$

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Kashiwara '81/'84, Mebkhout '84

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{reghol}}(\mathcal{D}_X) \xrightarrow[\sim]{\mathcal{S}ol_X = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(-,\mathcal{O}_X)} \mathrm{D}^{\mathrm{b}}_{\mathrm{constr}}(\mathbb{C}_X)$$

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\begin{array}{ccc}
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A basic example of an irregular D-module

$X = \mathbb{A}^1(\mathbb{C})$

reg $\mathcal{M}_1 = \mathcal{D}_X / \mathcal{D}_X (z^2 \frac{\mathrm{d}}{\mathrm{d}z} + z)$

solutions: $c \cdot \frac{1}{z} \rightsquigarrow$ local system $\mathbb{C}_{X \setminus \{0\}}$

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irreg $\mathcal{M}_2 = \mathcal{E}^{\frac{1}{z}} = \mathcal{D}_X / \mathcal{D}_X(z^2 \frac{\mathrm{d}}{\mathrm{d}z} + 1)$

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In fact: $\mathcal{S}ol_X(\mathcal{M}_1) \simeq \mathcal{S}ol_X(\mathcal{M}_2)$

Idea: Use a different solution functor that takes into account the growth of the solutions.

Tempered solutions

Idea (Kashiwara–Schapira): Replace $\mathcal{O}_X$ by $\mathcal{O}_X^{\mathrm{t}}$, defined by

$$\mathcal{O}_X^{\mathrm{t}}(U) := \left\{ f \in \mathcal{O}_X(U); \exists C, M \in \mathbb{R}_{>0} \, \forall z \in U : |f(z)| \leq \frac{C}{\operatorname{dist}(z, \partial U)^M} \right\}$$

(for $U \subset X$ relatively compact). $\mathcal{S}ol_X^{\mathrm{t}} := \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{O}_X^{\mathrm{t}})$

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Then for $U \subset X \setminus \{0\}$

$$\mathrm{H}^0 \mathcal{S}ol_X^{\mathrm{t}}(\mathcal{M}_1)(U) = \mathbb{C}$$

$$\mathrm{H}^0 \mathcal{S}ol_X^{\mathrm{t}}(\mathcal{M}_2)(U) = \begin{cases} \mathbb{C} & \text{if } \mathrm{Re}\, \frac{1}{z} \text{ is bounded on } U \\ 0 & \text{otherwise} \end{cases}$$

Technical problem: $\mathcal{O}_X^{\mathrm{t}}$ is not a sheaf!

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$\mathcal{O}_X^{\mathrm{t}}$ **is** a sheaf on the *subanalytic site*:

open sets: subanalytic relatively compact open subsets of X
coverings: finite coverings

In this framework, we can write

$$\mathrm{H}^0\mathcal{S}ol^{\mathrm{t}}(\mathcal{M}_1) \simeq \mathbb{C}_{X\setminus\{0\}}$$
$$\mathrm{H}^0\mathcal{S}ol^{\mathrm{t}}(\mathcal{E}^{\frac{1}{z}}) \simeq \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbb{C}_{\{z\in X\setminus\{0\};\mathrm{Re}\,\frac{1}{z}<a\}}$$

Enhanced ind-sheaves

A further construction is needed to obtain a fully faithful functor.
$\rightsquigarrow$ *Enhanced ind-sheaves* $\mathsf{E}^{\mathsf{b}}(\mathrm{I}\mathbb{C}_X) \approx$ Subanalytic sheaves on $X \times \overline{\mathbb{R}}$

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Theorem (D'Agnolo–Kashiwara '16)

Let X be a complex manifold. There is a fully faithful functor

$$\mathcal{S}ol_X^{\mathsf{E}} \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{hol}}(\mathcal{D}_X) \hookrightarrow \mathsf{E}^{\mathsf{b}}_{\mathbb{R}\text{-}\mathsf{c}}(\mathrm{I}\mathbb{C}_X)$$

extending the Riemann–Hilbert functor for regular holonomic $\mathcal{D}_X$-modules

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Six functors: $\overset{+}{\otimes}$, $\mathsf{E}f_*$, $\mathsf{E}f_{!!}$, etc.

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Galois descent – the case $\mathbb{C}/\mathbb{R}$

V: complex vector space

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There is a correspondence

isomorphisms		sub-$\mathbb{R}$-vector spaces
$\varphi\colon V \xrightarrow{\sim} \overline{V}$		$W \subset V$
such that	$\longleftrightarrow$	such that
$V \xrightarrow{\varphi} \overline{V} \xrightarrow{\overline{\varphi}} \overline{\overline{V}} = V$		$W \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$
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This generalizes to finite Galois extensions L/K.

Galois descent for enhanced ind-sheaves

Theorem (BHHS '23, H. '24)

Let L/K be a finite Galois extension. Let G be its Galois group. Then there is a correspondence between

- *objects of* $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)$ *and*
- *pairs* $(F, (\varphi_g)_{g\in G})$, *where* $F \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_X)$ *and* $\varphi_g \colon F \xrightarrow{\sim} \overline{F}^g$ *such that for any* $g, h \in G$ *the diagram*

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In other words, if we are given F and we can find such a collection of φ_g's, then there exists an object F_K such that $F \simeq L_X \otimes_{K_X} F_K$.

Hypergeometric differential equations

$X = \mathbb{G}_{\mathrm{m},q}$: one-dimensional complex algebraic torus, coordinate q.
For $\alpha_1, \ldots, \alpha_n, \beta_1 \ldots, \beta_m \in [0,1) \subset \mathbb{R}$, set

$$P := \prod_{i=1}^{n} (q\partial_q - \alpha_i) - q \prod_{j=1}^{m} (q\partial_q - \beta_j); \qquad \mathcal{H}(\alpha;\beta) := \mathcal{D}_X/(P).$$

We assume $\alpha_i \neq \beta_j$ for any i, j.

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Theorem (Barco–Hien–H.–Sevenheck 2023)

If the non-zero α_i come in pairs that sum up to 1 (and the same for the β_j), then there exists $F \in \mathsf{E}^{\mathsf{b}}_{\mathbb{R}\text{-}c}(\mathbb{R}_X)$ such that

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{H}(\alpha;\beta)) \simeq \mathbb{C}_X \otimes_{\mathbb{R}_X} F.$$

Idea of the proof

"Proposition" (close to reality)

$$\mathcal{H}(\alpha;\beta) \simeq p_+(\mathcal{E}^\varphi \otimes \mathcal{R})$$

$p\colon (\mathbb{G}_\mathrm{m})^{n+m} \times \mathbb{G}_{\mathrm{m},q} \to \mathbb{G}_{\mathrm{m},q}$ *projection,*

$\varphi = \frac{1}{x_1} + \ldots + \frac{1}{x_n} + x_{n+1} + \ldots + x_{n+m} + q \cdot x_1 \cdots x_{n+m}$,

$\mathcal{R}$ *regular with solutions* $c \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot x_{n+1}^{\beta_1} \cdots x_{n+m}^{\beta_m}$.

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$\mathcal{R}$ regular with solutions $c \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot x_{n+1}^{\beta_1} \cdots x_{n+m}^{\beta_m}$.

Hence

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{H}(\alpha;\beta)) \simeq \mathsf{E}p_{!!}(\mathbb{E}^\varphi \overset{+}{\otimes} \mathcal{L})$$

$\uparrow$ local system with monodromies $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_n}, e^{2\pi i\beta_1}, \ldots, e^{2\pi i\beta_m}$, id

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$$\overline{\mathcal{S}ol_X^{\mathsf{E}}(\mathcal{H}(\alpha;\beta))} \simeq \mathsf{E}p_{!!}(\mathbb{E}^{\varphi} \overset{+}{\otimes} \overline{\mathcal{L}})$$

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A more general version

Theorem (Barco–Hien–H.–Sevenheck 2023)

Let $L \subset \mathbb{C}$ be a field containing $e^{2\pi i\alpha_i}, e^{2\pi i\beta_j}$ for any i, j. Let $K \subset L$ be a finite Galois extension such that the natural action of the Galois group $\mathrm{Gal}(L/K)$ on L induces actions on the sets $\{e^{2\pi i\alpha_1}, \dots, e^{2\pi i\alpha_n}\}$, $\{e^{2\pi i\beta_1}, \dots, e^{2\pi i\beta_m}\}$. Then there exists $F \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)$ such that

$$\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{H}(\alpha;\beta)) \simeq \mathbb{C}_X \otimes_{K_X} F.$$

One can make this explicit for concrete examples of K, e.g. $K = \mathbb{Q}$.

Generalized monodromy data — Stokes data

$X =$ disk around the origin, $D = \{0\}$

Recall:

$$\left\{ \begin{array}{c} \text{meromorphic connections on } X \\ \text{with regular pole at } 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{local systems} \\ \text{on } X \setminus \{0\} \end{array} \right\}$$

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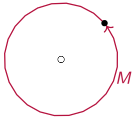

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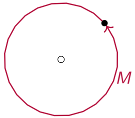
The general (i.e. not necessarily regular) case:

$$\left\{ \begin{array}{c} \text{meromorphic connections on } X \\ \text{with pole at } 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Stokes data} \\ \text{around } 0 \end{array} \right\}$$

Generalized monodromy data — Stokes data

$X =$ disk around the origin, $D = \{0\}$

monodromy



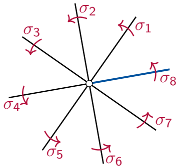
Recall:

$$\left\{ \begin{array}{c} \text{meromorphic connections on } X \\ \text{with regular pole at } 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{local systems} \\ \text{on } X \setminus \{0\} \end{array} \right\}$$

The general (i.e. not necessarily regular) case:

$$\left\{ \begin{array}{c} \text{meromorphic connections on } X \\ \text{with pole at } 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Stokes data} \\ \text{around } 0 \end{array} \right\}$$

Stokes data



Descent for Stokes data

Theorem (H. '24)

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. If $\mathcal{S}ol^{\mathsf{E}}(\mathcal{M})$ comes from an object over K, then its Stokes data can be represented by matrices with entries in K.

Related results

Theorem (H.–Schapira '23)

Let $f\colon X \to Y$ be a morphism of real analytic manifolds and L/K a field extension. Let F (resp. G) be a sheaf of K-vector spaces on X (resp. Y). Under weak constructibility assumptions on F, G and f, one has isomorphisms

$$f_*(F \otimes_{K_X} L_X) \xleftarrow{\sim} (f_*F) \otimes_{K_Y} L_Y,$$
$$f^!(G \otimes_{K_Y} L_Y) \xleftarrow{\sim} (f^!G) \otimes_{K_X} L_X.$$

Related results

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Theorem (H. '24)

Kashiwara's conjugation functor for $\mathcal{D}_X$-modules corresponds to complex conjugation on enhanced ind-sheaves via the Riemann–Hilbert functor

$$\mathcal{S}ol_X^{\mathsf{E}}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{hol}}(\mathcal{D}_X) \longrightarrow \mathsf{E}^{\mathsf{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X).$$

Thank You!

- D. Barco, M. Hien, A. Hohl and C. Sevenheck, *Betti Structures of Hypergeometric Equations*, Int. Math. Res. Not. **2023** (2023), 10641–10701.
- A. Hohl, *An introduction to field extensions and Galois descent for sheaves of vector spaces*, 2023, `arXiv:2302.14837`.
- A. Hohl and P. Schapira, *Unusual functorialities for weakly constructible sheaves*, 2023, `arXiv:2303.11189`.
- A. Hohl, *Kashiwara conjugation and the enhanced Riemann–Hilbert correspondence*, Port. Math. **81** (2024), 347–387.