Holomorphic Lax pairs of $\mathrm{P_{VI}}$ and $\mathrm{q\text{-}P_{VI}}$

Robert Conte [1,2]

1. Centre Borelli, École normale supérieure de Paris-Saclay, France
2. Dept of mathematics, 香港大學, Hong Kong

21 January 2026, French-Japanese online seminar

Outline

Motivation

Remove two restrictions of the matrix Lax pairs of Jimbo and Miwa (1981) for $\mathrm{P_{VI}}$ and of Jimbo and Sakai (1996) for $\mathrm{q\text{-}P_{VI}}$:

1. **They do not exist** when $\theta_\infty^2 = 1$ for $\mathrm{P_{VI}}$ and $\kappa_1 - \kappa_2$ for $\mathrm{q\text{-}P_{VI}}$, i.e. when the residue at infinity, assumed diagonal, has a double eigenvalue.
2. **Their elements are not rational** in $u(x), u'(x)$ (or $(u, \overline{u}, v, \bar{v})$), only their [discrete] logarithmic derivative is rational.

Results

Case $\mathrm{P_{VI}}$. Done. Both restrictions are removed (RC 2017) by the **moving frame of Bonnet surfaces**, the result being a nondiagonal matrix for the residue at infinity.

Case $\mathrm{q\text{-}P_{VI}}$. Done here. Assuming a nondiagonal matrix for the residue at infinity, we obtain matrix Lax pairs holomorphic in (κ_1, κ_2), rational in the two dependent variables.

Remark. Discrete scalar Lax pairs (Yamada 2010, Noumi, Tsujimoto, Yamada 2013, Nagao 2015) are discarded, because they always have an apparent singularity, as opposed to matrix Lax pairs.

Outline

PVI Lax pairs

Scalar Lax pair first written by Richard Fuchs 1905.
Matrix Lax pair first written by Jimbo and Miwa 1981.

$\mathrm{P_{VI}}$ scalar. Lazy Poincaré, happy Richard Fuchs

Poincaré 1884 Acta mathematica; R. Fuchs Comptes rendus 1905

Poincaré 1884 page 219:
On ne peut pas en général trouver une équation d'ordre supérieur au second, sans points à apparence singulière et qui admette un groupe donné. Il faut donc en général, si l'on veut construire une équation ayant un groupe donné, lui donner des points à apparence singulière. C'est pour éviter ces points qui compliqueraient notablement les résultats et les démonstrations que je me bornerai ici au cas des équations du second ordre. **Je supposerai** donc que j'ai affaire à une équation du **second ordre sans point à apparence singulière**.
Richard Fuchs 1905 did this prescribed computation and was the first to isolate $\mathrm{P_{VI}}$.

$\mathrm{P_{VI}}$ matrix pair. Lazy Schlesinger, happy Jimbo and Miwa

Schlesinger 1912

Schlesinger 1912 page 105

Indem wir diese Matrix α_{ik} *gleich* $(B_{ik}^{(\sigma+1)})^{-1}$ *nehmen, . . .*

$\mathrm{P_{VI}}$. Second order matrix Lax pair. The theory

Schlesinger 1912

$\mathrm{d}\psi = (L\mathrm{d}x + M\mathrm{d}t)\psi,\ L_t - M_x + [L, M] = 0,\ t$=spectral.
Fuchsian singularities: **4 nonapparent + 0 apparent**

$$M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x},\ L = -\frac{M_x}{t-x} + L_\infty,\ M_\infty = -M_0 - M_1 - M_x,$$

- $(M_\infty$ const.$) \Leftrightarrow (L_\infty / M_\infty =$ scalar $f(x))$. This $\not\Rightarrow M_\infty$ const
- If M_∞ is invertible, there exists a change of basis allowing one to set L_∞ to zero and therefore to make the Lax pair unique.

Matrix pair of Jimbo and Miwa, assumptions

Jimbo and Miwa, Physica D 1981; traceless form, Mahoux, Cargèse 1999

$$L_\infty = 0, M_\infty = \frac{1}{2}\begin{pmatrix} a_\infty & 0 \\ 0 & -a_\infty \end{pmatrix},$$

$$M_j = \frac{1}{2}\begin{pmatrix} z_j & (\theta_j - z_j)u_j \\ (\theta_j + z_j)/u_j & -z_j \end{pmatrix}, j = 0, 1, x,$$

constant $a_\infty \neq 0$ (otherwise only 3 singularities, P_{VI} impossible),
$z_0, z_1, u_0, u_1 =$ four unknown functions of three variables x, u, u',
u of $\mathrm{P}_{\mathrm{VI}} :=$ the unique zero $t = u$ of M_{12}.

Matrix pair of Jimbo and Miwa, result

Jimbo and Miwa 1981

$a_\infty = \theta_\infty - 1 \Longrightarrow$ Lax pair does not exist for $\theta_\infty^2 = 1$ (denominator a_∞ as expected in matrix elements).

Off-diagonal elements M_j are not algebraic, only their log deriv is.

Second order matrix Lax pair of Bonnet, result

RC Comptes rendus 2017, JMP 2017

$$M_\infty = \text{nondiagonal} = \frac{1}{4}\begin{pmatrix} 2a & -4 \\ a^2 - \theta_\infty^2 & -2a \end{pmatrix},\ a = \text{any constant},$$

$$L_\infty = -\frac{u-x}{x(x-1)} M_\infty,$$

$$M_{12} = \frac{t-u}{t(t-1)},$$

$$(L, M) = \text{affine}(\theta_\infty^2, \theta_0^2, \theta_1^2), \text{quadratic}(\theta_x),$$

$$(M_x)_{12} = 0.$$

The matrix Lax pair of $\mathrm{P_{VI}}$, rational(u, u'), polynomial(θ_j)

RC CRAS 2017, JMP 2017

$$
\begin{cases}
L = -\dfrac{M_x}{t-x} - \dfrac{u-x}{x(x-1)} M_\infty,\ M_\infty = \dfrac{1}{4}\begin{pmatrix} 2a & -4 \\ a^2-\theta_\infty^2 & -2a \end{pmatrix}, \\
M_0 = -\dfrac{1}{2(u-x)} \begin{pmatrix} e_0 & -2u(u-x) \\ \dfrac{e_0^2-\theta_0^2(u-x)^2}{2u(u-x)} & -e_0 \end{pmatrix}, \\
M_1 = \dfrac{1}{2(u-x)} \begin{pmatrix} e_1 & -2(u-1)(u-x) \\ \dfrac{e_1^2-\theta_1^2(u-x)^2}{2(u-1)(u-x)} & -e_1 \end{pmatrix}, \\
M_x = \dfrac{1}{2}\begin{pmatrix} -\Theta_x & 0 \\ 2M_{x,21} & \Theta_x \end{pmatrix}, \\
e = x(x-1)u' + \Theta_x u(u-1), \\
e_0 = e - (\Theta_x - a)u(u-x), \\
e_1 = e - (\Theta_x - a)(u-1)(u-x), \\
-4 \det M_j = \theta_j^2,\ j=\infty,0,1;\ -4\det M_x = \Theta_x^2,\ \Theta_x^2 = (\theta_x-1)^2,
\end{cases}
$$

$a =$ irrelevant arb. constant, e.g. $a = \Theta_x$. No residue ever vanishes.

Outline

q-$\mathrm{P_{VI}}$. Second order matrix Lax pair. The theory

Jimbo and Sakai 1996

x=indep var, z=spectral param.

A = monodromy matrix,

$$\begin{cases} \Psi(x,zq) = A(x,z)\Psi(x,z), \Psi(xq,z) = B(x,z)\Psi(x,z), \\ X \equiv B(x,zq)A(x,z) - A(xq,z)B(x,z) = 0, \end{cases} \quad (1)$$

Method of JS: discretize a second order matrix Lax pair of $\mathrm{P_{VI}}$.

Assumptions on the monodromy matrix:

1. $A(x,z) = A_2(x)z^2 + A_1(x)z + A_0(x)$;
2. $A_2(x) = \text{constant} = \mathrm{diag}(\kappa_1, \kappa_2)$;
3. $A_0(x)$ has two eigenvalues $t_1 x, t_2 x$ (t_j constants);
4. $\det A(x,z)$ has four zeros $z = a_1 x, a_2 x, a_3, a_4$ (a_j constants).

$t_1 t_2 = \kappa_1 \kappa_2 a_1 a_2 a_3 a_4$

q-P_{VI}. Second order matrix Lax pair. Result of JS

Jimbo and Sakai 1996

$$A(x,z) = \frac{1}{(\kappa_1-\kappa_2)^2}\begin{pmatrix} R_{11}(z,u,v) & R_{12}(z,u,v)w \\ R_{21}(z,u,v)/w & R_{22}(z,u,v)\end{pmatrix},$$

$$B(x,z) = \varphi(x)\frac{z(z\mathbb{I}+B_0(x))}{(z-qa_1x)(z-qa_2x)}, \varphi(x) = \text{arbitrary},$$

$$B_0(x) = \frac{1}{(\kappa_1-\kappa_2)^2}\begin{pmatrix} r_{11}(u,v,\overline{u},\overline{v}) & r_{12}(u,v,\overline{u},\overline{v})w \\ r_{21}(u,v,\overline{u},\overline{v})/w & r_{22}(u,v,\overline{u},\overline{v})\end{pmatrix} \quad (2)$$

$R_{ij}(z,u,v)$ polynomial in z, κ_1, κ_2, rational in u, v,
r_{ij} polynomial in κ_1, κ_2, rational in its four arguments,
w defined by its discrete logarithmic derivative,

$$\begin{cases} u\overline{u} = a_3a_4\dfrac{(\overline{v}-a_1a_2x/t_1)(\overline{v}-a_1a_2x/t_2)}{(\overline{v}-1/(\kappa_1 q))(\overline{v}-1/\kappa_2)}, \\ v\overline{v} = \dfrac{1}{\kappa_1\kappa_2 q}\dfrac{(u-a_1x)(u-a_2x)}{(u-a_3)(u-a_4)}, \quad \dfrac{\overline{w}}{w} = \kappa_1\kappa_2 q\dfrac{(\overline{v}-1/(\kappa_1 q))}{(\overline{v}-1/\kappa_2)}, \\ t_1t_2 = \kappa_1\kappa_2a_1a_2a_3a_4. \end{cases} \quad (3)$$

q-$\mathrm{P_{VI}}$. Search for a holomorphic, rational Lax pair

At least two possible approaches.

1. First define, then find a "**discrete Bonnet surface**", convert its moving frame to a matrix Lax pair of q-$\mathrm{P_{VI}}$. Seems uneasy.
2. In Jimbo and Sakai 1996, change the assumption $\mathrm{diag}(\kappa_1, \kappa_2)$ to "nondiagonal matrix" of (continuous) $\mathrm{P_{VI}}$. No big difficulty.

q-$\mathrm{P_{VI}}$. Nondiagonal assumption

Replace the assumption
$A_2(x) = \text{constant} = \mathrm{diag}(\kappa_1, \kappa_2)$ by
$A_2(x)$ has 2 constant eigenvalues κ_1, κ_2,
and represent $A_2(x)$ by a nondiagonal matrix

$$A_2(x) = \begin{pmatrix} (\kappa_1 + \kappa_2)/2 + c & -1 \\ c^2 - (\kappa_1 - \kappa_2)^2/4 & (\kappa_1 + \kappa_2)/2 - c \end{pmatrix}, c \text{ arbitrary constant.}$$

Then, in matrix $B(x, z)$, the
old numerator $z(z\mathbb{I} + B_0(x))$ must be changed to the
new numerator $z(zB_1(x) + B_0(x))$.

q-$\mathrm{P_{VI}}$. Result of the nondiagonal assumption
RC, 2025

All matrix elements of A are Laurent polynomials of two of them, $A_{1,12}$ and $A_{1,11}$, constrained by a single relation

$$(A_{1,12} - 2u)A_{1,11}^2 + P_2(A_{1,12})A_{1,11} + Q_2(A_{1,12}) = 0, \quad (4)$$

with P_2 and Q_2 second degree polynomials with coefficients Laurent polynomials in u, v and all the parameters.

Require solutions $A_{1,12}$ to be rational. **Only 8 solutions**:
$A_{1,12} = u + a_1 x,\ u + a_2 x,\ u + a_3,\ u + a_4,\ u,\ 2u$ and two others.
For each such $A_{1,12}$, the two roots $A_{1,11}$ are also rational.
Total 15 solutions.
The simplest one is $A_{1,12} = u$.

q-$\mathrm{P_{VI}}$. Result of the nondiagonal assumption. One solution

RC, 2025

For $A_{1,12} = u$, the q-$\mathrm{P_{VI}}$ is (notation z_1, z_2 of JS)

$$\begin{cases} u\overline{u} = \dfrac{1}{a_1 a_2} \dfrac{(\bar{v} - a_1 a_2 x/t_1)(\bar{v} - a_1 a_2 x/(q t_2))}{(\bar{v} - 1/(\kappa_1 q))(\bar{v} - 1/(\kappa_2 q))}, \\ v\bar{v} = \dfrac{1}{\kappa_1 \kappa_2 q^2} \dfrac{(u - a_1 x)(u - a_2 x)}{(u - a_3)(u - a_4)}, t_1 t_2 = \kappa_1 \kappa_2 a_1 a_2 a_3 a_4. \end{cases} \quad (5)$$

1. $A(x, z)$ is rational in u, v, and $B(x, z)$ is rational in $u, v, \overline{u}, \bar{v}$.
2. $A_{11} = k_1 z_1 z/u - (z - u) t_1 x/u$.
3. $A_{12} = -(z - u)z$ (second zero z independent of u).
4. $A_{22} = k_2 z_2 + (z - u)[k_2 z_2/u + (k_1 + k_2)z - t_2 x/u]$.
5. $B_{11} = (k_1 + k_2)z - q t_1 x/u + k_2 z_2 (q a_1 a_2 x^2 - zu)/(quv)$.
6. $B_{12} = z$.
7. $B_{22} = -z/q/\bar{v} + (a_1 a_2 q x^2/\bar{v} - q t_1 x)/\overline{u}$.
8. A_{22}, B_{22} not that short.

Outline

Conclusion

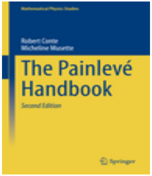
The continuum limit of this holomorphic, rational matrix Lax pair is (certainly) the moving frame of Bonnet surfaces.

This should be useful to build discrete Bonnet surfaces.

Outline

References

- O. Bonnet, J. École polytechnique **42** (1867) 1–151.
- RC, Comptes rendus **355** (2017); J. Math. Phys. **58** (2017).
- RC, M. Musette, *The Painlevé handbook*, 2nd ed. (Springer, 2020).



- M. Jimbo and T. Miwa, Physica D **2** (1981) 407–448.
- M. Jimbo and H. Sakai, Lett. math. phys. **38** (1996) 145–154.
- H. Poincaré, Acta math. **4** (1884) 201–312.
- L. Schlesinger, J. für die r. und angew. Math. **141** (1912) 96–145.