Hamiltonian aspects of the tt^*-Toda equations

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Plan

§1 Some Lie theory

§2 tt^*-Toda (higher order Painlevé)

§3 Results (with Chang-Shou Lin, Alexander Its, Nan-Kuo Ho)

§4 Hamiltonian aspects

§1 Some Lie theory

G = complex semisimple Lie group, $\pi_1 G = 0$, rank $= l$

eg $G = Sl_n\mathbb{C}$ $(l = n-1)$

Def $G^{reg} = \{ g \in G \mid \dim C(g) = \operatorname{rank} G \}$

Thm (Steinberg) (1) $G^{reg}/G \cong \mathbb{C}^l$ (conjugacy classes)

$$[g] \longleftrightarrow (\chi_1(g), \dots, \chi_l(g))$$

($\chi_1, \dots, \chi_l$ = characters of basic representations of G)

(2) Fix simple roots $\alpha_1, \dots, \alpha_l$. Then the map

$$\mathbb{C}^l \to G^{reg}, \quad (t_1, \dots, t_l) \longmapsto \prod_{i=1}^{l} \exp(t E_{\alpha_i})\, s_{\alpha_i}$$

is a cross-section of $G^{reg} \to G^{reg}/G \cong \mathbb{C}^l$.

[eg $G = Sl_{n+1}\mathbb{C}$ $(l = n)$ $\quad t \longmapsto \left(\begin{array}{ccc|c} t_1 & \cdots & t_n & 1 \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & \end{array}\right)$]

We denote the image by $S_{\alpha_1, \dots, \alpha_l}$ or S_{Δ_+}

(Steinberg cross-section)

§2 tt^* – Toda

tt^* = topological-antitopological fusion

Cecotti – Vafa 1991 : equations (p.d.e.) for deformations of SUSY QFT

Dubrovin 1993 : integrability of tt^* equations

$\Updownarrow$

$\exists$ zero-curvature formulation (Lax Pair)

and

$\exists$ isomonodromy formulation (o.d.e. in spectral parameter)

Eg :

tt^* of "Toda type" with $G = Sl_{n+1}\mathbb{C}$ is the system

$$2(w_i)_{t\bar{t}} = -e^{2w_{i+1}-2w_i} + e^{2w_i-2w_{i-1}}$$

for $w_0, \dots, w_n : \mathbb{C}^* \to \mathbb{R}$

with $w_i + w_{n-i} = 0$ and $w_i = w_i(|t|)$ $(\forall i)$

Zero-curvature formulation : $\quad d\alpha + \alpha\wedge\alpha = 0$

$$\alpha = \left[\begin{pmatrix} w_0 & & \\ & \ddots & \\ & & w_n \end{pmatrix}_t + \frac{1}{\lambda}\begin{pmatrix} 0 & & & e^{w_0-w_n} \\ e^{w_1-w_0} & \ddots & & \\ & \ddots & \ddots & \\ & & e^{w_n-w_{n-1}} & 0 \end{pmatrix} \right] dt \; +$$

$$\left[-\begin{pmatrix} w_0 & & \\ & \ddots & \\ & & w_n \end{pmatrix}_{\bar t} + \lambda \begin{pmatrix} 0 & e^{w_1-w_0} & & \\ & \ddots & \ddots & \\ & & \ddots & e^{w_n-w_{n-1}} \\ e^{w_0-w_n} & & & 0 \end{pmatrix} \right] d\bar t$$

Isomonodromy formulation : $\quad \hat\alpha$ is isomonodromic

$$\hat\alpha = \left[-\frac{t}{\lambda^2}\begin{pmatrix} 0 & & & e^{w_0-w_n} \\ e^{w_1-w_0} & \ddots & & \\ & \ddots & \ddots & \\ & & e^{w_n-w_{n-1}} & 0 \end{pmatrix} - \frac{x}{\lambda}\begin{pmatrix} w_0 & & \\ & \ddots & \\ & & w_n \end{pmatrix}_x + \bar t \begin{pmatrix} 0 & e^{w_1-w_0} & & \\ & \ddots & \ddots & \\ & & \ddots & e^{w_n-w_{n-1}} \\ e^{w_0-w_n} & & & 0 \end{pmatrix} \right] d\lambda$$

$$x = |t|$$

[i.e. Stokes matrices at $\lambda = 0$, Stokes matrices at $\lambda = \infty$, and "connection matrices" are independent of λ]

<u>Remark</u> For $n=1$: $\quad tt^*\text{-Toda} = P\,\mathrm{III}\;(D8)$

§3 Results

Thm (Guest-Ho) The monodromy data $\{S_k^{(0)}, S_k^{(\infty)}, E_k\}$ (corr. to a solution of tt^*-Toda) is equivalent to a pair $(M,E) \in G \times G$ where

$M \in S_{\Delta_+}$ (a particular Steinberg cross-section)

$ME = EM$

(A) M, E satisfy "anti-symmetry conditions" (corr. to $w_i + w_{n-i} = 0$)

(R) M, E satisfy "reality conditions" (corr. to $w_i \in \mathbb{R}$)

(Recall that Stokes matrices are defined by $\Psi_{k+1}^{(0)} = \Psi_k^{(0)} S_k^{(0)}$ and $\Psi_{k+1}^{(\infty)} = \Psi_k^{(\infty)} S_k^{(\infty)}$ where $\Psi_k^{(0)}, \Psi_k^{(\infty)}$ are canonical solutions on Stokes sectors ; connection matrices are defined by $\Psi_k^{(\infty)} = \Psi_k^{(0)} E_k$.)

(Brief explanation of M, E:

$E \iff E_k$'s

$M \iff$ Stokes data (and M^h = monodromy, where h = Coxeter number))

Riemann-Hilbert Correspondence:

meromorphic solution of tt^*-Toda $\longmapsto$ (M, E)

(injective ; probably surjective (?))

Smooth solutions :

Thm (Guest-Its-Lin, Mochizuki) $(G = Sl_{n+1}\mathbb{C})$

smooth on $\mathbb{C}^*$ (no pole on $\mathbb{C}^*$) $\iff$ $E = I$, M has all eigenvalues in S^1

$\uparrow$

$e^{\frac{2\pi i}{n+1}(m_j + \rho_j)}$ where

$\rho = \left(\frac{n}{2}, \frac{n}{2}-1, \dots, -\frac{n}{2}\right)$

$m_{i-1} - m_i + 1 \geq 0$

Asymptotics at $t=0$:

$w_i(|t|) \sim -m_i \log|t|$

Asymptotics at $t = \infty$:
[omitted]

(as conjectured by Cecotti - Vafa)

<u>Def</u> (Lusztig) $Z = \{(M,E) \in S_{\Delta_+} \times G \mid ME = EM\}$
is the <u>universal centralizer</u>

<u>Thm</u> (Finkelberg – Tsymbaliuk) Z is a complex manifold
with a (natural) holomorphic symplectic structure

"quasi-Hamiltonian reduction of $G \times G$ in the sense of
Alekseev – Malkin – Meinrenken" (Bezrukavnikov – Finkelberg – Mirković)

<u>Cor</u> (Guest – Ho) $\mathcal{S} \overset{\text{def}}{=} \{(M,E) \in Z \mid (A),(R) \text{ hold}\}$
has a (natural) <u>real</u> symplectic structure

cf. work of Boalch on moduli spaces of meromorphic connections
with 'irregular' singularities

§4 Hamiltonian aspects

$(G = SL_{n+1}\mathbb{C})$

(Ryosuke Odoi, arXiv 2022) (+ in progress)

<u>Def</u> $\mathcal{S}^{0<|t|<\epsilon} = \left\{ \begin{array}{l} \text{solutions of } tt^*\text{-Toda on intervals} \\ \text{of the form } 0<|t|<\epsilon \text{, with} \\ w_i(|t|) \sim -m_i \log|t| + l_i + o(1) \end{array} \right\}$

$\cong \left\{ \begin{array}{l} \text{diagonalizable } M, E \text{ with} \\ \text{eigenvalues } M_0, \dots, M_n,\ E_0, \dots, E_n \\ \text{and } |M_i| = 1,\ E_i > 0 \ \forall i \end{array} \right\}$

<u>Remark</u> $\mathcal{S}^{0<|t|<\infty} \subseteq \mathcal{S}^{0<|t|<\epsilon} \underset{\text{open}}{\subseteq} \mathcal{S}$

in this case l_i is determined by m_i
(and $E_i = 1$)

<u>Thm</u> The symplectic form on $\mathcal{S}^{0<|t|<\epsilon}$ is given explicitly by
$\sum_{i=0}^{n} \frac{dM_i}{M_i} \wedge \frac{dE_i}{E_i}$ (x constant). It follows (from the G-I-L theorem) that this is equal to $\sum_{i=0}^{n} dm_i \wedge dl_i$ (x constant).

(canonical coordinates)

<u>Application</u>: tau functions for $(w_0, \dots, w_n) \in \mathcal{S}^{0<|t|<\infty}$

(Brief explanation: the tt^*-Toda equations have a Hamiltonian formulation, with Hamiltonian $H = H(w_0, \dots, w_n, w_0', \dots, w_n')$. The tau function is defined by $(\log \tau)' = H$.)

From $w_i(|t|) \sim -m_i \log |t|$, $t \to 0$

$w_i(|t|) \sim$ [omitted], $t \to \infty$

we obtain the asymptotics of τ at $t=0$, $t=\infty$ up to constant factors C_0, C_∞

<u>Thm</u> C_0 / C_∞ = [explicit formula] (Odoi, arXiv 2022)

cf. Tracy

cf. Its - Prokhorov