Quantum cohomology, isomonodromic deformations, and derived categories

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Web-seminar on Painlevé Equations and related topics

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Main theme: to study relations between

- topology,
- enumerative geometry,
- complex geometry

of a complex smooth projective variety X.

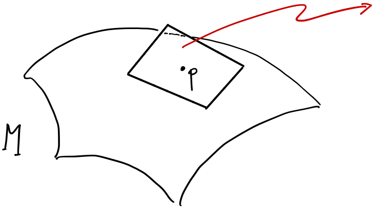
Cotti. *IMRN*, Vol.2022(2), 1454-1493
Cotti. In *Geometric Methods in Physics XXXVIII*, Trends in Mathematics, 2020
Cotti, Dubrovin, Guzzetti. arXiv:1811.09235
Cotti, Varchenko. In *Integrability, Quantization, and Geometry*, AMS, 2020

This is done via

- analysis of isomonodromic deformations of connections on $\mathbb{P}^1$.

Cotti. arXiv:2005.08262
Cotti, Dubrovin, Guzzetti. *Duke Math. Journal*, Volume 168, Number 6 (2019), 967-1108.
Cotti, Dubrovin, Guzzetti. SIGMA 16 (2020), 040, 105 pages.
Cotti, Guzzetti. *Random Matrices Theory Appl.*, Vol. 6 (2017), no. 4, 1740004, 36 pp.
Cotti, Guzzetti. *Random Matrices Theory Appl.*, Vol. 07 (2018), no. 4, 1840003, 27 pp.

Frobenius Manifolds are complex manifolds whose tangent spaces admit a *Frobenius algebra* structure.



FROBENIUS ALGEBRA

$$x(yz) = (xy)z,$$
$$xy = yx,$$
$$xe = x,$$
$$\eta(x, yz) = \eta(xy, z)$$

Examples coming from:

- Symplectic and Algebraic Geometry
- Singularity Theory

Milestones: Dubrovin, Hitchin, Kontsevich, Manin, Saito, Vafa, Witten, ...

What is enumerative geometry?

Let C be an algebraic curve in $\mathbb{P}^2$. It can be described in two ways:

$$f(x_0, x_1, x_2) = 0,$$
$$f \in \mathbb{C}[x_0, x_1, x_2], \quad \deg f = d,$$
$$g = \frac{(d-1)(d-2)}{2} - \#\text{nodes}.$$

$$x_0 = P(t),$$
$$x_1 = Q(t),$$
$$x_2 = R(t), \quad t \in \mathbb{C}.$$

C will be called rational if $P, Q, R \in \mathbb{C}(t)$.

Theorem: The curve C is rational iff $g = 0$.

Question: How many irreducible nodal rational curves of degree d in $\mathbb{P}^2$?

$$f(x_0, x_1, x_2) = \sum_{0 \le i+j \le d} a_{ij}\, x_0^i x_1^j x_2^{d-(i+j)}$$

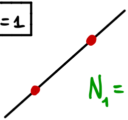
$$\dim \left\{ \begin{array}{c} \text{degree } d \text{ polynomials} \\ f(x_0, x_1, x_2) \end{array} \right\} = \binom{d+2}{2}$$

EACH NODE CUTS OUT A CODIMENSION 1 SUBSPACE

$$\dim \left\{ \begin{array}{c} \text{degree } d \text{ irr.} \\ \text{nodal rat.} \\ \text{curves} \end{array} \right\} = \binom{d+2}{2} - 1 - \frac{(d-1)(d-2)}{2}$$
$$= 3d - 1$$

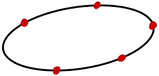
Expectation: #{nodal curves of degree d through $3d-1$ points} $< \infty$

$\boxed{d=1}$



$N_1 = 1$

$\boxed{d=2}$



$N_2 = 1$

$$f(x,y) = \begin{vmatrix} 1 & x & y & x^2 & y^2 & xy \\ 1 & a_1 & a_2 & a_1^2 & a_2^2 & a_1a_2 \\ 1 & b_1 & b_2 & b_1^2 & b_2^2 & b_1b_2 \\ 1 & c_1 & c_2 & c_1^2 & c_2^2 & c_1c_2 \\ 1 & d_1 & d_2 & d_1^2 & d_2^2 & d_1d_2 \\ 1 & e_1 & e_2 & e_1^2 & e_2^2 & e_1e_2 \end{vmatrix}$$

- STEINER (1848), case $d=3$: $N_3 = 12$
- ZEUTHEN (1873), case $d=4$: $N_4 = 620$

d	1	2	3	4	5
N_d	1	1	12	620	...

XV HILBERT'S PROBLEM : TO FIND RIGOROUS FOUNDATIONS OF ENUMERATIVE GEOMETRY

Revolution: Geometry $\longleftrightarrow$ Physics

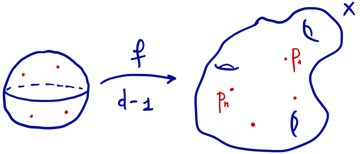
M. Kontsevich 1994

$$N_d = \sum_{a+b=d} N_a N_b\, a^2 b^2 \left[\binom{3d-4}{3a-2} - a \binom{3d-4}{3a-1} \right]$$

using Frobenius Manifolds !!

Gromov-Witten theory attach to any smooth complex projective variety X a Frobenius manifold, its Quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X

$\#\Big\{$ 

$\Big\}$ / automorphisms of $\mathbb{P}^1$ $\quad GW_{n,d}(X, \underline{p})$

GROMOV-WITTEN INVARIANTS of X

COLLECT GW INVARIANTS IN A GENERATING FUNCTION F_X

$$F_X(\underline{t}) = \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} \sum_{\alpha_1 \dots \alpha_n = 1}^{N} \frac{t^{\alpha_1} \dots t^{\alpha_n}}{n!} GW_{n,d}(X, p_1 \dots p_n)$$

ASSUMPTION: NON-EMPTY DOMAIN Ω of CONVERGENCE of F_X

Frobenius manifold $\rightsquigarrow \Omega$

Gromov-Witten theory attach to any smooth complex projective variety X a Frobenius manifold, its Quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X

Example : $X = \mathbb{P}^1$

$\dim_{\mathbb{C}} H^\bullet(X, \mathbb{C}) = 2$

Δ_0, Δ_1 basis of $H^\bullet(X, \mathbb{C})$

t_0, t_1 dual coordinates

$$F_X(\underline{t}) = \frac{1}{2} t_0^2 t_1 + e^{t_1} - \left(1 + t_1 + \frac{t_1^2}{2}\right)$$

2-dimensional Frobenius manifold

$$c_{\alpha\beta\gamma} := \frac{\partial^3 F_X}{\partial t_\alpha \partial t_\beta \partial t_\gamma}, \quad \eta(A,B) := \int_{\mathbb{P}^1} A \cup B$$

$$\frac{\partial}{\partial t_\alpha} * \frac{\partial}{\partial t_\beta} := \sum_\lambda c^\lambda_{\alpha\beta} \frac{\partial}{\partial t_\lambda}$$

FAMILY OF FROBENIUS ALGEBRAS

Gromov-Witten theory attach to any smooth complex projective variety X a Frobenius manifold, its Quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X

... THIS is just ONE HALF of THE STORY ...

Singularity Theory $\rightsquigarrow$ another class of Frobenius manifolds

Mirror Symmetry is an isomorphism of Frobenius manifolds.

$$QH^\bullet(X) \cong (V, f \colon V \to \mathbb{C})$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

Its solutions are multivalued, and they manifest a Stokes phenomenon.

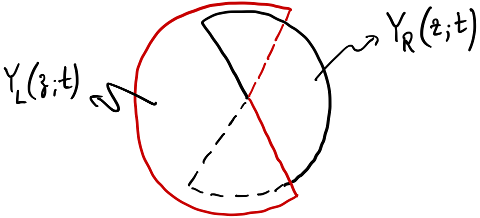
$\to$ Monodromy data of *qDE*

FUCHSIAN SINGULARITY $z=0$

IRREGULAR SINGULARITY $z=\infty$

$Y_o(z;t) \rightsquigarrow$ solution at $z=0$

$Y_F(z;t) \rightsquigarrow$ formal solution with exponential expansion at $z=\infty$



$$Y_R(z;t) = Y_o(z;t)\cdot C(t), \qquad Y_L(z;t) = Y_R(z;t)\cdot S(t)$$

- Isomonodromic property: monodromy data do not depend on t.

Results

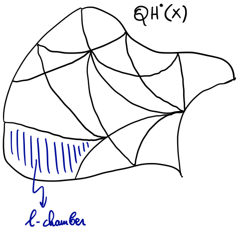
Cotti, Dubrovin, Guzzetti. arXiv:1811.09235

$$\begin{array}{c}\text{Symplectic and Enumerative Geometry of } X\text{: } QH^\bullet(X)\\ \Big\downarrow{\scriptstyle qDE}\\ \text{Complex geometry of } X\text{: } \mathcal{D}^b(X)\end{array}$$

The monodromy data of the *qDE* of X are determined by

- the topology of X (dimension, characteristic classes),
- characteristic classes of exceptional collections in $\mathcal{D}^b(X)$

$$(E_i)_{i=1}^n, \quad \mathrm{Hom}^\bullet(E_i, E_i) \cong \mathbb{C}, \quad \mathrm{Hom}^\bullet(E_j, E_i) = 0, \quad j > i.$$

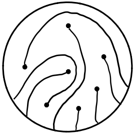


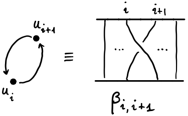
TO EACH CHAMBER CORRESPONDS an EXCEPTIONAL COLLECTION

$$(S^{-1})_{ij} = \chi(E_i, E_j),$$

$$C_k = \underbrace{\frac{1}{(2\pi\sqrt{-1})^{\frac{\dim X}{2}}}\,\hat{\Gamma}_X^-\, e^{-\pi\sqrt{-1}\,c_1(X)}}_{\text{topological invariant of } X}\,\mathrm{Ch}(E_k)$$

WALL-CROSSING PHENOMENON:
BRAID GROUP B_n - ACTION ON BOTH (S,C) AND EXC. COLL.





DICTIONARY

$$\left\{\begin{array}{c}\text{Analysis of}\\ \text{iso-def. of qDE}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{Geometry of}\\ \mathcal{D}^b(X)\end{array}\right\}$$

Known cases: Grassmannians, Hirzebruch surfaces, via explicit computations.

IMPORTANT REMARK: $\mathcal{D}^b(X) \rightsquigarrow$ Riemann-Hilbert-Birkhoff boundary value problem $\rightsquigarrow$ Reconstruction of GW-theory of X

Results

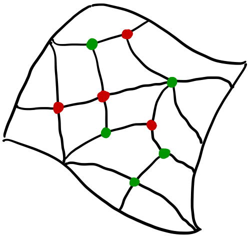
Cotti, Dubrovin, Guzzetti. Duke Math. Journal, Volume 168, Number 6 (2019), 967-1108.

Cotti, Dubrovin, Guzzetti. SIGMA 16 (2020), 040, 105 pages.

Theory of non-generic Isomonodromic Deformations

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad U = \mathrm{diag}(u_1(t), \dots, u_n(t))$$

- Main problem: extend the analytical theory when $u_i(t) = u_j(t)$, $i \neq j$.
- Results: Formal solutions, Asymptotics, Stokes phenomenon, (isomonodromic) deformation theory...
- Applications: Frobenius manifolds, Painlevé transcendents, Riemann-Hilbert problems...



- COALESCING POINTS with SEMISIMPLE FROB. ALGEBRA
- COALESCING POINTS with NON-SEMISIMPLE FROB. ALG.

IT MAY HAPPEN THAT THE FROBENIUS STRUCTURE is KNOWN ONLY AT ● POINTS !!

FRAMEWORK of CDG - Results

GENERAL DEFORMATIONS THEORY BOTH ISOMONODROMIC AND NOT for SYSTEMS $\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t)\right) Y$

PHILOSOPHY OF MAIN CDG - MAIN RESULTS :

To find sharp conditions on the coeff.'s of the def. so that the analysis is TAME

FROBENIUS MANIFOLDS CASE:

All sharp conditions are surprisingly implied by the axioms defining the Frobenius Structure !!!

The analysis at ●-points is well-behaved!

Natural questions : is it really so interesting ??

How much often coalescences arise ?

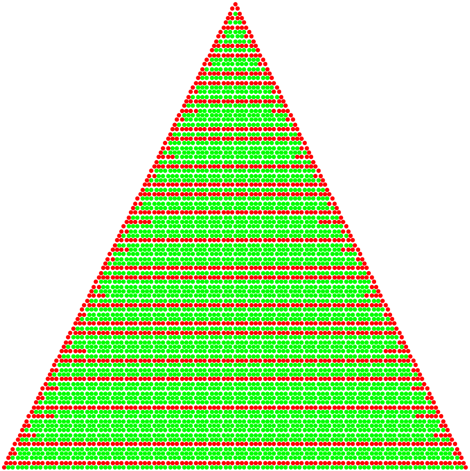


Figure: Surprising connection with prime number distribution: equivalent formulations of RH. Cotti. *IMRN*, doi: 10.1093/imrn/rnaa163, 2020

QUANTUM SATAKE PRINCIPLE

$G(k,n)$ is COALESCING iff $\pi_1(n) \leq k \leq n - \pi_1(n)$.

Results

Cotti, Varchenko. In *Integrability, Quantization, and Geometry*, AMS, 2020

Equivariant framework: let G act on X.

$$QH^{\bullet}_{G}(X) \overset{}{\underset{qDE+qKZ}{\cdots\cdots\cdots\succ}} \mathcal{D}^{b}_{G}(X)$$

Maulik, Okounkov, Tarasov, Varchenko: the qDE admits a compatible system of difference equations

$$Y(t, z_1, \dots, z_i - 1, \dots, z_m) = K_i(t, z)Y(t, z), \quad i = 1, \dots, m.$$

$$\begin{tikzcd}
\text{Equivariant K-theory} \arrow[rr, "\text{Б}"] \arrow[dr, bend right, "TV"'] & & \text{Equivariant Cohomology} \arrow[dl, bend left, "S_0"] \\
& \text{Solutions of qDE} &
\end{tikzcd}$$

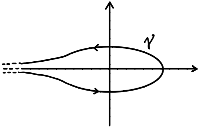
$$\text{Б}(E) := e^{\pi\sqrt{-1}\,c_1(X)}\,\hat{\Gamma}^{+}_{X}\,Ch(E)$$

$S_0(\alpha)$ is a suitable generating function of GW-invariants with gravitational descendants

Explicit computations for $X = \mathbb{P}^{n-1}$ and $G = \mathbb{T}^n$.

Results: perfect equivariant lifts of my previous results!

A very recent result



Cotti. arxiv:arXiv:2005.08262

Borel-Laplace (α, β)-multitransform

$$\mathscr{B}_{\alpha,\beta}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i} \int_\gamma \prod_{j=1}^{h} \Phi_j \left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j} \right) e^{\lambda} \frac{d\lambda}{\lambda},$$

$$\mathscr{L}_{\alpha,\beta}\left[\Phi_1, \dots, \Phi_h\right](z) := \int_0^\infty \prod_{i=1}^{h} \Phi_i(z^{\alpha_i \beta_i} \lambda^{\beta_i}) e^{-\lambda} d\lambda,$$

$\rightarrow$ integral representations of solutions of qDE's

- wide class of varieties (Fano complete intersections),
- Mellin-Barnes integral representations,
- advantages w.r.t. Landau-Ginzburg oscillatory integrals,
- explicit asymptotic analysis.

Important example: Dubrovin's conjecture for Hirzebruch surfaces.

Framework

- X Fano smooth projective variety
- E vector bundle over X
- Y zero locus of a regular section of E

$$\begin{array}{ccc} E & & \\ \downarrow & & \\ X & \hookleftarrow & Y \end{array}$$

Main Problem: Assume we know integral representations of sol.s of qDE of X. Is it possible to construct integral representations for Y?

Main Results: YES!! At least under two "SPLITTING ASSUMPTIONS"

A. $E = \bigoplus$ fractional powers of $\det TX$

B. $X = \prod_{i=1}^{N} X_i$, X_i Fano, $E = \boxtimes_{i=1}^{N} (\det TX_i)^{p_i/q_i}$

For example: in case A, we have

$$\Phi \quad \begin{array}{l}\text{sol.}\\ \text{qDE of } X\end{array} \rightsquigarrow \tilde{\Phi}(z) = e^{-cz} \int_0^\infty \dots \int_0^\infty \Phi\left(z^{\frac{\ell - \sum_i d_i}{\ell}} \prod_i \zeta_i^{\frac{d_i}{\ell}} \right) e^{-\sum_i \zeta_i} \prod_i d\zeta_i$$

$\rightsquigarrow$ solutions of qDE of Y!!!

Application: Dubrovin's conjecture for Hirzebruch surfaces,

$$\mathbb{F}_k := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k)), \quad k \in \mathbb{Z}.$$

Recall: $\mathbb{F}_k \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ hypersurface of bidegree $(1,k)$

- Case of $\mathbb{F}_{2k}$: it easily follows from $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$;
- Case of $\mathbb{F}_{2k+1}$: it is reduced to the case of $\mathbb{F}_1 := \widetilde{\mathbb{P}^2}$.

The *qDE* of $\widetilde{\mathbb{P}^2}$ can be reduced to the scalar equation:

$$(283z-24)\vartheta^4\Phi + \left(283z^2-590z+24\right)\vartheta^3\Phi + \left(-2264z^2+192z+3\right)\vartheta^2\Phi$$
$$-4z^2\left(2547z^2+350z-104\right)\vartheta\Phi + z^2\left(-3113z^3-9924z^2+1476z+192\right)\Phi = 0.$$

$$\mathcal{S}(\mathbb{P}^1) := \left\{\Phi\colon\ \vartheta^2\Phi = 4z^2\Phi\right\}, \quad \mathcal{S}(\mathbb{P}^2) := \left\{\Phi\colon\ \vartheta^3\Phi = 27z^3\Phi\right\}$$

$$\mathscr{P}\colon \mathcal{S}(\mathbb{P}^1)\otimes\mathcal{S}(\mathbb{P}^2) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

$$\mathscr{P}(\Phi_1\otimes\Phi_2;\ z) := e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1,\Phi_2;\ z]$$
$$= e^{-z}\int_0^\infty \Phi_1\left(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right)\Phi_2\left(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}\right)e^{-\lambda}d\lambda,$$

$\Phi_1 \in S(\mathbb{P}^1)$

$$\Phi_1(z) = \sum_{m=0}^{\infty} \left(A_{m,1} + A_{m,0} \log z \right) \frac{z^{2m}}{(m!)^2}$$

$\Phi_2 \in S(\mathbb{P}^2)$

$$\Phi_2(z) = \sum_{m=0}^{\infty} \left(B_{n,2} + B_{n,1} \log z + B_{n,2} \log^2 z \right) \frac{z^{3n}}{(n!)^3}$$

$\to$ $(A_{0,i}\, B_{0,j})$ with $i=0,1$ and $j=0,1,2$ are coordinates on $S(\mathbb{P}^1) \otimes S(\mathbb{P}^2)$.

$$\mathcal{H}: \begin{cases} A_{00}\, B_{00} = 0 \\ 4 A_{0,1} B_{0,0} = 3 A_{0,0} B_{0,1} \end{cases}$$

$\mathcal{H}$ is iso to solutions of qDE of $\mathbb{F}_1$

$\to$ Reconstruction of Stokes bases of solutions

The central connection matrix of $\mathbb{F}_{2k+1}$ is

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i + \frac{\gamma}{\pi} & -i - \frac{\gamma}{\pi} \\ \frac{\gamma - 2\gamma k - i\pi}{2\pi} & -\frac{\gamma - 2\gamma k + i\pi}{2\pi} & \frac{-2\gamma k - i(2\pi k + \pi) + \gamma}{2\pi} & \frac{(2k-1)(\gamma + i\pi)}{2\pi} \\ \gamma\left(-i + \frac{2\gamma}{\pi}\right) & \gamma\left(-i - \frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma + i\pi)}{\pi} & -\frac{2(\gamma + i\pi)^2}{\pi} \end{pmatrix}.$$

Theorem

Dubrovin conjecture holds true for all Hirzebruch surfaces.

The matrix C_k is the matrix associated with the morphism

$$Д^-_{\mathbb{F}_{2k+1}} : K_0(\mathbb{F}_{2k+1})_{\mathbb{C}} \to H^\bullet(\mathbb{F}_{2k+1}, \mathbb{C}), \quad [\mathscr{F}] \mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{F}_{2k+1}} \cup e^{-\pi i c_1(\mathbb{F}_{2k+1})} \cup \mathrm{Ch}(\mathscr{F}),$$

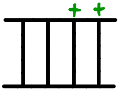
w.r.t. an exceptional basis $\mathfrak{E} := (E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}$.

$\mathbb{F}_\kappa \equiv \mathcal{O}(-\kappa) \cup \infty$ section

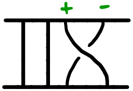
fiber of $\mathcal{O}(-\kappa)$

∞-section

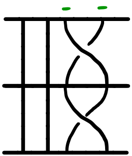
The exceptional collection $\mathfrak{E}$ is obtained from $(\mathcal{O}, \mathcal{O}(\Sigma_2), \mathcal{O}(\Sigma_4), \mathcal{O}(\Sigma_2 + \Sigma_4))$ by applying the following elements of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$:



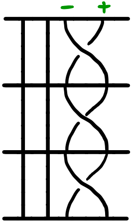
$\mathbb{F}_1$



$\mathbb{F}_3$



$\mathbb{F}_5$



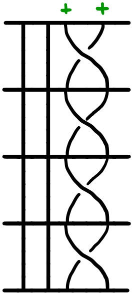
$\mathbb{F}_7$

Different Complex Structures

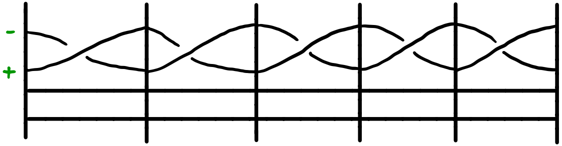
$\Downarrow$

increasing powers of the SAME BRAID

β_{34}^{κ}



$\mathbb{F}_9$



$\mathbb{F}_{11}$