On the Fourier transform of Stokes data of irregular connections

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General context

Wild character varieties = moduli spaces of generalised monodromy data (Stokes data) of meromorphic connections with irregular singularities.

They have rich geometric structures (symplectic, hyperkähler)

They are moduli spaces for different objects:

- Irregular connections
- Stokes data (via the Riemann-Hilbert-Birkhoff correspondence)
- Irregular Higgs bundles (via the nonabelian Hodge correspondence)

Via isomonodromic deformations, they give rise to many interesting systems of nonlinear ODE, e.g. Painlevé equations.

Motivation: isomorphisms of WCVs

A WCV depends on a choice of "wild Riemann surface": a curve Σ together with singularity data.

WCVs coming from different wild Riemann surfaces can be isomorphic.

A manifestation of this is the existence of dualities between different isomonodromy systems, e.g. the existence of several Lax pairs for Painlevé equations

More specifically: there is a notion of Fourier transform for connections on $\mathbb{P}^1$ which induces such isomorphisms.

Question: how does the Fourier transform act on generalised monodromy data?

Today: Describe this in some cases (j.w. A. Hohl)

Outline

Meromorphic connections

- Let Σ be a smooth complex algebraic curve. Here $\Sigma = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
- Consider (E, ∇) vector bundle with algebraic connection on $\Sigma \setminus \{a_1, \ldots, a_m\}$.
- In a local trivialization and with a choice of coordinate z:

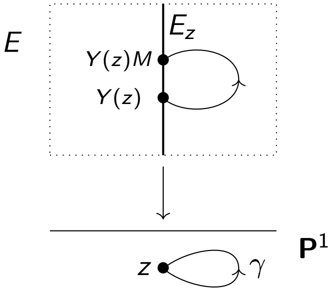
$$\nabla = d - A(z)dz,$$

  with $A(z)$ $n \times n$ matrix having poles at singular points.
- This corresponds to the system of linear differential equations

$$\frac{dY}{dz} = A(z)Y.$$

Monodromy

Consider a solution Y of the equation. If we go around one singularity: $Y(z) \mapsto Y(z)M$, with $M \in GL_n(\mathbb{C})$.

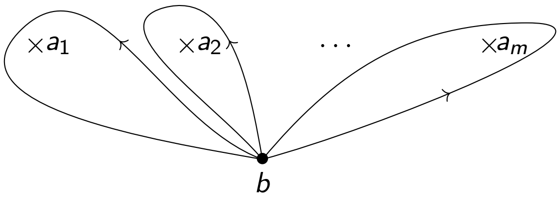


Example: if $\nabla = d - \frac{\lambda}{z}dz$, solution $y(z) = z^\lambda$, monodromy $e^{2i\pi\lambda}$.

Here ∇ is flat so this only depends on the homotopy class of γ. To ∇ we associate its monodromy representation $\rho : \pi_1(\Sigma) \to GL_n(\mathbb{C})$.

Moduli spaces of monodromy data: character varieties

Choose some paths $\gamma_1, \ldots, \gamma_m$ around a_i generating $\pi_1(\Sigma^o, b)$



Let $M_i = \rho(\gamma_i) \in G = GL_n(\mathbb{C})$.

The moduli space of monodromy data is the *character variety*

$$\mathcal{M}_B(\Sigma, \mathbf{a}) = \{M_1, \ldots, M_m \,|\, M_1 \ldots M_m = 1\}/G.$$

It is a Poisson manifold (Atiyah-Bott, Goldman).

Regular Riemann-Hilbert correspondence

Case of regular singularities (i.e basically simple poles)

de Rham moduli space:

$$\mathcal{M}_{dR}(\Sigma, \mathbf{a}) = \{\text{connections with regular singularities on } \Sigma \setminus \mathbf{a}\} / \sim$$

Here $\sim$ corresponds to *gauge transformations* i.e. changes of trivialisation $g : \Sigma^o \to GL_n(\mathbb{C})$, doing

$$A \mapsto gAg^{-1} - dg\, g^{-1}.$$

For the system $Y' = AY$, it corresponds to change of variable $Z = g(z)Y$.

Riemann-Hilbert correspondence (Deligne):

$$\boxed{\mathcal{M}_{dR}(\Sigma, \mathbf{a}) \simeq \mathcal{M}_B(\Sigma, \mathbf{a})}$$

Regular vs irregular singularities

Irregular singularities: higher order poles

$$\nabla = d - A(z)dz, \qquad A(z) = \frac{A_s}{z^s} + \dots + \frac{A_1}{z} + \dots$$

Monodromy is not enough to reconstruct the connection.

Example:

- Regular $\nabla = d - \frac{\lambda}{z}dz$, monodromy $e^{2i\pi\lambda}$.
- Irregular $\nabla = d - dq - \frac{\lambda}{z}dz$, with $q \in z^{-1}\mathbb{C}[z^{-1}]$ has monodromy $e^{2i\pi\lambda}$ for any q.

$\Rightarrow$ need generalised monodromy data for a topological description of irregular connections

Formal data

Turritin-Levelt theorem: it is possible to "diagonalise" ∇ using formal gauge transformations to a normal form

$$\nabla^0 = d - dQ - \frac{\Lambda}{z}dz, \quad Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in z^{-1/r}\mathbb{C}[z^{-1/r}],$$

where

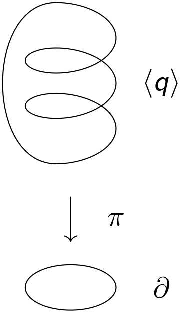
- q_i: exponential factors of ∇,
- Q: irregular type of ∇, r ramification order, Q is untwisted if $r = 1$.
- Regular singularity if $Q = 0$.
- Λ: exponent of formal monodromy (constant and block diagonal with blocks corresponding to the distinct q_i)

A fundamental solution of ∇^0 is $e^Q z^\Lambda$.

Exponential factors as Stokes circles

∂ : circle of directions around singularity $z = 0$

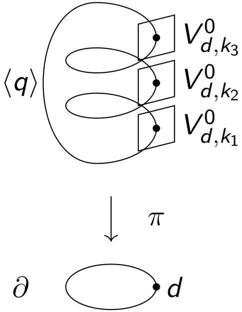
Exponential factors q as germs of functions on ∂, sections of the exponential local system $\pi : \mathcal{I} \to \partial$.



Connected components of $\mathcal{I}$: Stokes circles $\langle q \rangle$

The map $\langle q \rangle \to \partial$ is $r : 1$ with r=ramification index of q (=3 on the picture)

Geometric description of formal data

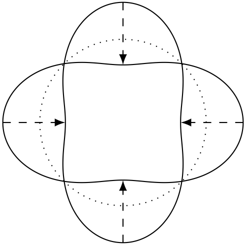


- Irregular class $\Theta = \sum_i n_i \langle q_i \rangle$, $n \geq 1$.
- Local system of formal solutions $V^0 \to \partial$, with a grading $V^0_d = \bigoplus_{\pi(i)=d} V^0_{d,i}$, and $\dim(V^0_{d,k}) = n_i$ if $k \in \langle q_i \rangle$.
- The formal monodromy correponds to the monodromy of V^0.

Stokes phenomenon

Regular singularities: formal solutions are actually convergent

Irregular singularities: when resumming formal solutions, we get analytic solutions which jump at singular (or anti-Stokes) directions.



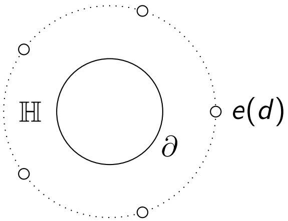
Stokes diagram: draw growth rate $\mathrm{Re}(q_i(z))$ for $|z| \to 0$ as a function of the direction (here $q_1 = z^{-2}$, $q_2 = -z^{-2}$)

Stokes arrow $i \leftarrow j$ at d if $e^{q_i - q_j}$ has maximal decay when $z \to 0$ along d.

Gluing formal and analytic solutions

Modified surface $\tilde{\Sigma}(\Theta)$:

- Take the real blow up at $z = 0$ (i.e. replace the singularity by ∂)
- Add tangential puncture $e(d)$ for each singular direction d



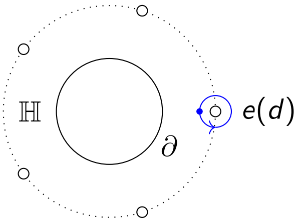
Consider:

- On the halo $\mathbb{H}$: Formal local system V^0
- Outside: Local system of analytic solutions V

$\Rightarrow$ Canonical way to glue them except at tangential punctures (Martinet-Ramis, Loday-Richaud).

Stokes local systems

One gets a "Stokes local system" (Boalch) $\mathbb{V}$ on $\tilde{\Sigma}(\Theta)$.



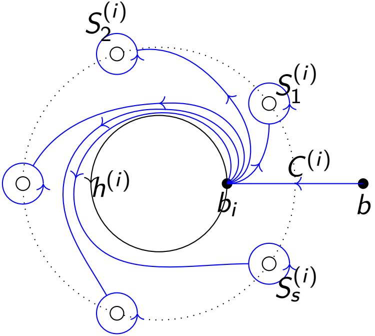
Properties: if ρ is the parallel transport in $\mathbb{V}$,

- For γ_d loop around $e(d)$, $\rho(\gamma_d)$ belongs to the Stokes group $\mathrm{Sto_d} \subset \mathrm{GL}(V_d^0)$
  - Identity blocks on the diagonal
  - Other nontrivial blocks $V_{d,j}^0 \to V_{d,i}^0$ for each Stokes arrow $i \leftarrow_d j$.
- Formal monodromy $\rho(\partial)$ compatible with grading of V^0.

Explicit description

Doing this for each singularity a_i, get global modified surface $\tilde{\Sigma}(\boldsymbol{\Theta})$

Choosing a basepoint b, get wild monodromy representation $\rho : \pi_1(\tilde{\Sigma}(\boldsymbol{\Theta}), b) \to G$.



The monodromy around a_i is the product $M_i = {C^{(i)}}^{-1} h^{(i)} S_{k_i}^{(i)} \dots S_1^{(i)} C^{(i)}$.

Wild character varieties

Get the wild character variety

$$\mathcal{M}_B(\mathbf{a},\boldsymbol{\Theta}) = \left\{ (C^{(i)}, h^{(i)}, S_k^{(i)}) \;\middle|\; \prod_i (C^{(i)^{-1}} h^{(i)} S_{k_i}^{(i)} \dots S_1^{(i)} C^{(i)}) = 1 \right\} / G \times \mathbf{H}$$

where $\mathbf{H} = \prod_i H_i$ and $H_i \subset G$ corresponding to changes of graded framings of $\mathbb{V}_{b_i}$.

It has a quasi-Hamiltonian structure (Boalch)

Riemann-Hilbert-Birkhoff correspondence (Deligne-Malgrange):

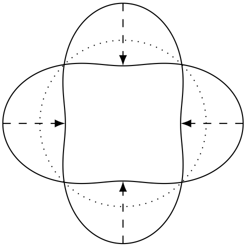
$$\boxed{\mathcal{M}_{dR}(\mathbf{a},\boldsymbol{\Theta}) \cong \mathcal{M}_B(\mathbf{a},\boldsymbol{\Theta}).}$$

Untwisted example (Pure gaussian case)

- Singularity at infinity, two exponential factors $q_1 = z^2$, $q_2 = -z^2$, 4 singular directions, 4 Stokes matrices.
- Moduli space

$$\mathcal{M}_B = \{hS_4S_3S_2S_1 = 1\}/H$$

with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

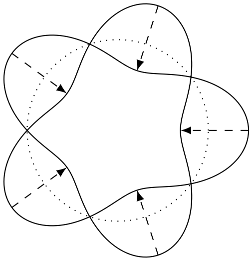


Twisted example (Painlevé I)

- Singularity at infinity, one exponential factor $z^{5/2}$, 5 singular directions, 5 Stokes matrices.
- Moduli space (of dimension 2)

$$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\}/H$$

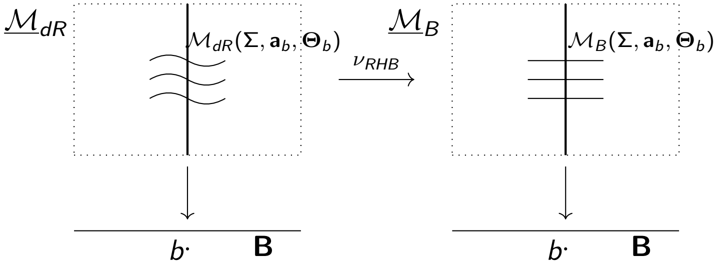
with $S_{2i+1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $S_{2i} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, $h = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$



Geometric POV on (irregular) isomonodromic deformations

Let's move the positions of singularities and irregular classes (regular and irregular "times"): how should ∇ change for the Stokes data to remain constant?

This can be viewed as a flat (Ehresmann) connection on an admissible family of wild character varieties $(\mathcal{M}_{dR}(\Sigma, \mathbf{a}_b, \mathbf{\Theta}_b))_{b\in\mathbf{B}}$.



$\mathbf{B}$: space of "times"

All Painlevé equations can be obtained in that way

The Fourier transform

- Weyl algebra $A_1 = \mathbb{C}[z]\langle\partial_z\rangle$ of differential operators with $[\partial_z, z] = 1$.
- Fourier transform: automorphism of the Weyl algebra:

$$\begin{cases} z & \mapsto -\partial_z \\ \partial_z & \mapsto z \end{cases}$$

- If M is a module over the Weyl algebra $\rightarrow$ Fourier transform $\mathcal{F}M$
- Connections on the affine line $\mathbb{C} = \mathbb{P}^1 \setminus \infty$ are closely related to A_1-modules, and this induces (with a few restrictions) a transformation of connections on $\mathbb{C}$.
- More generally: we can act with any matrix $A \in SL_2(\mathbb{C})$

$$\begin{cases} z & \mapsto az + b\partial_z \\ \partial_z & \mapsto cz + d\partial_z \end{cases}, \qquad \text{with } ad - bc = 1.$$

The stationary phase formula [Malgrange 91, Fang 09, Sabbah 08]

It relates the formal data of a connection and its Fourier transform.

Heuristic idea:

- Solutions are linear combinations of terms of the form

$$f(z) = e^{q(z)} g(z),$$

- The Fourier transform is an integral $\widehat{f}(\xi) = \int_\gamma e^{q(z)-\xi z} g(z)$
- The behaviour of the integral when $\xi \to \infty$ is determined by the critical point of the exponential factor, i.e. z_0 such that

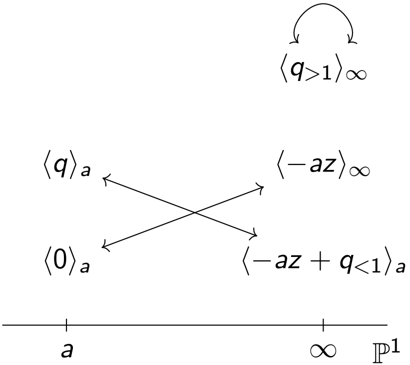
$$\frac{\partial q}{\partial z}(z_0) = \xi.$$

- New exponential factor $\widetilde{q}(\xi) = q(z_0(\xi)) - \xi z_0(\xi)$
$\Rightarrow$ Legendre transform of q.

The stationary phase formula

Different types of circles:

1. The *pure circles* at infinity, of the form $\langle \alpha z \rangle_\infty$, with $\alpha \in \mathbb{C}$.
2. Other circles of slope ≤ 1 at infinity, of the form $\langle \alpha z + q \rangle_\infty$, with $\alpha \in \mathbb{C}$, and $q \neq 0$ of slope < 1,
3. Circles $\langle q \rangle_\infty$ of slope > 1 at infinity,
4. Irregular circles at finite distance $\langle q \rangle_a$, with $q \neq 0$, $a \in \mathbb{C}$.
5. The tame circles $\langle 0 \rangle_a$, $a \in \mathbb{C}$ at finite distance.



Fourier transform of Stokes data: some history

Well-known case (Balser-Jurkat-Lutz, Malgrange, Boalch, d'Agnolo-Hien-Morando-Sabbah)

- One singularity of order 2 at ∞,
- Regular singularities at finite distance.

Other known case: "pure gaussian type": (Sabbah, Hohl)

- Just one singularity at infinity
- All exponential factors of the form $q = az^2$.

In general, not many explicit examples.

In the "simply-laced case" (one pole of order less than 3 at infinity + regular singularities at finite distance), some symplectic isomorphisms obtained (Boalch), but unclear if there are the ones induced by Fourier.

General approaches (Malgrange 1991, T. Mochizuki 2010, 2018): general results but not very explicit

The setting

Joint work with A. Hohl: we use results of T. Mochizuki to obtain explicit isomorphisms in a large class of cases.

In brief:

- Translate a class of cases of Mochizuki's "Stokes shells and Fourier transform" (2018) into the language of Stokes local systems
- Get explicit formulas for the transformation of Stokes matrices

Assumption:

- Only Stokes circles of slope >1 at ∞
- Circles of pure level $r/s > 1$ with s, r coprime $q_i = a_i z^{s/r}$.
- Extra hypothesis $|a_i| = 1$.

Stronger version of Legendre transform

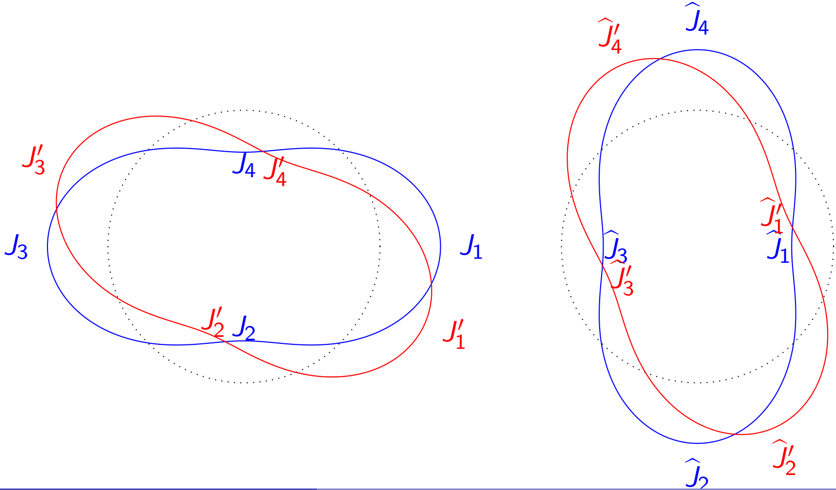
Main idea: the Legendre transform as an homeomorphism between circles $\ell : \langle q \rangle \cong \langle \hat{q} \rangle$.

One can use ℓ to transport the nontrivial entries of Stokes data (up to signs)

Distinguished intervals

On each Stokes circle, intervals J_i where q is either increasing or decreasing when $|z| \to 0$.

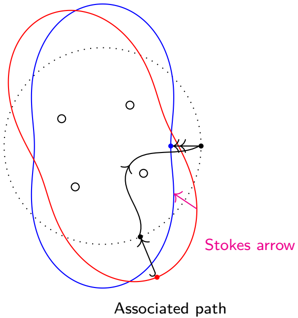
Increasing intervals are sent by ℓ to decreasing ones and vice versa



Stokes paths

Nontrivial entry of Stokes matrix $\leftrightarrow$ entry of parallel transport in Stokes local system along a path $\gamma_{i \to j}$

If the Stokes arrow goes from sector I to J, $i, j \in \partial$ are the midpoints of I, J.



The Stokes local system can be reconstructed from these entries of the parallel transport along these Stokes paths

The algorithm

Start with connection (E, ∇) on $\mathbb{C}$ with irregular class Θ, formal local system V^0, Stokes local system $\mathbb{V}$.

The corresponding objets $\widehat{\Theta}$, $\widehat{V}^0$, $\widehat{\mathbb{V}}$ for the Fourier transform are determined as follows:

- Formal part: $\widehat{V}^0$ obtained from $\ell_* V^0$ by adding some signs when passing from one distinguished sector to the next
- Stokes data: for any Stokes path $\gamma_{i \to j}$, the parallel transport is

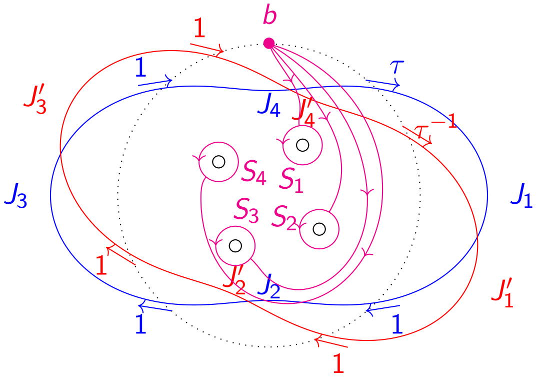
$$\widehat{\rho}(\gamma_{i \to j}) = \pm \rho(\gamma_{\ell^{-1}(i) \to \ell^{-1}(j)})$$

  with an explicitly determined sign.

The nontrivial entries of Stokes matrices are exactly the "deformation data" considered by Mochizuki

Example : pure gaussian case

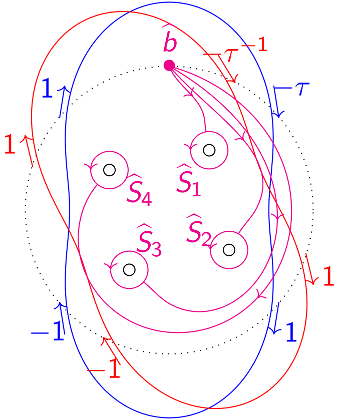
Initial irregular class $\Theta = \langle z^2 \rangle + \langle \frac{1+i}{\sqrt{2}} z^2 \rangle$.



$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & s_4 \\ 0 & 1 \end{pmatrix} \quad h = \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$$

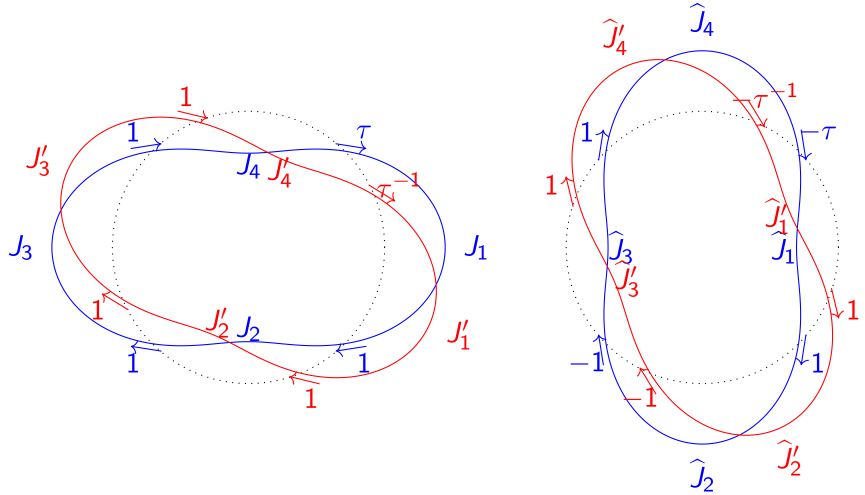
$$\mathcal{M}_B(\Theta) = \{h, S_1, S_2, S_3, S_4 \mid hS_4S_3S_2S_1 = 1\}/H.$$

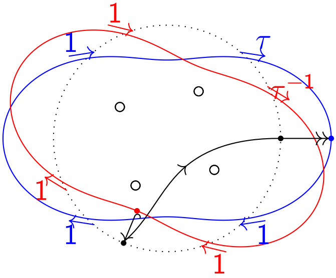
New irregular class $\widehat{\Theta} = \langle -z^2 \rangle + \langle \frac{-1+i}{\sqrt{2}} z^2 \rangle$.



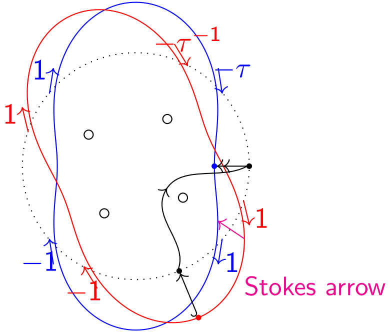
$$\mathcal{M}_B(\widehat{\Theta}) = \{\widehat{h}, \widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4 \mid hS_4S_3S_2S_1 = 1\}/H.$$

Correspondence between distinguished intervals and transformation of the formal data





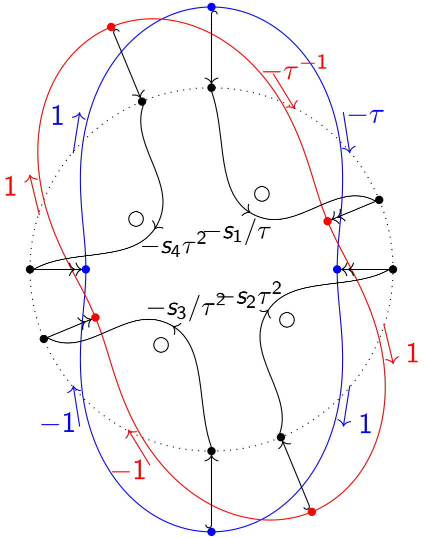
Inverse image $\ell^{-1}(\gamma)$



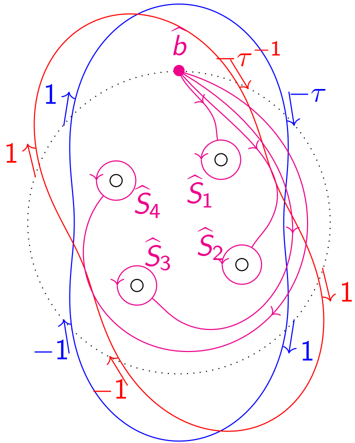
Associated Stokes path γ

With the Legendre transform, transport γ to the initial Stokes diagram

One obtains the entries of the parallel transport along the Stokes paths



Finally we get the new Stokes matrices

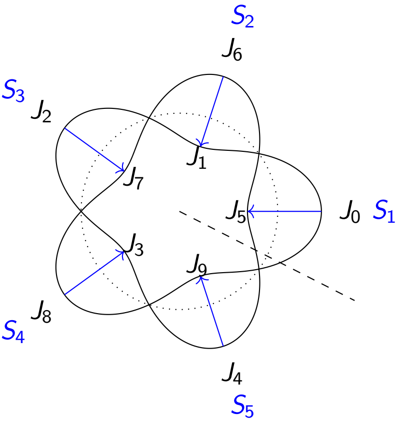


$$\boxed{\widehat{S}_i = S_i, \qquad \widehat{h} = h.}$$

(consistent with Sabbah, Hohl)

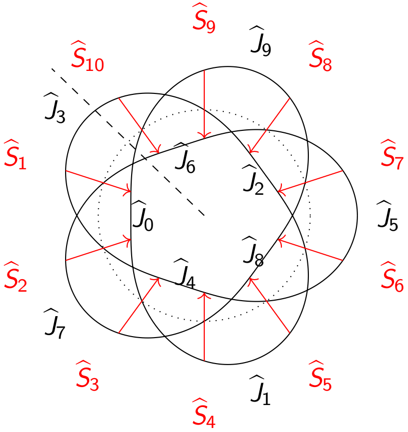
Example: Painlevé I case

$\Theta = \langle -z^{5/2} \rangle$



$\mathcal{M}_B = \{hS_5S_4S_3S_2S_1 = 1\}$ with $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}, \ldots$

Fourier transform $\widehat{\Theta} = \langle z^{5/3} \rangle$.



$$\mathcal{M}'_B = \left\{ \hat{h}\hat{S}_{10} \ldots \hat{S}_1 = 1 \right\} \text{ with } \hat{h} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \hat{S}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & 0 & 1 \end{pmatrix}, \hat{S}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}$$

$\mathcal{M}_B \simeq \mathcal{M}'_B$ via $\Phi : (s_1, s_2, s_3, s_4, s_5) \mapsto (s_3, -s_5, -s_2, s_4, s_1, -s_3, s_5, s_2, -s_4, -s_1)$

Computation of the isomorphism

coefficient	Stokes arrow	Stokes matrix entry	extra sign
t_1	$3 \to 0$	$-s_5$	$+$
t_2	$7 \to 0$	$-s_2$	$+$
t_3	$7 \to 4$	s_4	$+$
t_4	$1 \to 4$	$-s_1$	$-$
t_5	$1 \to 8$	s_3	$-$
t_6	$5 \to 8$	$-s_5$	$-$
t_7	$5 \to 2$	s_2	$+$
t_8	$9 \to 2$	$-s_4$	$+$
t_9	$9 \to 6$	$-s_1$	$+$
t_{10}	$3 \to 6$	$-s_3$	$-$

The isomorphism is symplectic!

Questions

Conjecture: the isomorphisms induced by the Fourier transform preserve the symplectic structure of the WCVs.

Further questions:

- Can we show this?
- Obtain explicit isomorphisms for more general situations (several irregular singularities, several levels...)?
- How these isomorphisms behave in families: can we relate the corresponding spaces of times and isomonodromy systems?