Wild Monodromy Groupoids and Affine del Pezzo Surfaces, a new look on Painlevé equations

Jean-Pierre Ramis, in collaboration with Emmanuel Paul

Abstract

During 2012 spring, in a lecture at a conference in Wuhan university in China, I proposed a program around some (unformal) conjectures for the 9 "irregular" Painlevé equations (P_J, $J \neq VI$). In particular, I proposed the idea of non linear natural dynamics on the Okamoto varieties of initial conditions and conjectured that they are algebraized into rational dynamics on the character varieties by the (wild) Riemann-Hilbert coreespondance. The first decisive step was realized by Martin Klimes in 2016 (cf. Painlvé seminar 2021), with a very complete study of the confluence $P_{VI} \rightarrow P_V$. He proved in particular that the confluence of the corresponding character varieties C_{VI} and C_V (interpreted as affine cubic surfaces) can be expressed by a one parameter family of birational morphisms: $\psi_\kappa : C_V \rightarrow C_{VI,\kappa}$. Later, with E. Paul, we recovered the ψ_κ by "duality", from a one parameter family of very simple natural morphisms $\phi_\kappa : \pi_1^{VI,\kappa} \rightarrow \pi_1^V$ of *wild fundamental groupoids*. These groupoids are not the "classical" wild groupoids used by several authors (as Philip Boalch). We grafted on these classical groupoids two "algebraic tori of loops" and this is crucial.

1 Confluence of the 10 character varieties from morphisms between the 10 wild groupoids

In a first part of my lecture, I will describe (roughly) the 10 wild groupoids of the Painlevé linear associated equations and the corresponding morphisms (or one parameter families of morphisms) according to the Ohyama-Okumura confluence diagram of the Painlevé equations:

$$
\begin{tikzcd}
 & & \pi_1^{III} \arrow[r] \arrow[dr] & \pi_1^{III-D7} \arrow[r] \arrow[dr] & \pi_1^{III-D8} \\
\pi_1^{VI} \arrow[r] & \pi_1^{V} \arrow[ur] \arrow[r] \arrow[dr] & \pi_1^{V,deg} \arrow[ur] \arrow[dr] & \pi_1^{II-JM} \arrow[r] & \pi_1^{I} \\
 & & \pi_1 IV \arrow[ur] \arrow[r] & \pi_1^{II-FN} \arrow[ur] &
\end{tikzcd}
$$

Using (conditioned) representations, we get, by “duality”, a diagram of birational symplectic maps between the character varieties:

$$
\begin{tikzcd}
 & & C_{III} \arrow[dl] & C_{III-D7} \arrow[l] \arrow[dl] & C_{III-D8} \arrow[l] \\
C_{VI} & C_{V} \arrow[l] & C_{V,deg} \arrow[l] & C_{II-JM} \arrow[ul] \arrow[dl] \arrow[r, leftrightarrow] & C_{I} \arrow[ul] \arrow[dl] \\
 & & C_{IV} \arrow[ul] & C_{II-FN} \arrow[l] \arrow[ul] &
\end{tikzcd}
$$

Each confluence ”remove a line from a character variety”. (It fits with a KPR conjecture).

Using this diagram, we can pull the classical dynamics of P_{VI} into rational symplectic dynamics on the other equations. The blue arrows represent in fact one parameter families, therefore the dynamics ‘increase” during the process. At the end of the ”cascade”, for P_I the symplectic dynamics is ”as great as possible”.

2 Cubic surfaces and Segre surfaces

In the second part of my lecture, I will describe some recents results of N. Joshi, M. Mazzocco and P. Roffelsen on the variety of monodromy data of q-P_{VI} and on the character varieties of the 10 Painlevé equations.

The space of monodromy data of q-P_{VI} (in Birkhoff line) was studied by Ohyama-Ramis-Sauloy. Later N. Joshi and P. Roffelsen endowed this set of a structure of Segre surface $\mathcal{Y}$. later Ramis-Sauloy endowed the same set with a geometric quotient structure $\mathcal{F}$ and proved that the natural bijective map $\mathcal{F} \to \mathcal{Y}$ is an isomorphism of algebraic varieties.

JMR proved that, when $q \to 1$, a model of $\mathcal{Y}$ in $\mathbb{C}^6$ tends to a Segre surface and that this Segre surface is isomorphic to the Fricke cubic affine surface of P_{VI}. There is also generically a ”known” isomorphism between the Fricke surface and a Segre surface obtained by a blowdown of one line of the triangle at infinity. This suggested an extension of the mechanism and JMR obtained for each Painlevé (wild) character variety (an affine cubic surface) one (or several) isomorphism with an affine Segre surface. They used ”generalized blow-downs” defined explicitely in coordinates.

In several cases they are not classical blow-downs because the blow-downed line contains a singularity.

3 Affine del Pezzo equations

In the third part of my lecture, I will propose some tools to unify the two preceding parts. My approach is based on del Pezzo surfaces. The singularities are essential, therefore I need a singular version of del Pezzo surfaces (Dolgachev). Their minimal desingularisations are the weak del Pezzo surfaces.

An *affine del Pezzo surface on anticanonical form* is the data of a a quasi-projective surface $AS \subset P^d(\mathbb{C})$, where $d \geq 3$, such that there exists a del Pezzo surface $S \subset P^d(\mathbb{C})$ on anticanonical form and an hyperplane section HS of S (an effective anticanonical divisor) such that $AS = S \setminus HS_{red}$. We denote (AS, S, HS). By definition its *degree* is $d = \deg S$.

An *affine del Pezzo surface* is the data of a normal algebraic surface AS with (at most) RDP singularities and of a symplectic form ω on AS, defined up to a scaling, such that there exists an affine del Pezzo surface on anticanonical form (AS_1, S_1, HS_1) and an isomorphism $\rho : AS_1 \rightarrow AS$ such that HS_1 is the divisor defined by $\rho^*\omega$.

This formalism allows a simple and rigorous definition of generalized blow-ups and generalized blow-downs even when there are singularities. I will use them to propose a large generalizations of JMR results and a possible interpretation of my joint results with E. Paul (cf. the first part) by a simple mechanism of ”removing a line in an affine del Pezzo surface”.