The Geometric P=W conjecture and Thurston’s compactification

Mohammad F. Tehrani
Joint with C. Frohman and A. A. Kutteri

Webinar on Painlevé Equations and related topics

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Character Varieties of Surfaces

- $\Sigma_{g,n}$: genus g surface with n punctures

  G: Algebraic Reductive Lie group

$$\boxed{\mathcal{X}_{g,n}(G) = \mathrm{Hom}\left(\pi_1(\Sigma_{g,n}), G\right) /\!\!/ G}$$

- $n = 0 \implies$ I will write $\mathcal{X}_g(G)$

  $G = SL(2,\mathbb{C}) \implies$ I will write $\mathcal{X}_g$

- In arXiv: 2305.12306, Frohman & I studied punctured case
  I will review the results at the end

Dolbeault moduli space & P=W conjecture

- $\mathcal{M}_g(G)$ = Moduli space parametrizing flat Principal G- Higgs bundles on Σ_g

- non-abelian Hodge correspondence

$$\Psi : \mathcal{M}_g(G) \longrightarrow \mathcal{X}_g(G)$$

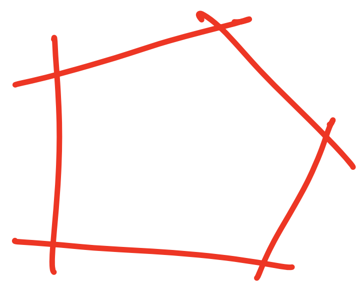
(analytic but not algebraic)

[Hitchin, Simpson, Corlette, Donaldson,...
see Eper- Szabo for a survey of results
and an extensive list of references]

- Geometric P=W conjecture aims to understand asymptotic behaviour of $\mathcal{F}$

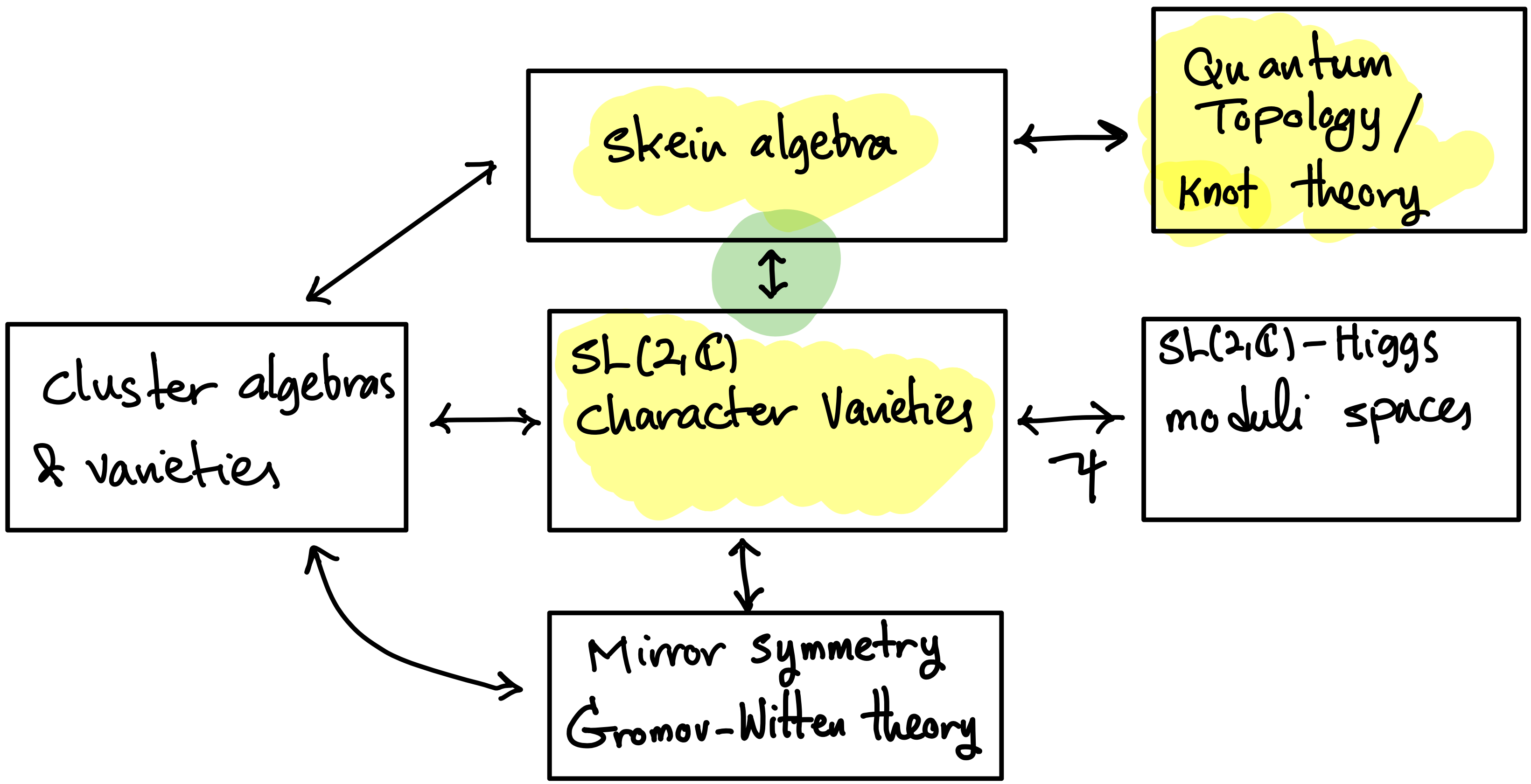
Conjecture [Mauri-Mazzon-Stevenson]

$\mathcal{X}_g$ admits a dlt log CY compactification such that the dual intersection complex of the boundary divisors is a polyhedral complex homeomorphic to a sphere (i.e. S^{6g-5})



Focus of this talk:

Constructing a projective compactification of $\mathcal{X}_g$ whose dual intersection complex is a sphere and even better!



$\boxed{1}$ $SL(2,\mathbb{C})$ – character varieties vs. Skein algebra

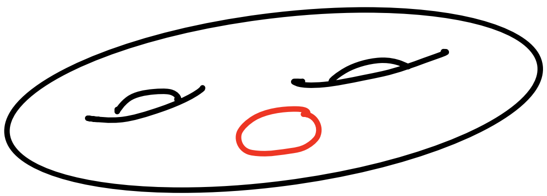
- Theorem (Bullock-Przytycki-Sikora, Charles-Marché + Barrett)

$$\underbrace{\mathbb{C}[X_{g,n}]}_{\text{Ring of regular functions}} \cong \underbrace{sk(\Sigma_{g,n})}_{\text{classical limit } q=1 \text{ of skein algebra}}$$

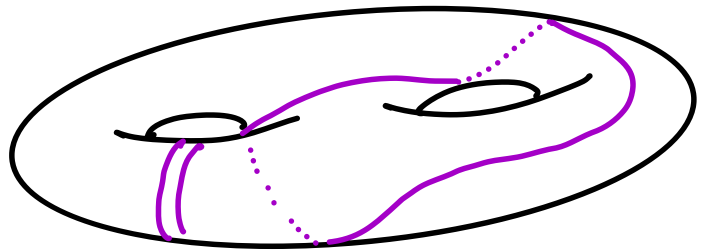
- $Sk(\Sigma_{g,n})$:= Algebra generated as a vector space by multicurves on $\Sigma_{g,n}$, with multiplication $*$ coming from taking union and resolving intersections

[Lets dig into this a bit]

- **Multicurve:** (Isotropy class of) a disjoint union of simple closed curves on $\Sigma_{g,n}$

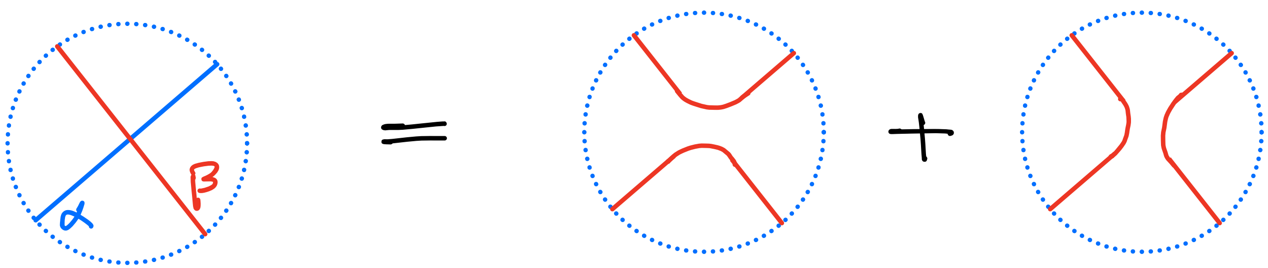


$= -2$



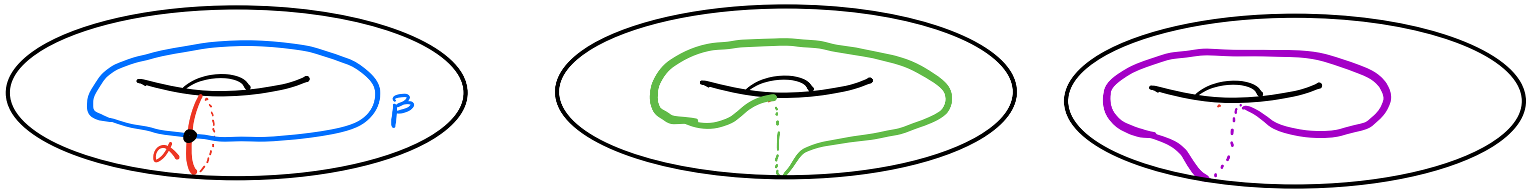
Convention: trivial curve = -2

- Product * :

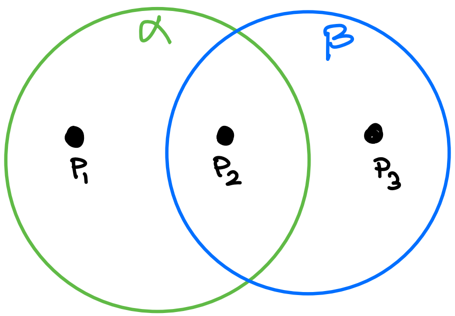


$\alpha * \beta$

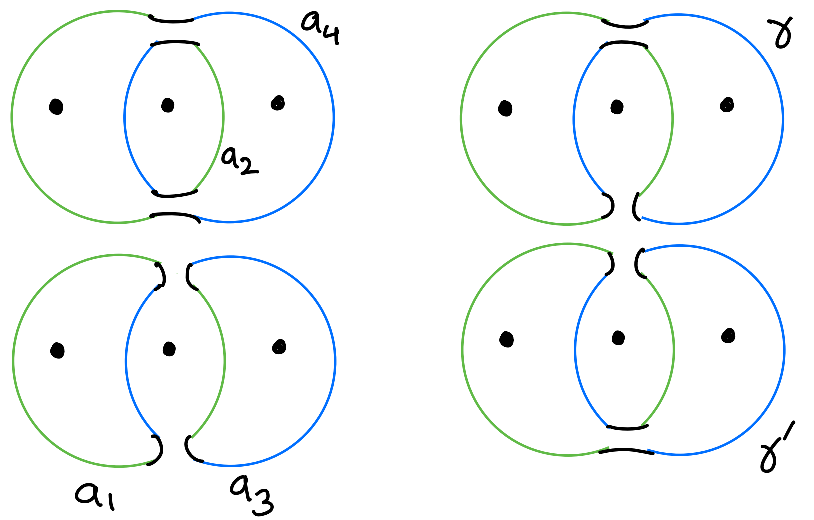
Example:



$$\alpha * \beta = \gamma + \gamma'$$



$\alpha * \beta$ =



$$= a_1 a_3 + a_2 a_4 + \gamma + \gamma'$$

$\boxed{2}$ Filtration $\to$ compactification

- There is a one-to-one correspondence

Projective Compactifications $\overline{X_g}$ $\longleftrightarrow$ Projective Filtrations on $\mathbb{C}[X_g]$

Given $F_0 \subseteq F_1 \subseteq \cdots \subseteq \bigcup_{d \geq 0} F_d = \mathbb{C}[X_g]$

we get a compactification

formal variable

$$\overline{X_g} = \mathrm{Proj}\left(\mathbb{C}[X_g][u] = F_0 \oplus F_1 u \oplus F_2 u^2 \oplus \cdots \right)$$

with $\partial \overline{X_g} = (u=0) = \mathrm{Proj}\left(\mathbb{C}[X_g]^{gr} \right)$

$$\mathbb{C}[X_g]^{gr} = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots$$

- idea: use Geometric intersection number with a collection of carves to define a filtration

- Geometric intersection number:

$$i : \underset{\alpha}{\text{Multicurves}} \times \underset{\beta}{\text{Multicurves}} \longrightarrow \underset{i(\alpha,\beta)}{\mathbb{N}}$$

$i(\alpha,\beta)$ = Minimum # intersection points of two transverse representatives

- Fix k curves $c_1 \cdots c_k$ $\Longrightarrow$

map any multicurve m to $(i(m,c_1), \dots, i(m,c_N)) \in \mathbb{N}^k$

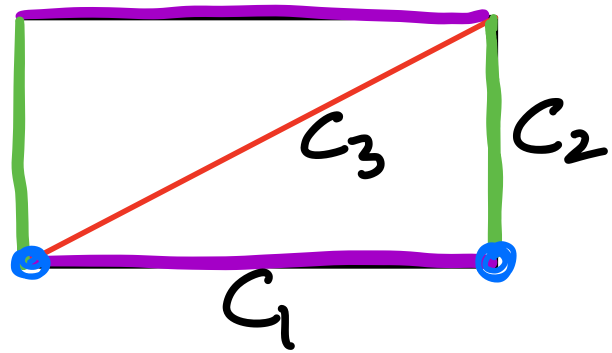
- $$\boxed{|m| = \sum_{j=1}^{k} i(m, c_j)}$$

$$\boxed{F_d = \left\{ \sum a_s m_s \ : \max(|m_s|) \leq d \right\} \subseteq Sk(\Sigma_g)}$$

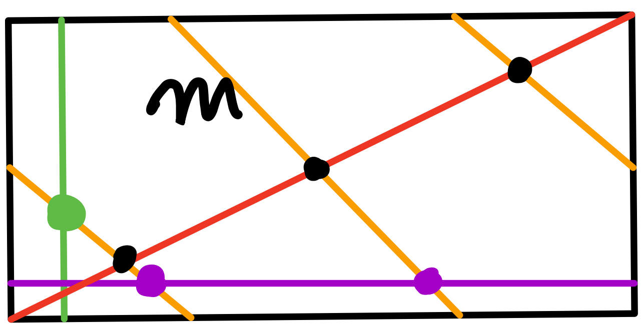
- Example :

Torus

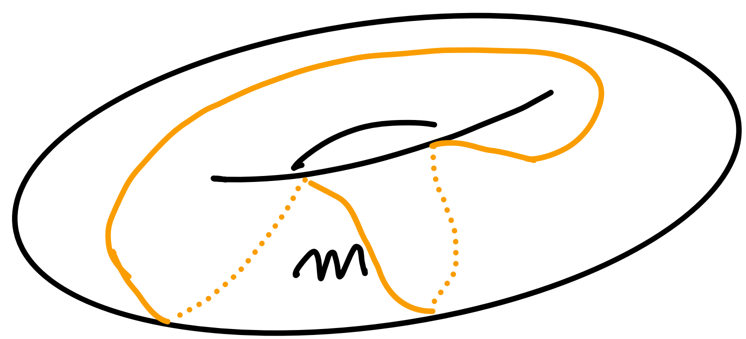
$g=1$



$|c_1| = |c_2| = |c_3| = 2$



$|m| = 6$



$\boxed{3}$ **Question**: What collection of curves results in a nice compactification

$\downarrow$

Look into quantum topology & the construction of Thurston's compactification

- On Σ_g:

  $\mathcal{S}$: isotopy classes of simple closed curves

  $\mathcal{S}'$: isotopy classes of multicurves

  $\mathcal{MF}$: space of measured Foliations

  $\mathcal{T}$: Teichmüller space

- $S \subset S' \subset MF$

  This is like closure of S' in some sense

- **Three embeddings**

$$\iota : S' \longrightarrow \mathbb{N}^{S} \qquad m \longrightarrow (i(m,c))_{c\in S}$$

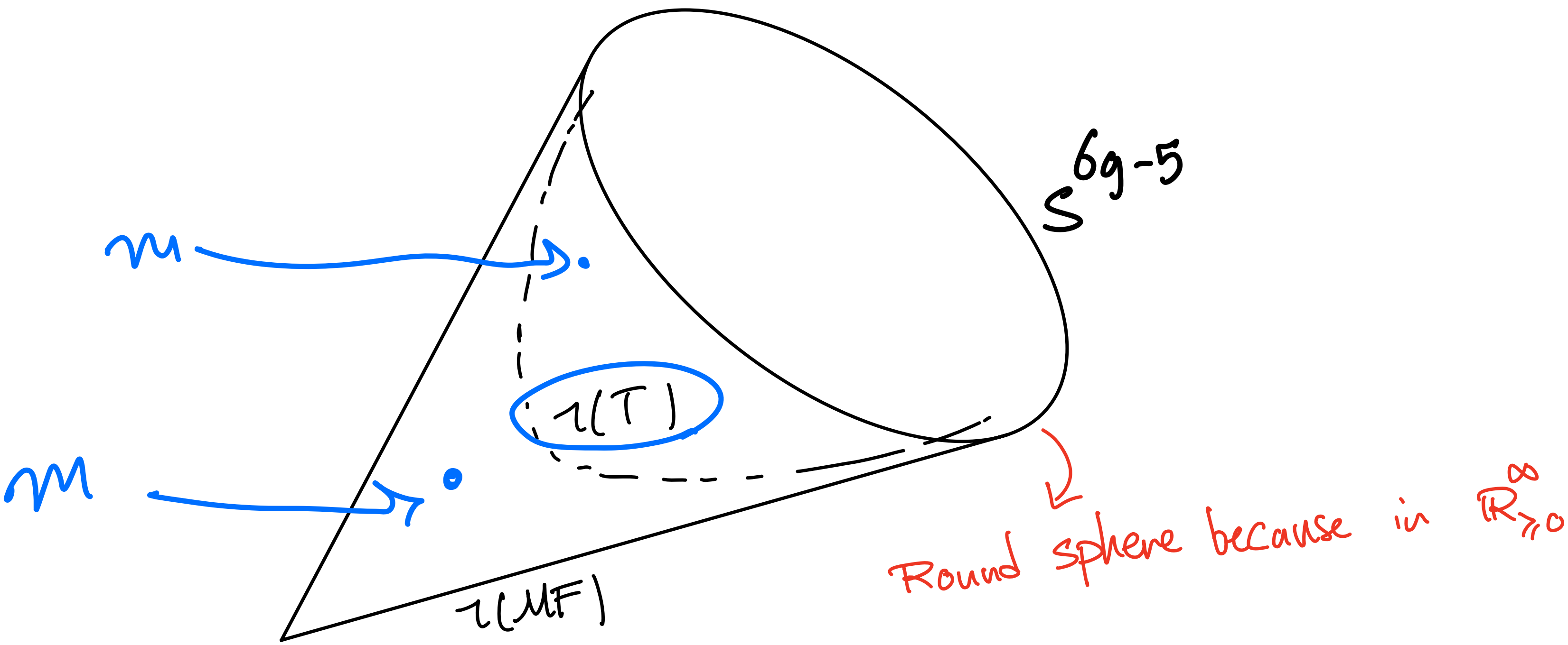
$$\iota : MF \longrightarrow \mathbb{R}_{\geq 0}^{S} \qquad f \longrightarrow (f(c))_{c\in S}$$

$$\iota : T \longrightarrow \mathbb{R}_{+}^{S} \qquad \rho \longrightarrow (\ell_{\rho}(c))_{c\in S}$$

length of unique geodesic in class c

Theorem: $T \cup MF \longrightarrow \mathbb{P}(\mathbb{R}^{S}_{\geq 0})$ is an embedding that realizes the union as a compactification of T homeomorphic to a disk.

In particular $\iota(MF)$ is a cone over S^{6g-5} in $\mathbb{R}^{S}_{\geq 0}$



4 Reduction

Go from the entire S to a finite collection $\{c_1 - c_k\}$

such that $\iota: MF \longrightarrow \mathbb{R}^k_{\geq 0}$ is still an embedding

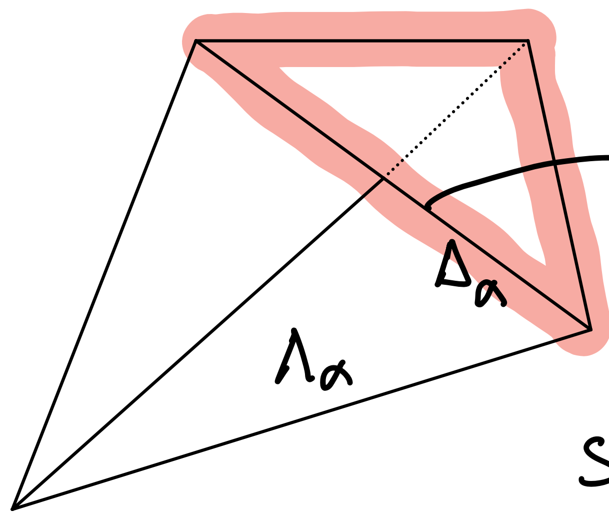
- **Theorem.** There are 9g-9 curves for which the result above is true

  Infact: thats how the theorem before is proved

- Finite dimensional Picture

$$\iota(MF) = \Lambda = \bigcup_a \Lambda_a$$

Union of rational 6g-6 dimensional rational polyhedral cones



$$S^{6g-5} = \bigcup_\alpha \Delta_a$$

- $\iota : S' \longrightarrow \Lambda$ ← $(6g-6)$-dim Cone in $\mathbb{R}_{\geq 0}^{9g-9}$

  $m \longrightarrow \underbrace{(i(m, C_1), \cdots, i(m, C_{9g-9}))}_{\text{These intersection numbers uniquely specify m}}$

- The cones Λ_α are extremely hard to describe

Our main result in a nutshell (in preparation)

For suitable choice of $9g-9$ curves, the map

$\iota : S' \longrightarrow \Lambda$ gives an isomorphism

between $Sk(\Sigma_g)^{gr}$ and $\mathbb{C}[\Lambda']$, s.t $\underbrace{\Lambda' = \iota(S') \subseteq \Lambda}_{\text{finite index submonoid of } \Lambda \cap \mathbb{N}^{9g-9}}$.

Corollary The compactification coming from the associated filtration has toric boundary divisors with moment polytopes Δ_α

$\Rightarrow$ The dual intersection complex of $\partial \overline{X}_g$ is S^{6g-5}

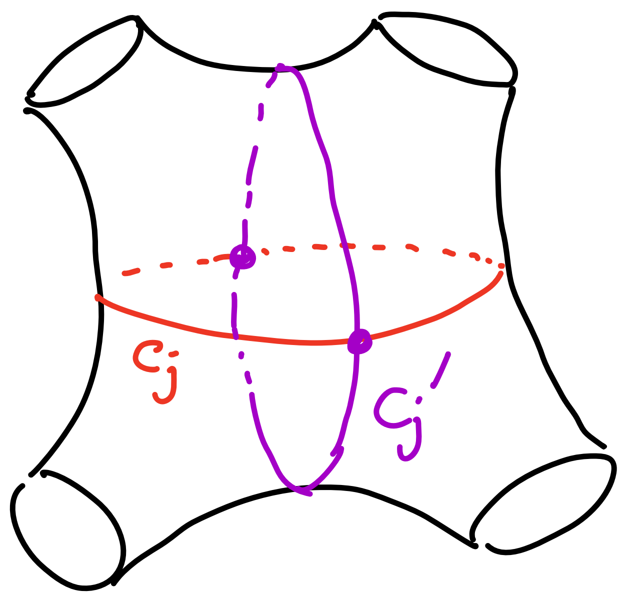
dual of the sphere in Thurston compactification

Comments on steps of the proof

1. what are those 9g−9 curves :

We have $\{(c_j, c_j', c_j'')\}_{j=1}^{3g-3}$ such that

$\{c_j\}_{j=1}^{3g-3}$ defines a pants decomposition, c_j' are as shown in the picture and c_j'' are Dehn Twists of c_j' w.r.t c_j



2. For two multicurves m, m' let

$$m * m' = \sum_s m_s$$

To prove the theorem we show

1) if $\iota(m)$ & $\iota(m')$ do not belong to the same Λ_α, for some α, then

$$|m_s| < |m| + |m'| \quad \forall s$$

$\Rightarrow$ $m * m' = 0$ in the graded algebra

2) if $\tau(m), \tau(m') \in \Lambda_\alpha$ for some α

$\rightsquigarrow$ $\exists!$ m_s denoted by $m \# m'$ such that

$\tau(m \# m') = \tau(m) + \tau(m')$ and the rest

of m_s are of lower degree

$\Rightarrow$ In graded algebra $\underline{m * m' = m \# m'}$

3 It is easier to describe multicurves using Dehn-Thurston coordinates

$$DT(m) = (n_1 \text{---} n_{3g-3}, t_1 \text{---} t_{3g-3})$$

$$\begin{cases} n_j = i(m, c_j) \\ t_j = \text{twist parameter along } c_j \end{cases}$$

(This is discrete version of FN coordinates on T)

We find explicit function F for expressing $\iota(m)$ as a function of $DL(m)$

F is a piece-wise linear function defining different Δ_α

<u>Rest of talk</u> : A quick review of the result in the punctured setting

Taking trace at punctures defines a fibration

$$\begin{array}{c} \mathcal{X}_{g,n} \\ \pi \big\downarrow \\ \mathbb{C}^n \end{array}$$

Geometric P=W is expected for slices

$$\mathcal{X}_{g,n}^{t} = \pi^{-1}(t)$$

Main Results:

1. Ideal triangulations Δ of $\Sigma_{g,n}$ determine explicit compactification $\overline{X}_{g,n}$ and relative compactifications $\overline{X}^{rel}_{g,n} \xrightarrow{\pi} \mathbb{C}^n$

2. $D = \partial \overline{X}_{g,n}$ is an irr toric variety with an explicitly defined moment polytope P_Δ

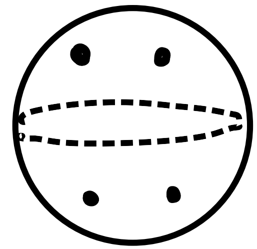
3. $D_t = \partial \overline{X}^t_{g,n}$ is a toric sub-variety of D independent of t $\Rightarrow$ we call it D^{rel}

4. The moment polytope complex of D^{rel} is a sphere

5. Generators of $\mathbb{C}[X_{g,n}]$ can be described explicitely

Well-Known example:

$$(g,n) = (0,4)$$



$$\mathcal{X}_{0,4} \cong \{ y_1 y_2 y_3 = y_1^2 + y_2^2 + y_3^2 + f_1(t)\, y_1 + f_2(t)\, y_2 + f_3(t)\, y_3 + f(t) \}$$

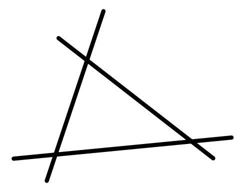
$$\subseteq \mathbb{C}^3_y \times \mathbb{C}^4_t$$

Relative Compactification $\longrightarrow$

$$\overline{\mathcal{X}}^{rel}_{0,4} \subseteq \mathbb{CP}^3 \times \mathbb{C}^4_t$$

Semi-universal family of cubic surfaces

$$D_t = \{ y_1 y_2 y_3 = 0 \} \subseteq \mathbb{CP}^2$$



A cycle of (-1)-curves

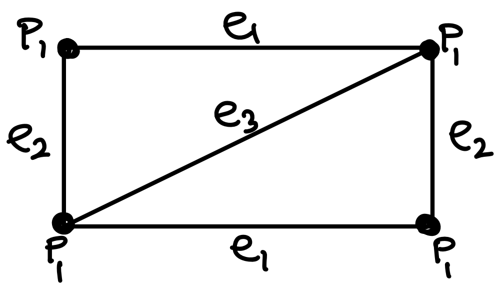
Putting Coordinates on Skein Algebra of punctured surfaces

- **Dfn**: An ideal triangulation Δ of a punctured surface $\Sigma_{g,n}$ (with $2g+n \geq 3$) is a triangulation of Σ_g whose set of vertices V is the set of punctures and the set of edges E is a maximal collection of disjoint non-isotopic arcs between the punctures

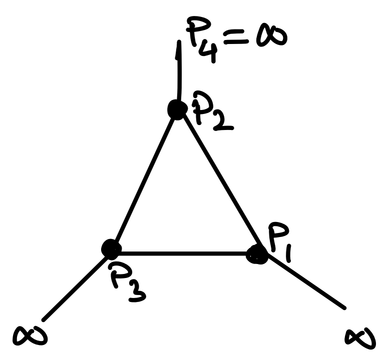
$$|V| = n \qquad |E| = 3(2g+n-2) = \dim_{\mathbb{C}} \mathcal{X}_{g,n}$$

- Examples:

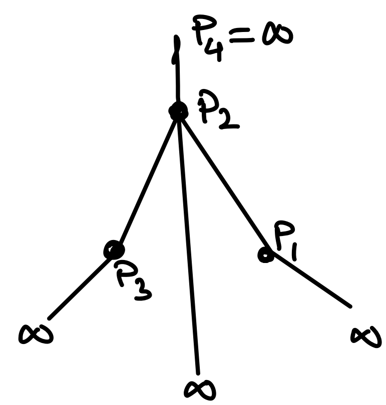
$(g,n) = (1,1)$



$(g,n) = (0,4)$



$(g,n) = (0,4)$



Intersection Coordinates:

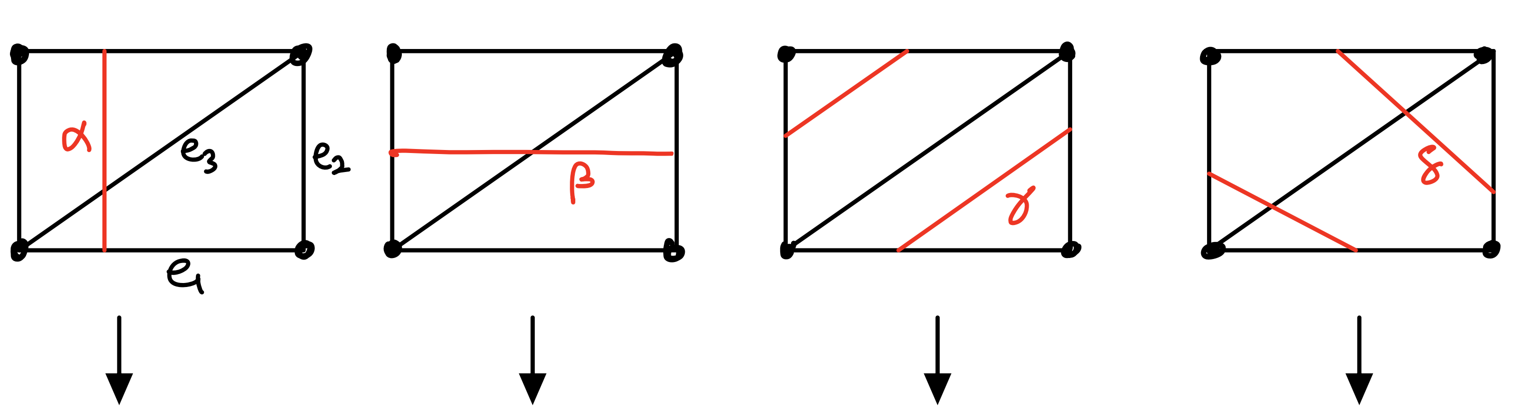
- Multi-curve $m \longrightarrow v = (n_e)_{e \in E} \in \mathbb{N}^E$

$n_e =$ minimal intersection number of m with the arc e

- Fact: the intersection number $|m \cap e|$ is minimum if there is no bigon in the complement of $m \cup e$

(Bigon Criterion)

- Example:

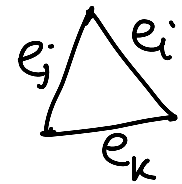


$v(\alpha) = (1,0,1)$ $\quad v(\beta) = (0,1,1)$ $\quad v(\gamma) = (1,1,0)$ $\quad v(\delta) = (1,1,2)$

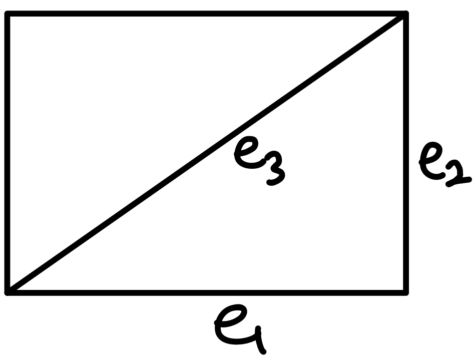
Notice that $v(\delta) = v(\alpha) + v(\beta)$ Also $\alpha * \beta = \delta + \gamma$

Theorem (Well-Known): The map $m \longrightarrow v(m) \in \mathbb{N}^E$ gives a 1-to-1 correspondence between the set of multi-curves on Σ and the elements of the monoid

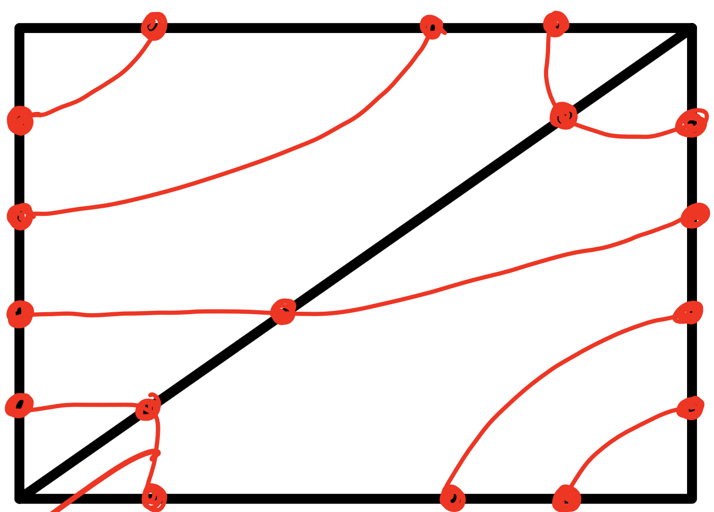
$$\Lambda = \left\{ v = (n_e)_{e\in E} \in \mathbb{N}^E : \begin{array}{l} (1)\ n_{e_i} \leq n_{e_j} + n_{e_k} \\ \\ (2)\ 2 \mid n_{e_i} + n_{e_j} + n_{e_k} \end{array} \quad \forall \right\}$$



Example:



$\Longrightarrow$



$m(v)$

$v = (3, 4, 3)$

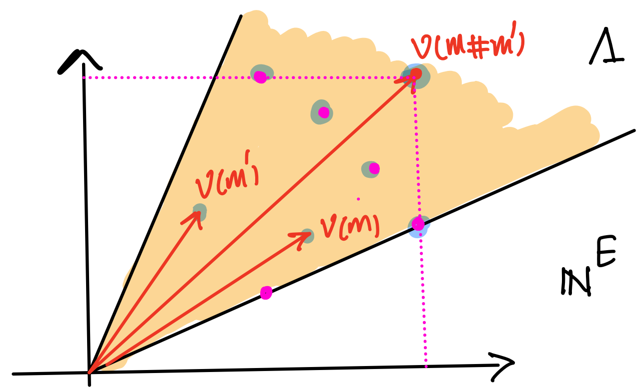
$3+4+3 = 10$

Leading Term :

- Theorem (Abdiel - Frohman 2017) $\forall\, m \,\&\, m'$ we have

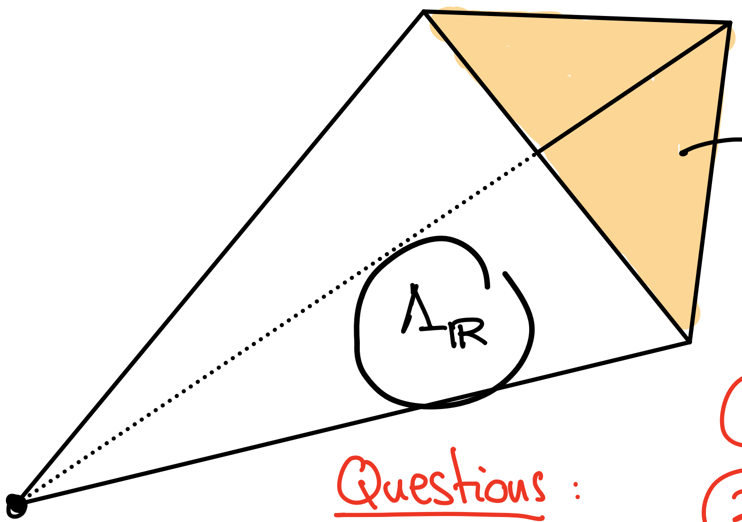
$$m * m' = \underbrace{m \# m'}_{\text{lead - term}} + \text{lower order terms w.r.t } |-|$$

$$\nu(m) + \nu(m') = \nu(m \# m')$$



- Corollary: The graded algebra $\mathbb{C}[X_{g,n}]^{gr}$ is isomorphic to the semi-group algebra $\mathbb{C}[\Lambda]$

- Therefore, the boundary divisor D of the compactification determined by $\mathcal{F}$ is a toric variety with moment polytope P_Δ = slice of the real cone $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$



$$P_\Delta = \{ v \in \Lambda_{\mathbb{R}} : v \cdot \vec{I} = \text{constant } c \}$$

Questions :

① What are the $\mathbb{Z}$-generators of Λ?

② What are the generators of $\Lambda_{\mathbb{R}}$?
(i.e. vertices of P_Δ)

③ How does the closure of $X^t_{g,n}$ in $\overline{X_{g,n}}$ look like?

Compactification of Slices / relative Compactification $\overline{\mathcal{X}}^{rel}_{g,n} \xrightarrow{\pi} \mathbb{C}^n$

- $\mathbb{C}[\mathcal{X}^t_{g,n}]$ can be identified, as a vector space, with

$$Sk(\Sigma_{g,n}) \supset Sk^{rel}(\Sigma_{g,n}) = \left\langle \begin{array}{l} \text{multi-curves that don't contain the} \\ \text{peripheral curves } a_1 - a_n \text{ around } \underbrace{p_1 \cdots p_n}_{\text{Punctures}} \end{array} \right\rangle$$

- The product structure $*_t$ on $\mathbb{C}[\mathcal{X}^t_{g,n}]$ Corresponds to replacing a_i with t_i whenever a_i shows up in the outcome of the usual product $m * m'$

- The filtration on $Sk(\Sigma_{g,n})^{rel}$ is the restriction of F

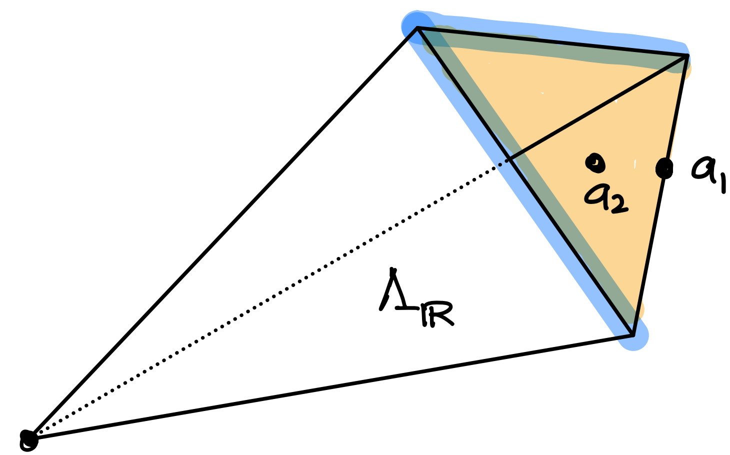
- In the graded algebra :

$$m *_t m' = \begin{cases} 0 & \text{if } m \# m' \text{ contains one of } a_1 \cdots a_n \\ m \# m' & \text{otherwise} \end{cases}$$

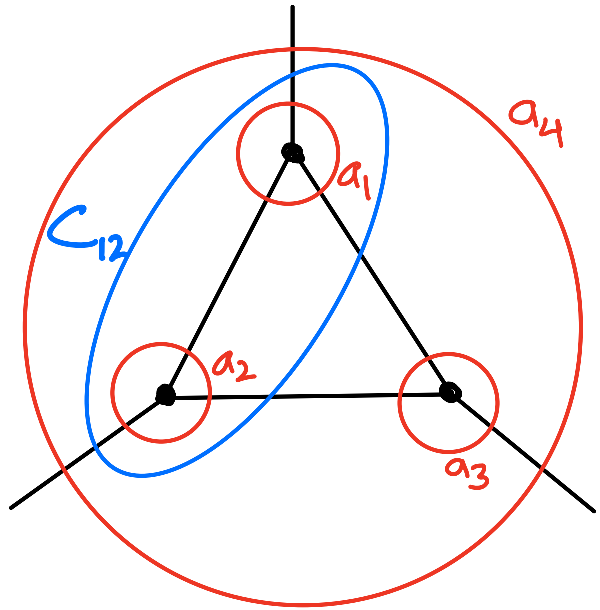
- $\partial \overline{X^t_{g,n}}$ is a toric subvariety independent of t

- Moment polytope complex P^{rel}_Δ of $D^{rel} = \partial \overline{X^t_{g,n}}$ is the union of those faces in P_Δ that do not contain the points corresponding to $a_1 - a_n$

$$\boxed{\text{Why is } P^{rel}_\Delta \text{ a sphere?}}$$



Some Examples :



- $(g,n) = (0,4)$ revisited

— $v(a_1)$ $v(a_2)$ $v(a_3)$ $v(a_4)$
$v(c_{12})$ $v(c_{13})$ $v(c_{23})$
generate $\Lambda \subseteq \mathbb{N}^6$ and $\Lambda_{\mathbb{R}}$

— $|a_i| = 3$ $\quad |c_{ij}| = 4$

— The only relation is
$$c_{12}\, c_{13}\, c_{23} = c_{12}^2 + c_{23}^2 + c_{13}^2 + f_{12|34}\, c_{12} + f_{13|24}\, c_{13} + f_{23|14}\, c_{23} + f$$

$$f_{ij|kl} = a_i a_j + a_k a_l$$
$$f = a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4$$

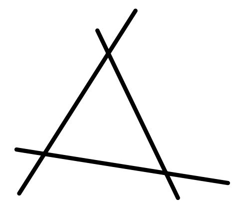
$\longrightarrow$ $\overline{\mathcal{X}_{0,4}}$: $\left(c_{12} c_{13} c_{23} = a_1 a_2 a_3 a_4 + u F \right)$

$\longrightarrow \quad D = (u=0) \; : \; \underline{\left(c_{12}c_{13}c_{23} = a_1a_2a_3a_4\right)} \subseteq \mathbb{P}(4:4:4:3:3:3:3)$

Corresponds to the linear relation

$v(c_{12}) + v(c_{13}) + v(c_{23}) = v(a_1) + v(a_2) + v(a_3) + v(a_4)$ in $\mathbb{Z}^6$

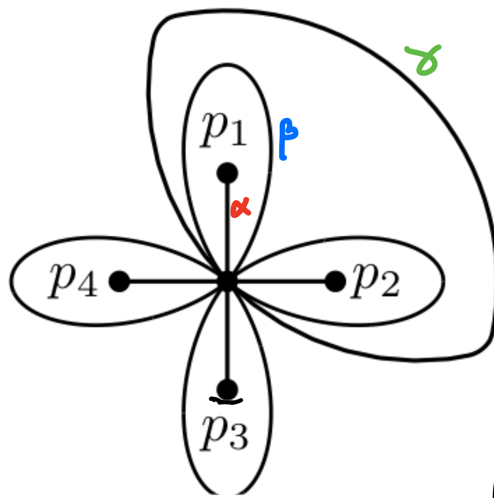
$\longrightarrow \quad D^{rel} = D \cap (a_i = 0) \; : \quad c_{12}c_{13}c_{23} = 0$



Proof of sphere property P_Δ^{rel}

1. We prove it for a particular (type of) Δ
2. we show the property is preserved under mutations

Flower triangulations:



- α_i : arc from P_i to P_n $\quad 1 \leq i \leq n-1$
- β_i : arc from P_n to P_n going around P_i
- γ-curves : the rest of edges needed to triangulate the surface

- $v = (v^\alpha, v^\beta, v^\gamma) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \times \mathbb{Z}^{6g+n-4} = \mathbb{Z}^E$
- It is easy to see that $v(a_i) = (\vec{1}_i, 0, 0)$ for $i = 1, \ldots, n-1$ and $v(m) = (v_1^\alpha, 2v_1^\alpha, v^\gamma)$ for multicurves that do not include $a_1 \cdots a_{n-1}$

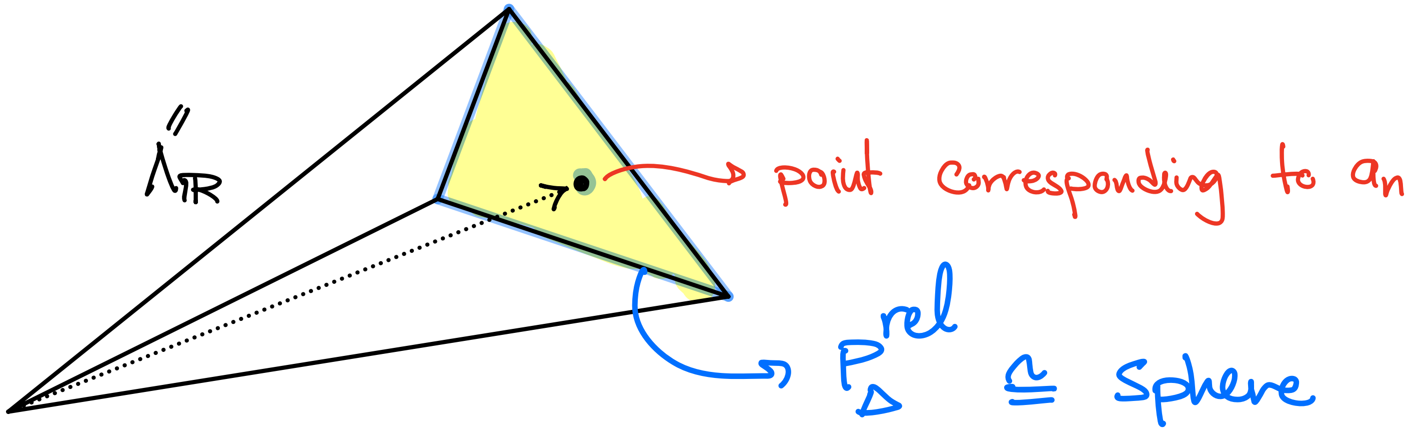
$\Lambda' =$ monoid generated by $v(a_1) \ldots v(a_{n-1})$

$\Lambda'' =$ " " " m that do not contain $a_1 \ldots a_{n-1}$

$\rightarrow$ (i) $\Lambda = \Lambda' \oplus \Lambda''$

(ii) if $v(a_n) = v'(a_n) \oplus v''(a_n) \longrightarrow v''(a_n)$ belongs to the interior of $\Lambda''_{\mathbb{R}}$

$\longrightarrow$ Let $P''_\Delta =$ slice of $\Lambda''_{\mathbb{R}} \longrightarrow P^{rel}_\Delta = \underbrace{\partial P''_\Delta}_{\text{Sphere}}$



Look at the paper for the example of 5-punctured sphere & the explicit triangulated S^3 we get

The End.