Quantization of discrete Painlevé/Garnier system via affine quantum group

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Painlevé equations have rich symmetry known as "Bäcklund transformations" which in fact generate affine Weyl groups.

In the VIth case, the group is of type $D_4^{(1)}$ and there is a symmetry-preserving time discretization by Jimbo and Sakai.

We will quantize this, i.e. construct a non-commutative version of their discrete VIth equation.

Two ways to quantize: the symmetry point of view and the monodromy-preserving point of view.

It turns out that these two provides the same equation.

§1. Introduction. The Painlevé equations and the difference PVI
§2. Warming up : Symmetry based quantization [0703036]
§3. Lax form (=isomonodoromy) based quantization[1210.0915]
§4. Recent progress [2211.16772, 2309.15364] and Garnier case

§1. Introduction : Painlevé equations $\mathrm{P_I} - \mathrm{P_{VI}}$.

(I) $y'' = 6y^2 + t$ $\qquad (y = y(t), \quad ' = \frac{d}{dt}$

(II) $y'' = 2y^3 + ty + \alpha$ $\qquad \alpha, \beta \cdots$: parameters.)

(III) $y'' = \dfrac{y'^2}{y} - \dfrac{y'}{t} + \dfrac{\alpha y^2 + \beta}{t} + \gamma y^3 + \dfrac{\delta}{y}$

(IV) $y'' = \dfrac{y'^2}{2y} + \dfrac{3}{2}y^2 + 4ty^2 + 2(t^2 - \alpha)y + \dfrac{\beta}{y}$

(V) $y'' = \left(\dfrac{y'^2}{2y} + \dfrac{y'^2}{y-1}\right) - \dfrac{y'}{t} + \dfrac{(y-1)^2}{t^2}(\alpha y + \dfrac{\beta}{y}) + \gamma\dfrac{y}{t} + \delta\dfrac{y(y+1)}{y-1}$

(VI) $$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{y'^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Characteristic features of Painlevé equations

• Nonlinear ODE of order 2 ; Solutions have no 'moving' singularities other than poles $\longrightarrow$ six of them. [Painlevé^{1900}, R.Fuchs^{1907}]

• Arise from monodromy preserving deformations. [Riemann, Fuchs]

• Solutions are transcendental in general : i.e. cannot obtained by the finite iteration of integration of elementary functions. [Umemura]

• Nonautonomous Hamiltonian system; $\exists$ affine Weyl group symmetry. (Bäcklund transformations)

eq	I	II	III	IV	V	VI
Symmetry	$-$	$A_1^{(1)}$	$A_1^{(1)} \times A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$D_4^{(1)}$

[Okamoto]

Reminder : Affine Weyl groups

• $C = [C_{i,j}]_{i,j=0,\cdots,l} \in \mathrm{Mat}_{l+1,l+1}(\boldsymbol{Z})$: Generalised Cartan matrix iff (i) $C_{ii} = 2, C_{ij} \le 0 (i \ne j)$ (ii) $C_{ij} = 0 \Leftrightarrow C_{ji} = 0$. $\leftrightarrow$ Dynkin diag.

ex. $C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \leftrightarrow$ ○—○ or 0—1 (type $A_2, \leftrightarrow sl_3(\boldsymbol{C})$)

• C is of finite type $:\Leftrightarrow \det C > 0 \leftrightarrow$ fin. dim. simple Lie algebras$_{/\boldsymbol{C}}$

C is of affine type $:\Leftrightarrow \mathrm{Ker} C \simeq \mathbf{C} \leftrightarrow$ affine Lie algebras

ex. $C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \leftrightarrow$
```
  0
 / \
1 - 2
```
$(A_2^{(1)}, \leftrightarrow sl_3(\boldsymbol{C}[t, t^{-1}]) \oplus \boldsymbol{C}c)$

Assume $C_{ij} = 0$ or $-1 (i \ne j)$ for simplicity ('simply laced'). Then

• $W = W(C) := \langle s_i | i = 0, \cdots, l \rangle$ with relations (i)$s_i^2 = 1$,

(ii) $s_i s_j = s_j s_i$ (if $C_{ij} = 0$) $s_i s_j s_i = s_j s_i s_j$ (if $C_{ij} = -1$)

is called an affine Weyl group if C is of affine type.

$\Rightarrow W(C) \simeq W_{\mathrm{fin}} \ltimes \boldsymbol{Z}^l$, where $W_{\mathrm{fin}} = W([C_{ij}]_{i,j \ne 0})$

The Hamiltonian and the symmetry of PVI.

- Okamoto Hamiltonian for $\mathrm{P_{VI}}$:$\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=0, \{z,y\}=1$,

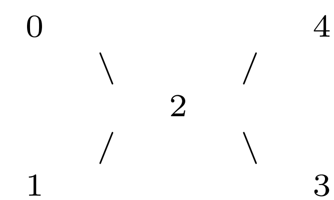
$$H := \frac{y(y-1)(y-t)}{t(t-1)}\left[z^2-\left\{\frac{\alpha_0-1}{y-t}+\frac{\alpha_3}{y-1}+\frac{\alpha_4}{y}\right\}z+\frac{(\alpha_1+\alpha_2)\alpha_2}{y(y-1)}\right]$$

Then $\mathrm{P_{VI}}$ *for* $y(t) \Longleftrightarrow \frac{dy}{dt}=\frac{\partial H}{\partial z}=\{H,y\}, \quad \frac{dz}{dt}=-\frac{\partial H}{\partial y}=\{H,z\}.$

- PVI is invariant under the rational $W(D_4^{(1)})$- action:

$s_i(\alpha_j)=\alpha_j-C_{ij}\alpha_i$ (C_{ij} :the generalized Cartan matrix of type $D_4^{(1)}$),

i	0	1	2	3	4
$s_i(y)=$	$y-\frac{\alpha_0}{z-t}$	y	y	$y-\frac{\alpha_3}{z-1}$	$y-\frac{\alpha_4}{z}$
$s_i(z)=$	z	z	$z+\frac{\alpha_2}{y}$	z	z

[$D_4^{(1)}$: ]

- s_i's preserve the Poisson bracket: $\{s_i(z), s_i(y)\}=\{z,y\}=1.$

Symmetry preserving time discretization of PVI

- qPVI [Jimbo-Sakai'96] $y \leftrightarrow y(t), x \leftrightarrow z(t)$ in PVI.

$$y(t)y(pt) = p^2 t^{-2} \frac{x(t) + a_1^2 p^{-1} t}{x(t) + a_0^{-2} p t^{-1}} \cdot \frac{x(t) + a_1^2 p^{-1} t}{x(t) + a_0^2 p t^{-1}},$$

$$x(t)x(p^{-1}t) = t^{-2} \frac{y(t) + a_4^{-2} t^2}{y(t) + a_5^{-2} t^{-1}} \cdot \frac{y(t) + a_4^2\, t}{y(t) + a_5^{-2} t^{-1}}.$$

(p : step of time, $a_i = e^{\alpha_i}$: parameters.)

- The symmetry is generated by : $s_i(a_j) = a_i^{-C_{ij}} a_j = e^{s_i(\alpha_j)}$,

$$s_2(x(t)) := \frac{a_0 a_1^{-1} y(t) + a_2^2}{a_0 a_1^{-1} a_2^2 y(t) + 1} x(t), \quad s_j(x(t)) := x(t) \quad (j \neq 2)$$

$$s_3(y(t)) := \frac{a_3^2 a_4 a_5^{-1} x(t) + 1}{a_4 a_5^{-1} x(t) + a_3^2} y(t) \quad s_j(y(t)) := y(t) \quad (j \neq 3)$$

AIM: Symmetry preserving quantization

- $W(D_4^{(1)})$-invariant Poisson structure $\rightarrow$ why not quantize?

PVI case	differential	difference
classical	P_{VI}[Painlevé]	qP_{VI}[Jimbo-Sakai]
quantum	$\widehat{P_{VI}}$[H.Nagoya]	$\widehat{qP_{VI}}$[THIS TALK]

Q.q-difference and Weyl group symmetry : something to do with quantum groups? (especially the origin of the Hamiltonian)

cf.
[M.Noumi-Y.Yamada$'98$]discrete Painlevé $\Leftarrow$ Lie theory language
[T.Tsuda-T.Masuda$'06$]q-Painlevé VI eq. from 'q-UC' hierarchy
[H.Nagoya$'05$] symmetry preserving quantization of P_{VI} ($\widehat{P_{VI}}$)
[Reshetikhin$'92$]: The KZ eq. $=$ quantized isomonodromy equation

§2. Warming up: Symmetry based quantization.

Quick review of Noumi-Yamada:

- $\exists$ affine Weyl group (W) action, rational in indeterminates (below)
- Choose a lattice element $T \in W$. Regard T-action as the discrete time evolution. $\Rightarrow$ reproduces/generalises (some of) the above list.
- Symmetry. $W^T = \{w \in W | wT = Tw\}$ is again an affine Weyl group: $W^T \simeq \langle s_\alpha | \alpha \perp T\rangle$, s_α - reflection w.r.t. the root α

- Let $K := \mathbf{C}(a_0, \cdots, a_l, f_0, \cdots, f_l)$ and let

$$u_{ij} : u_{ji} = -C_{ij} : C_{ji} \ (u_{ij} = 0 \text{ if } i = j \text{ or } C_{ij} = 0).$$

$\Rightarrow$ $s_i(a_j) := a_j a_i^{-a_{ij}}, s_i(f_j) := f_j \left(\frac{a_j - f_i}{1 - a_i f_i}\right)^{u_{ij}}$ give $W \overset{hom}{\to} \mathrm{Aut}(K)$, which is Poisson wrt. $\{f_i, f_j\} = u_{ij} f_i f_j, \quad \{a_i, a_j\} = \{a_i, f_j\} = 0.$

$\Rightarrow$ The case $A_2^{(1)}$ gives the discrete Painlevé III of [Kajiwara-Kimura]

- Quantization: $\{f, g\} \to [f, g] = fg - gf$

Quantizing the Weyl group action : Type A case

- $Q = \boldsymbol{Z}\alpha_0 + \cdots + \boldsymbol{Z}\alpha_l$: $A_l^{(1)}$ root lattice, $\alpha_0, \cdots, \alpha_l$: simple roots
- $\boldsymbol{C}[Q] = \boldsymbol{C}[e^{\alpha_0}, \cdots, e^{\alpha_l}]$: the group algebra, $\exists\ W = \langle s_i \rangle_{i=0}^{l}$ -action

$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i, \quad s_i(a_j) = a_i^{-c_{ij}} a_j \ (a_j := e^{\alpha_j})$$

This extends to $\boldsymbol{K}$:= (the quotient field of $\boldsymbol{C}[Q]$) $= \boldsymbol{C}(a_0, \cdots, a_l)$.

- Let $\boldsymbol{K}\langle F_0, \cdots, F_l \rangle$ be a noncomutative $\boldsymbol{K}$- alg. gen. by $F_0, \cdots, F_l$

Assume : $\exists \boldsymbol{F}$:= (the quotient skew field of $\boldsymbol{K}\langle F_0, \cdots, F_l \rangle$).

Theorem The W action on $\boldsymbol{K}$ extends to $\boldsymbol{F}$ by

$$s_i(F_{i-1}) := \frac{1 + a_i F_i}{a_i + F_i} F_{i-1}, \quad s_i(F_{i+1}) := F_{i+1} \frac{a_i + F_i}{1 + a_i F_i}.$$

and $s_i(F_j) := F_j \ (i - j \not\equiv \pm 1)$.

$\Rightarrow s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i (j \not\equiv i \pm 1).$ Straightforward

Hamiltonians for the W-action

• Let $\partial_0, \cdots, \partial_l$ be $[\partial_j, \alpha_k] := c_{jk} \Rightarrow s_j(\alpha_k) = \rho_j \alpha_k \rho_j^{-1}$, where $\rho_j := e^{\pi\sqrt{-1}\alpha_j \partial_j}$: W- action on $\boldsymbol{K}$ is (essentially) inner (=Hamiltonian).

• Let $l \geq 2$ and $F_{i+1}F_i = qF_iF_{i+1},\ \ F_iF_j = F_jF_i\ (i-j \not\equiv \pm 1) : (*)$
$\Rightarrow$ Poisson bracket $\{\phi, \psi\} := \lim_{q\to 1} \frac{1}{q-1}[\phi, \psi]$ on $\boldsymbol{F} \bmod (q-1)$ reads

$$\{F_i, F_{i+1}\} \equiv \frac{1}{q-1}(F_iF_{i+1} - F_{i+1}F_i) \overset{(*)}{=} -F_iF_{i+1}.$$

Theorem (1) s_j preserves the relations (*).

(2) $s_i(\phi) = S_i \phi S_i^{-1}$ with $S_i := \Psi(a_i, F_i) \circ \rho_i$. where

$$(x)_\infty := \prod_{m=0}^{\infty}(1 + xq^m),\ \Psi(a, F) := \frac{(qF)_\infty (F^{-1})_\infty}{(qaF)_\infty (aF^{-1})_\infty} \in \boldsymbol{F}((q)).$$

∵Faddeev-Volkov'93: $(F_i)_\infty (F_{i+1})_\infty = (F_{i+1})_\infty (F_iF_{i+1})_\infty (F_i)_\infty$,
$\Psi(x, F_i)\Psi(xy, F_{i+1})\Psi(y, F_i) = \Psi(y, F_{i+1})\Psi(xy, F_i)\Psi(x, F_{i+1})$. □

Quantizing discrete PIII (qPIII) of Kajiwara-Kimura

- ω : $a_i \mapsto a_{i+1}$, $\quad F_i \mapsto F_{i+1 \bmod l+1}$ (the diagram autom.) and s_i generate the extended affine Weyl group $\tilde{W}(A_l^{(1)})$ action on $\boldsymbol{F}((q))$.

- Commuting elements in $\tilde{W}(A_l^{(1)}) = \langle \mathcal{S}_l, \text{weight lattice} \rangle$:

$$T_1 := s_1 s_2 \cdots s_l \omega^{-1},\ T_2 := s_1^{-1} T_1 s_1,\ T_3 := s_2^{-1} T_2 s_2, \cdots, T_l.$$

- Take T_1 as a discrete time evolution operator, then the subgroup $\langle s_0 s_1 s_0, s_2, s_3, \cdots, s_l \rangle \simeq W(A_{l-1}^{(1)})$ commutes with the evolution.

ex. $l = 2$. $a_0 a_1 a_2 =: p$ is invariant under $\tilde{W}(A_2^{(1)})$; The same holds for $F_0 F_1 F_2 =: c$ ($[c, \text{others}] = 0$) if $F_{i+1} F_i = q F_i F_{i+1}$.

Put $T_1 = s_1 s_2 \omega^{-1}$: $T_1(a_0) = p^{-1} a_0,\ T_1(a_1) = p a_1,\ T_1(a_2) = a_2$;

$$\Rightarrow \quad F_0 T_1(F_0) = c \frac{1 + a_1 F_1^{-1}}{1 + a_1 F_1}, \quad T_1^{-1}(F_1) F_1 = c \frac{1 + a_0^{-1} F_0}{1 + a_0^{-1} F_0^{-1}} \quad : \mathbf{q\widehat{P}_{III}}$$

Hamiltonian for the diagram automorphism

• T_1 commutes with $\langle s_0 s_1 s_0, s_2 \rangle \simeq W(A_1^{(1)})$ — quantization of discrete $\mathrm{P_{III}}$ system studied by Kajiwara-Kimura.

But we used the diagram automorphism ω

• Put Ω $:=$ $\left(\theta(F_0^{-1}F_1)\theta(qF_1)^2\theta(F_2^{-1}F_0^{-1})\right)^{-1} \times p^{-\partial_1'}\rho_1\rho_2$, where $\theta(X) := (X,q)_\infty(qX^{-1},q)_\infty$, $[\partial_1', \alpha_0] = -1, [\partial_1', \alpha_1] = 1, [\partial_1', \alpha_2] = 0$.

(Note : If we realize the $A_2^{(1)}$ root system in $\boldsymbol{R}^3 \oplus \boldsymbol{R}\delta$ by $\alpha_0 = e_3 - e_1 + \delta, \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$, where e_1, e_2, e_3 are the orthonormal basis of $\boldsymbol{R}^3$ and δ the canonical null root so that $p = e^\delta$, then $\partial_1' =$ the derivation corresponding to e_1.)

$$\Longrightarrow \Omega a_i \Omega^{-1} = \omega(a_i), \Omega F_i \Omega^{-1} = \omega(F_i) \; ; T_1 = \mathrm{Ad}(S_1 S_2 \Omega^{-1}),$$

$$S_1 S_2 \Omega^{-1} = \Psi(a_1, F_1)\Psi(a_1 a_2, F_2)\theta(F_0^{-1}F_1)\theta(qF_1)^2\theta(F_2^{-1}F_0^{-1})p^{\partial_1'}.$$

Other types : reduces to B_2, G_2.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} (B_2), \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} (G_2).$$

- Need relations. $\boldsymbol{F} := \boldsymbol{C}(a_1, a_2)(F_1, F_2)$, where a_i : central and

$$F_2F_1 = q^2F_1F_2. \ (B_2), \qquad F_2F_1 = q^3F_1F_2. \ (G_2).$$

Theorem. $\rho_j := e^{\pi\sqrt{-1}\alpha_j\partial_j}$, where $[\partial_j, \alpha_k] = c_{jk}$. Then:

- B_2 case. $S_1 := \Psi_q(a_1, F_1)\rho_1, \quad S_2 := \Psi_{q^2}(a_2^2, F_2)\rho_2$

$$\Longrightarrow Ad(S_j)^2 = id, \ S_1S_2S_1S_2 = S_2S_1S_2S_1.$$

- G_2 case. $S_1 := \Psi_q(a_1, F_1)\rho_1, \quad S_2 := \Psi_{q^3}(a_2^3, F_2)\rho_2$

$$\Longrightarrow Ad(S_j)^2 = id, \quad S_1S_2S_1S_2S_1S_2 = S_2S_1S_2S_1S_2S_1.$$

$\mathbf{B}_2$ case.

$$
\begin{aligned}
S_1 F_2 S_1^{-1} &= \Psi_q(a_1, F_1) F_2 \Psi_q(a_1, F_1)^{-1} \\
&= F_2 \Psi_q(a_1, q^{-2} F_1) \Psi_q(a_1, F_1)^{-1} \\
&= F_2 \frac{(F_1 q^{-1}, q)_\infty (F_1^{-1} q^2, q)_\infty}{(a_1 F_1 q^{-1}, q)_\infty (a_1 F_1^{-1} q^2, q)_\infty} \frac{(a_1 F_1 q, q)_\infty (a_1 F_1^{-1}, q)_\infty}{(F_1 q, q)_\infty (F_1^{-1}, q)_\infty} \\
&= F_2 \frac{(1 + F_1 q^{-1})(1 + F_1)}{(1 + a_1 F_1 q^{-1})(1 + a_1 F_1)} \frac{(1 + a_1 F_1^{-1})(1 + a_1 F_1^{-1} q)}{(1 + F_1^{-1})(1 + F_1^{-1} q)} \\
&= F_2 \frac{(a_1 + F_1)(a_1 + F_1 q^{-1})}{(1 + a_1 F_1)(1 + a_1 F_1 q^{-1})}.
\end{aligned}
$$

$\because$Thm. $(S_1 S_2 S_1 S_2 = S_2 S_1 S_2 S_1)$: Introduce $\sqrt{-F_2}$

$\Longrightarrow \Psi_{q^2}(a_2^2, F_2) = \Psi_q(a_2, \sqrt{-F_2}) \Psi_q(a_2, -\sqrt{-F_2})$; reduces to A case!

Details

- Write $a{:=}a_1$, $b{:=}a_2$, $\Psi^a_1{:=}\Psi_q(a,F_1)$ $\Psi^b_\pm{:=}\Psi_q(b,\pm\sqrt{-F_2})$ for short
- $S_1S_2S_1S_2 = S_2S_1S_2S_1$

$$\Longleftrightarrow \Psi^a_1(\Psi^{ab}_+\Psi^{ab}_-)\Psi^{ab^2}_1(\Psi^b_+\Psi^b_-) = (\Psi^b_+\Psi^b_-)\Psi^{ab^2}_1(\Psi^{ab}_+\Psi^{ab}_-)\Psi^a_1.$$

- Can use $\Psi^x_\pm\Psi^{xy}_1\Psi^y_\pm = \Psi^y_1\Psi^{xy}_\pm\Psi^x_1$, $\Psi^x_+\Psi^y_- = \Psi^y_-\Psi^x_+$ (x,y central):

$$\begin{aligned}
\mathrm{LHS} &= \Psi^a_1\Psi^{ab}_+\underline{\Psi^{ab}_-\Psi^{ab^2}_1\Psi^b_-}\Psi^b_+ \\
&= \underline{\Psi^a_1\Psi^{ab}_+\Psi^b_1}\Psi^{ab^2}_-\Psi^{ab}_1\Psi^b_+ \\
&= \Psi^b_+\Psi^{ab}_1\underline{\Psi^a_+\Psi^{ab^2}_-}\Psi^{ab}_1\Psi^b_+ = \Psi^b_+\Psi^{ab}_1\Psi^{ab^2}_-\underline{\Psi^a_+\Psi^{ab}_1\Psi^b_+} \\
&= \Psi^b_+\underline{\Psi^{ab}_1\Psi^{ab^2}_-\Psi^b_1}\Psi^{ab}_+\Psi^a_1 \\
&= \Psi^b_+\Psi^b_-\Psi^{ab^2}_1\Psi^{ab}_-\Psi^{ab}_+\Psi^a_1 = \mathrm{RHS}. \square
\end{aligned}$$

G_2 case is similar (Use cubic roots of $-F_2$) — These Weyl group actions quantizes the geometric crystal of Berenstein-Kazhdan.

Main theme: quantize discrete PVI.

- Use $D_5^{(1)}$: ($\{e_j\}$:orthonormal $\subset \boldsymbol{R}^5 \perp \delta$: the null root)

$$
\begin{array}{ccccccccccc}
\alpha_0 & & & & \alpha_5 & & \delta-e_1-e_2 & & & & e_4+e_5 \\
 & \backslash & & / & & & & \backslash & & / & \\
 & & \alpha_2 \; - \; \alpha_3 & & & = & & & e_2-e_3 \; - \; e_3-e_4 & & \\
 & / & & \backslash & & & & / & & \backslash & \\
\alpha_1 & & & & \alpha_4 & & e_1-e_2 & & & & e_4-e_5
\end{array}
$$

Let $q \in \mathbf{C}^\times, |q| < 1$. Write $a_j := q^{\alpha_j}, p := q^\delta = a_0 a_1 a_2^2 a_3^2 a_4 a_5$.

$$\Rightarrow \exists W = W(D_5^{(1)}) \text{ action : } s_i(a_j) = a_i^{-C_{ij}} a_j \quad (s_i(p) = p, \ \forall i)$$

- Diagram automorphisms. $\sigma : a_j \longleftrightarrow a_{5-j}^{-1} (j = 0, \cdots, 5)$,

$$\sigma_{01} : a_0 \leftrightarrow a_1^{-1}, a_j \mapsto a_j^{-1} (j{\neq}0, 1), \ \sigma_{45} : a_4 \leftrightarrow a_5^{-1}, a_j \mapsto a_j^{-1} (j{\neq}4, 5)$$

define the extended affine Weyl group action : $\tilde{W} = \langle W, \sigma_{01}, \sigma_{45}, \sigma \rangle$.

- $\boldsymbol{K} := \boldsymbol{C}(a_0, \cdots, a_5)\langle F, G \rangle$ [F, G : noncomm., $FG = qGF$ later]

Theorem: $\tilde{W}(D_5^{(1)})$**-action.**

- $\exists\ \langle W, \sigma_{01}\sigma_{45}\rangle \overset{hom}{\to} \mathrm{Aut}(\boldsymbol{K})$ given by $\sigma_{01}\sigma_{45} : F \leftrightarrow F^{-1}, G \leftrightarrow G^{-1}$,

$$s_2(F) := F\frac{a_0a_1^{-1}G+a_2^2}{a_0a_1^{-1}a_2^2G+1}, \quad s_j(F) := F \quad (j \neq 2)$$
$$s_3(G) := \frac{a_3^2a_4a_5^{-1}F+1}{a_4a_5^{-1}F+a_3^2}G, \quad s_j(G) := G \quad (j \neq 3)$$

- $FG = qGF \Rightarrow$ This action is Hamiltonian: $\exists \Sigma, S_j$ such that

$$\sigma_{01}\sigma_{45}(\phi) = \Sigma\phi\Sigma^{-1}, \quad s_j(\phi) = S_j\phi S_j^{-1} (: \forall j, \forall\phi \in \boldsymbol{K}).$$

Explicitely, let $(x)_\infty := \prod_{m=0}^{\infty}(1-q^mx)$, $\Psi(a,x) := \frac{(qx)_\infty(x^{-1})_\infty}{(aqx)_\infty(ax^{-1})_\infty}$,

$\Rightarrow \Sigma := (FG)_\infty(qG^{-1}F^{-1})_\infty(G^{-1}F)_\infty(qF^{-1}G)_\infty(F)_\infty^2(qF^{-1})_\infty^2(G)_\infty^2(qG^{-1})_\infty^2$,

$S_2 := \Psi(a_2, a_0a_1^{-1}G)\mathrm{e}^{\frac{\pi i}{2}\alpha_2\partial_2}$, $S_3 := \Psi(a_3, a_5a_4^{-1}G)\mathrm{e}^{\frac{\pi i}{2}\alpha_3\partial_3}$,

$S_j := \mathrm{e}^{\frac{\pi i}{2}\alpha_j\partial_j}(j \neq 2,3)$, where $\partial_j(\alpha_k) = C_{j,k}$.

[Remark: $\exists$ Quantization of the (Hamiltonian) affine Weyl group action in KNY for all Dynkin types, but the above is somehow different.]

$\widehat{qP_{VI}}$

[Tsuda-Masuda]: in $q = 1$ case, $T_3 := s_2 s_1 s_0 s_2 s_3 s_4 s_5 s_3 \sigma_{01} \sigma_{45} : e_j \mapsto e_j - \delta_{j,3}\delta,\ \delta \mapsto \delta$ reproduces the qP_{VI} of Jimbo-Sakai.

Theorem/Def. T_3 action is given as follows, which commutes with $W(D_4^{(1)}) \simeq \langle s_0, s_1, s_2 s_3 s_2, s_4, s_5 \rangle$. Put $t = q^{2e_3} = a_3^2 a_4 a_5$.

$$T_3(a_0, a_1, t, a_4, a_5) = (a_0, a_1, t/p, a_4, a_5),$$

$$T_3(F) = p^2 t^{-2} \frac{G + p^{-1} a_1^{-2} t}{G + p^3 a_0^{-2} t^{-1}} \cdot \frac{G + p^{-1} a_1^2 t}{G + p^3 a_0^2 t^{-1}} F^{-1},$$

$\widehat{qP_{VI}}$

$$T_3^{-1}(G) = t^2 \frac{F + a_4^2\, t}{F + a_5^2\, t^{-1}} \cdot \frac{F + a_4^{-2}\, t}{F + a_5^{-2}\, t^{-1}} G^{-1}.$$

- $FG = qGF \Rightarrow$ By construction, $\widehat{qP_{VI}} (= T_3)$ has the Hamiltonian:

$$T_3 = Ad(\mathcal{H}), \mathcal{H} := S_2 S_1 S_0 S_2 S_3 S_4 S_5 S_3 \Sigma.$$

§3. Lax form based quantization

Review: Isomonodromy deformation of the 2×2 singular connection

$$\nabla = L(z)dz = \sum_{j=1}^{n} \frac{L^{(j)}}{z-t_j} dz. \ (\infty : \text{pole, Res}_\infty = -\sum_1^n L^{(j)} =: L^{(\infty)})$$

- Let $Y(z)$ - fundamental solution of $\frac{dY}{dz} = L(z)Y(z)$,

  $C_j \in \pi_1(\mathbf{P}^1, *)$- the contour around $t_j \Rightarrow {C_j}_*(Y)(z) = Y(z)M_j$

  $M_j = e^{2\pi i L^{(j)}}$ - the monodromy matrix, $M_1 \cdots M_n M_\infty = 1$

FACT $\{M_j\}$:constant (isomonodromy)if $\frac{\partial Y}{\partial t_j} Y^{-1} = -\frac{L^{(j)}}{z-t_j} (=: B_j)$.

- Compatibility $\Rightarrow [\partial_z - L(z, \vec{t}), \partial_j - B_j(z, \vec{t})] = 0$ (Schlesinger eq.),

$\Leftrightarrow \partial_j L (= [B_j, L] + \partial_z B_j) = \{h_j, L\} + \partial_z B_j$, $h_j = \sum_{k \neq j} \frac{\mathrm{Tr}(L^{(j)} L^{(k)})}{t_j - t_k}$

where $\{L^{(i)}_{k,k'}, L^{(j)}_{m,m'}\} := -\delta_{ij}(\delta_{k'm} L^{(j)}_{km'} - \delta_{km'} L^{(j)}_{mk'})$.

- $n=3, (t_1, t_2, t_3, \infty) = (0, 1, t, \infty) \Rightarrow P_{VI}$; $y(t) \sim$off-diag element of $L^{(3)}$.

Reshetikhin: General n (Garnier system) $\rightsquigarrow$ quantization = KZ eq

Lax form : Difference case (Birkhoff)

- $\frac{Y(qz)-Y(z)}{qz-z} = L(z)Y(z) \quad (L(z) = \frac{L^{(1)}}{z} + \frac{L^{(2)}}{z-1} + \frac{L^{(3)}}{z-t}$, generic)

$\Leftrightarrow Y(qz) = \{1+(q-1)zL(z)\}Y(z)$

$\Leftrightarrow \mathcal{Y}(qz) = \mathcal{L}(z)\mathcal{Y}(z), \mathcal{L}(z) := (z-1)(z-t)\{1+(q-1)zL(z)\}$:poly. in z

($\because$multiply $(z-1)(z-t) = \frac{\gamma(qz)}{\gamma(z)}$ $(\exists\gamma)$ and put $\mathcal{Y} = \gamma Y$)

- Now singularities are $0, \infty$ ($1, t$ can be detected as the zero of $\det \mathcal{L}$)

$\exists\mathcal{Y}_0(z) = z^{\mathcal{L}(0)} \times$(power series in z)-local solution at 0, similar for ∞:

$\exists\mathcal{Y}_\infty(z) \rightsquigarrow$ 'connection' matrix $\mathcal{M}(z) := \mathcal{Y}_0(z)^{-1}\mathcal{Y}_\infty(z) = \mathcal{M}(qz)$

- $\exists$ Deformation preserving $\mathcal{M}(z) = \mathcal{M}(z,t)$

$\Leftrightarrow \exists\mathcal{B}(z,t)$ s.t. $\mathcal{Y}(z,qt) = \mathcal{B}(z,t)\mathcal{Y}(z,t)$

Compatibility $\Leftrightarrow \mathcal{L}(z,qt)\mathcal{B}(z,t) = \mathcal{B}(qz,t)\mathcal{L}(z,t) \overset{[JS]}{\rightsquigarrow} qP_{VI}$.

$\rightarrow$ Realize this(discrete Schlesinger) in the noncommutative setting!

Hint: Faddeev-Volkov quantization of discrete sine-Gordon eq.

Nonautonomous modification + Periodic reduction $\rightsquigarrow \widehat{qP_{VI}}$.

$U_q^\pm \subset U_q(A_1^{(1)})$ and its representation $\rho^\pm$

• Let $e_i^\pm (i{=}0,1), k = q^{h_1}, c, d$ be the Chevalley generators of the upper/lower part $U_q^\pm \subset U_q(A_1^{(1)})$: $ke_1^\pm k^{-1}{=}q^{\pm 2}e_1^\pm, ke_0^\pm k^{-1}{=}q^{\mp 2}e_0^\pm$

• Let $c^\pm \in \mathbf{C}$, and let $\rho^\pm$ be the representation of $U_q^\pm$ on the space $V^\pm := \boldsymbol{C}[e^{\pm\alpha_0}, e^{\pm\alpha_1}]$ defined respectively by

$$e_i^\pm \mapsto -(q-q^{-1})e^{\pm\alpha_i} =: E_i^\pm,\ \ h_i \mapsto h_i\ (i=0,1),\ \ c \mapsto c^\pm \in \boldsymbol{C}$$

By the definition h_i acts as the derivation $[h_i, e^{\pm\alpha_j}] = \pm\alpha_j(h_i)e^{\pm\alpha_j}$.

• Write $\Delta^\pm = E_0^\pm E_1^\pm$. We have $E_0^\pm E_1^\pm = \Delta^\pm \in \mathcal{Z}({U_q^\pm}')$ (=center of the derived algebra), and $q^d\Delta^\pm q^{-d}{=}q^{\pm 1}\Delta^\pm$.

• Let $\mathcal{R} \in U_q^+ \widehat{\otimes} U_q^-$ be the universal R matrix of $U_q(A_1^{(1)})$, and $\square_z$ be the two dimensional evaluation representation of $U_q(A_1^{(1)})$.

Nonautonomous Izergin-Korepin Lax op.

- $\rho^{\pm}$ - representation of $U_q^{\pm}$, $E_0^{\pm}E_1^{\pm} = \Delta^{\pm} \in \mathcal{Z}(U_q^{\pm\prime})$ Put

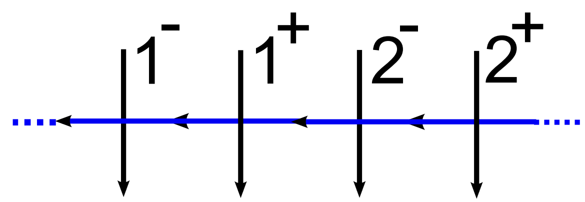
$$L_z^+(\Delta^+) := (\rho^+ \otimes \square_z)(\mathcal{R}) = \frac{(q^4 z^{-1}\Delta^+, q^4)_\infty}{(q^2 z^{-1}\Delta^+, q^4)_\infty} \begin{bmatrix} 1 & \frac{1}{z}E_0^+ \\ E_1^+ & 1 \end{bmatrix} \begin{bmatrix} k^{-\frac{1}{2}} & 0 \\ 0 & k^{\frac{1}{2}} \end{bmatrix} q^{-c^+ d}$$

$$L_z^-(\Delta^-) := (\square_z \otimes \rho^-)(\mathcal{R}) = \frac{(q^4 z\Delta^-, q^4)_\infty}{(q^2 z\Delta^-, q^4)_\infty} \begin{bmatrix} 1 & E_1^- \\ zE_0^- & 1 \end{bmatrix} \begin{bmatrix} k^{-\frac{1}{2}} & 0 \\ 0 & k^{\frac{1}{2}} \end{bmatrix} q^{-c^- d}$$

Here $\square_z$- 2 dim evaluation rep., $\mathcal{R}$-the universal R matrix(solves YBE).

- $“\begin{vmatrix} 1 & \frac{1}{z}E_0^+ \\ E_1^+ & 1 \end{vmatrix}” = 1 - \frac{\Delta^+}{z} \sim \left(\frac{(q^4 z^{-1}\Delta^+, q^4)}{(q^2 z^{-1}\Delta^+, q^4)}\right)^{-2}$; $q^d \Delta^{\pm} q^{-d} = q^{\pm 1}\Delta^{\pm}$.

- $\mathcal{L}_z(1^-1^+2^-2^+) := L_z^-(\Delta_1^-)^{-1} L_z^+(\Delta_1^+) L_z^-(\Delta_2^-)^{-1} L_z^+(\Delta_2^+)$

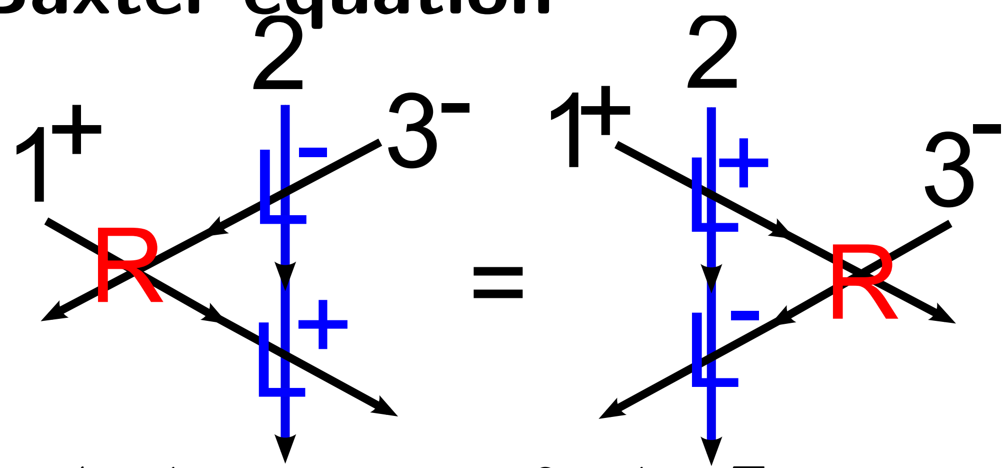
$=$ 

$$\in \mathrm{End}(\mathbf{C}^2 \otimes V^{1-} \otimes V^{1+} \otimes V^{2-} \otimes V^{2+}).$$

The Yang-Baxter equation

- $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$
  $\in U_q^+ \otimes U_q \otimes U_q^-$ implies

$L_z^+(\Delta^+)R(\Delta^+,\Delta^-)L_z^-(\Delta^-)$
$= L_z^-(\Delta^-)R(\Delta^+,\Delta^-)L_z^+(\Delta^+)$,
where $R := (\rho^+ \otimes \rho^-)(\mathcal{R})$
$= (qE_1 \otimes F_1, q^2)^{-1}(q^2\Delta^+ \otimes \Delta^-, q^4)^{-1}(qE_0 \otimes F_0, q^2)^{-1}q^{-T}$,
$T := \frac{1}{2}h \otimes h + c^+ \otimes d + d \otimes c^-$.

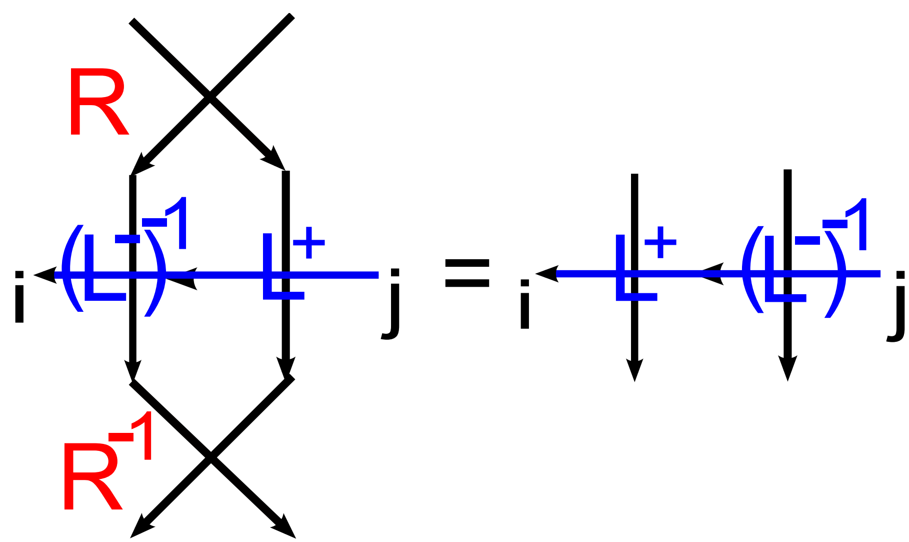


$\Updownarrow$

- exchange dynamics

$R^{-1}\left[(L_z^-)^{-1}L_z^+\right]_{ij} R$

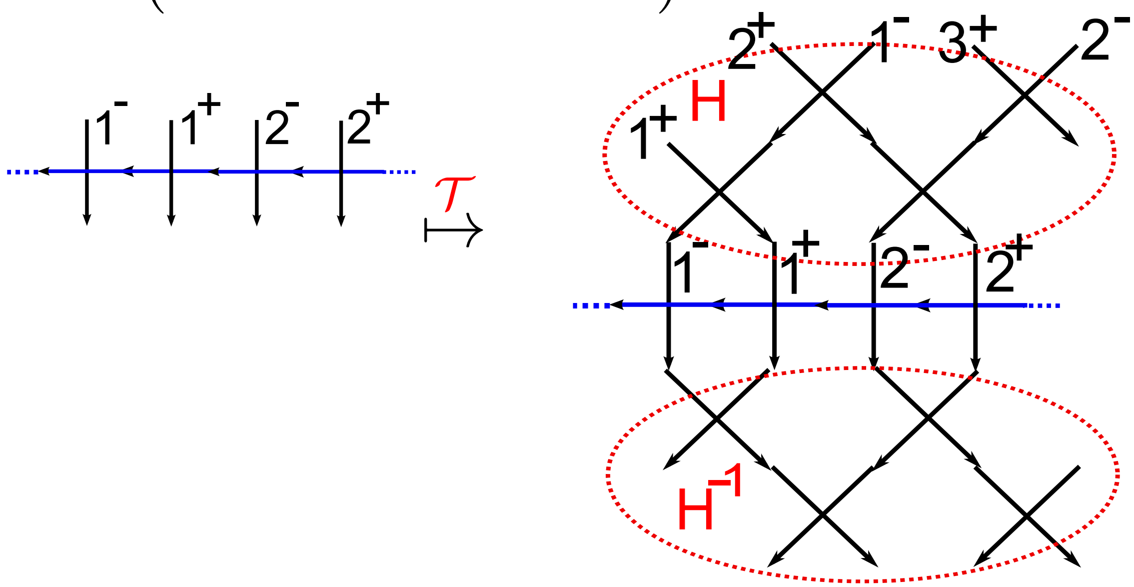
$= \left[L_z^+(L_z^-)^{-1}\right]_{ij}$



$\Updownarrow$ in fact (for $A_\ell^{(1)}$ case)
periodic Toda lattice

Nonautonomous Faddeev-Volkov system

$\mathcal{T} := Ad(\mathcal{H}^{-1})$, $\mathcal{H} = \mathcal{H}_e\mathcal{H}_o$: acts on the alg. gen. by matrix elements of $\mathcal{L}(\cdots 1^-1^+2^-2^+\cdots)$:



$\mathcal{H}_o := \cdots R_{2^+1^-}R_{3^+2^-}\cdots,$

$\mathcal{H}_e := \cdots R_{1^+1^-}R_{2^+2^-}\cdots,$

$\mathcal{L}$

$\mathcal{H}_e^{-1}$

$\mathcal{H}_o^{-1}$

- For $S = (1^+, 1^-)$ let $\Delta(S) := \Delta_1^+ \otimes \Delta_1^-$, $w(S) := E_1^{(1)} \otimes F_1^{(1)}$,

For $E := (1^+, 2^-)$ let $w(E) := E_1^{(1)} \otimes F_1^{(2)}$ etc

$\Rightarrow w(N) := \mathcal{H}w(S)\mathcal{H}^{-1}$ is locally

determined by w, Δ at W, S, E:

$$
\begin{array}{ccc}
 & N(2^-1^+) & \\
W(1^-0^+) & \Uparrow & E(2^-1^+) \\
 & S(1^-1^+) &
\end{array}
$$

Periodic reduction to $\widehat{qP_{VI}}$

We have: $\mathcal{T}(w(1^-1^+))w(0^-2^+)=\frac{w(0^-1^+)-q\Delta(0^-1^+)}{w(0^-1^+)-q}\cdot\frac{w(1^-2^+)-q\Delta(1^-2^+)}{w(1^-2^+)-q}$,

$$w(2^-1^+)\mathcal{T}^{-1}(w(1^-2^+))=\frac{w(1^-1^+)-q\Delta(1^-1^+)}{w(1^-1^+)-q}\cdot\frac{w(2^-2^+)-q\Delta(2^-2^+)}{w(2^-2^+)-q}.$$

If $c^{i\pm}=c^{(i+2)\pm}$, $\mathcal{T}$ preserves the condi. $w(i^-j^+)=w((i+2)^-(j+2)^+)$

$\Rightarrow$ Write

$\lambda:=\frac{w(0^-0^+)}{w(1^-1^+)}, \mu:=\frac{w(1^-0^+)}{w(0^-1^+)}, \alpha^2:=\frac{\Delta(0^-0^+)}{\Delta(1^-1^+)}, \beta^2:=\frac{\Delta(1^-0^+)}{\Delta(0^-1^+)}$ (:central),

$t:=\Delta(0^-0^+)\Delta(1^-1^+)$ (:$T(t)=q^{2c}t$, $c=c^{0-}+c^{1-}-c^{0+}-c^{1+}$),

$F:=w(0^-1^+), G:=w(1^-1^+)$ (:$FG=qGF$) , then

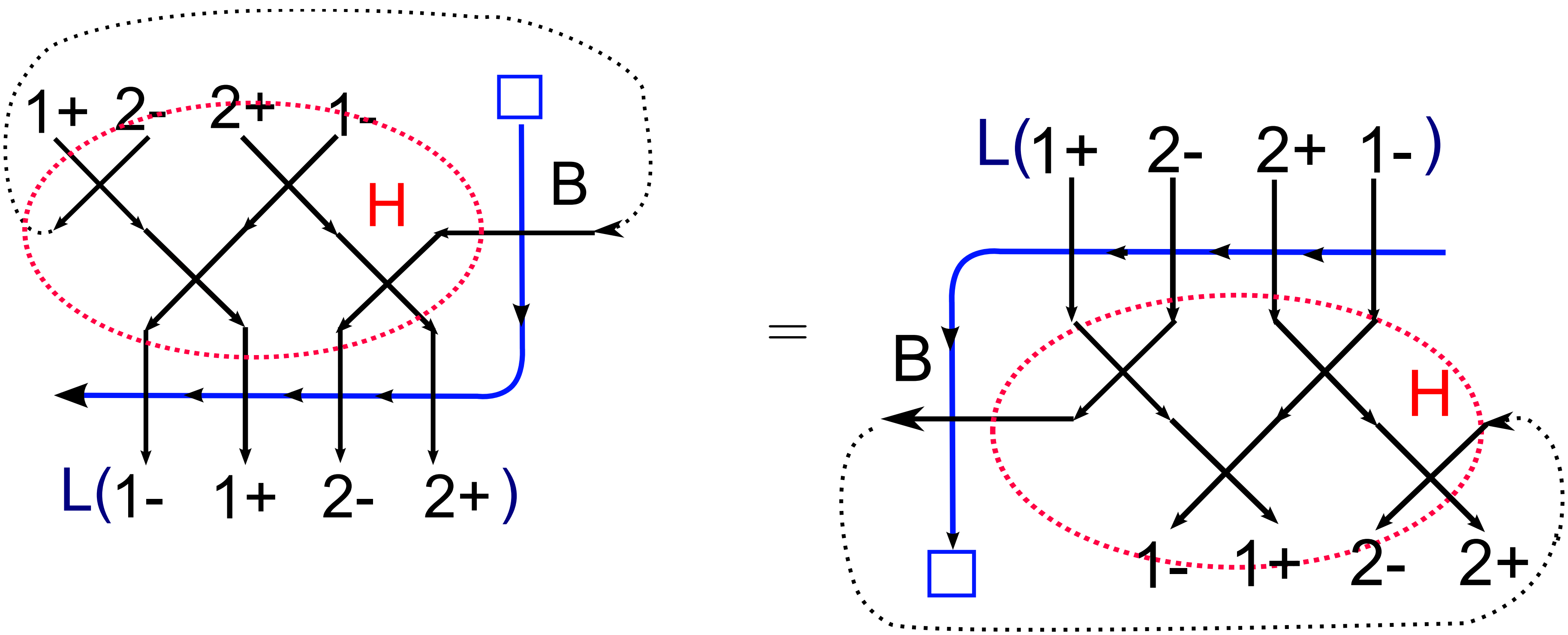
Theorem ($\widehat{qP_{VI}}$ again):

$$\begin{cases} \mathcal{T}(G)G = \lambda^{-1}\dfrac{F-qt/\beta}{F-q}\cdot\dfrac{F-qt\beta/\mu}{G-q/\mu}, \\ F\mathcal{T}^{-1}(F) = \mu^{-1}\dfrac{G-qt/\alpha}{G-q}\cdot\dfrac{G-qt\alpha/\lambda}{G-q/\lambda}. \end{cases}$$

Quantized discrete Schlesinger eq.

Theorem $\mathcal{L}(1^+2^-2^+1^-) = \mathcal{B}(zq^{-c})\mathcal{L}(1^-1^+2^-2^+)\mathcal{B}(z)^{-1}$
where $\mathcal{B}(z)$ is as below.

$\because$ Use YBE:



Thus we have quantized the isomonodromy equation (Lax form).

The Weyl group action

$$\sigma : \begin{cases} w_0^{11} \mapsto q^{-2} w_1^{11}, \\ w_0^{21} \mapsto q^{-2} w_1^{21}, \end{cases}$$

$$s_0 : \begin{cases} w_0^{11} \mapsto w_0^{11}, \\ w_0^{21} \mapsto w_0^{10}, \end{cases} \qquad s_5 : \begin{cases} w_0^{11} \mapsto w_0^{22}, \\ w_0^{21} \mapsto w_0^{21} \end{cases}$$

$$s_2 : \begin{cases} w_0^{11} \mapsto w_0^{11} \frac{w_0^{21}-1}{w_0^{21}-\Delta^{21}}, \\ w_0^{21} \mapsto \frac{1}{w_1^{21}}, \end{cases} \qquad s_3 : \begin{cases} w_0^{11} \mapsto \frac{1}{w_1^{11}}, \\ w_0^{21} \mapsto \frac{w_0^{11}-1}{w_0^{11}-\Delta^{11}} w_0^{10}, \end{cases}$$

$$s_1 : \begin{cases} w_0^{11} \mapsto w_0^{11}, \\ w_0^{21} \mapsto w_0^{21}, \end{cases} \qquad s_4 : \begin{cases} w_0^{11} \mapsto w_0^{11}, \\ w_0^{21} \mapsto w_0^{21} \end{cases}$$

§4. Generalization to the Garnier case

- The generalization of the Izergin-Korepin Lax matrix for the quantum discrete sine-Gordon equation is known :
Kashaev-Reshetikhin's work on quantum discrete Toda field equation.

- Non-autonomous generalization can be obtained by the universal construction

- The periodic reduction gives a (part of) quantized discrete Garnier system

- On the other hand, $\widehat{qP_6}$ appears in 5 dimensional Nekrasov partition function : [Shakirov 2111.07939](conjecture), [Awata-H-Kanno-Ohkawa-Shakirov-Shiraishi-Yamada 2211.16772, 2309.15364](proof)

- Higher rank and/or multi-points case in progress

Recent finding [Shakirov][AHKOSSY]

Define 5 dimensional gauge theory Nekrasov partition function by

$$\sum_{\nu_1,\nu_2,\mu_1,\mu_2\in \mathsf{P}} x_1^{|\nu_1|+|\nu_2|} x_2^{|\mu_1|+|\mu_2|} \prod_{a,b=1}^{2} \frac{\mathbb{N}_{\emptyset,\nu_b}(v\frac{f_a^+}{n_b})\mathbb{N}_{\nu_a,\mu_b}(\frac{n_a}{m_b})\mathbb{N}_{\mu_a,\emptyset}(v\frac{m_a}{f_b^-})}{\mathbb{N}_{\nu_a,\nu_b}(\frac{n_a}{n_b})\mathbb{N}_{\mu_a,\mu_b}(\frac{m_a}{m_b})}$$

$=: Z(x_1,x_2) \in \mathbb{C}[[x_1,x_2]]$ with parameters $(q,\tilde{q},Q,(T_i)_{i=1}^4)$, where

$\mathsf{P} = \{$Young diagrams$\}$, $m_1 = n_1 = 1, m_2 = qn_2 = Q$,

$v = (q/\tilde{q})^{1/2}$, $(f_1^+, f_2^+, f_1^-, f_2^-) = (QT_1/q, T_2^{-1}, T_3^{-1}, QT_4)$, and

$\mathbb{N}_{\lambda,\mu}(u) := \prod_{(i,j)\in\lambda}(1-uq^{\lambda_i-j}\tilde{q}^{\mu'_j-i+1})\prod_{(k,l)\in\mu}(1-uq^{-\mu_k+l-1}\tilde{q}^{-\lambda'_l+k})$.

Theorem We have $\mathcal{H}^\circ Z(Qx_1,\tilde{q}Q^{-1}x_2) = Z(x_1,x_2)$ under suitable identification of variables, where $\mathcal{H}^\circ$ be the infinite product part of $\mathcal{H}$.

The identification (rough sketch)

Recall that $\mathcal{H}$ acts on $\mathcal{V} = V^{1+} \otimes V^{1-} \otimes V^{2+} \otimes V^{2-}$ by definition. More precicely it acts on subspace $\mathcal{V}'$ generated by

$e^{\alpha} \otimes e^{-\alpha} \otimes 1 \otimes 1, 1 \otimes e^{-\beta} \otimes e^{\beta} \otimes 1, 1 \otimes 1 \otimes e^{\gamma} \otimes e^{-\gamma}$, $\alpha, \beta, \gamma \in \mathfrak{h}^*$.

It is easy to see $\mathcal{V}'$ is spanned by $e^{\kappa} \otimes e^{-\lambda} \otimes e^{\mu} \otimes e^{-\nu}$, where $\lambda + \nu = \kappa + \mu$.

$\Rightarrow$ We identify $e^{\alpha_i} \otimes e^{-\alpha_i} \otimes e^{\alpha_i} \otimes e^{-\alpha_i} = w_i^{11} w_i^{22}$ with x_i^2, $i = 0, 1 \bmod 2$.

Write $|b\rangle = (e^{\Lambda_1} \otimes e^{-\Lambda_1} \otimes e^{\Lambda_1} \otimes e^{-\Lambda_1})^b$ for $b \in \mathbb{C}$, we have

$$q^{-T^{11} - T^{11}} x_1^n |b\rangle = q^{-T^{21} - T^{12}} x_1^n |b\rangle = \widehat{x_1}^{\,n} |b\rangle, \quad \widehat{x_1} := q^{4b} x_1 q^{2\partial_1}$$

so we identiy $Q = q^{4b}$ (and similarly $\tilde{q} = q^{c^{1+} - c^{1-} + c^{2+} - c^{2-}}$) and get

$$q^{-T^{11} - T^{12} - T^{21} - T^{22}} Z(x_1, x_2)|b\rangle = Z(Qx_1, \tilde{q} Q^{-1} x_2)|b\rangle,$$

$$\mathcal{H} Z(x_1, x_2)|b\rangle = \mathcal{H}^{\circ} Z(Qx_1, \tilde{q} Q^{-1} x_2)|b\rangle,$$

where

$$\mathcal{H}^{\circ} = (T_1 \sqrt{q\tilde{q}} x_1; q^2)^{-1} (T_3 \sqrt{q\tilde{q}} \Lambda / x_1; q^2)^{-1} (T_2 \sqrt{q\tilde{q}} x_1; q^2)^{-1} (T_4 \sqrt{q\tilde{q}} \Lambda / x_1; q^2)^{-1} \cdot (Q^{-1} \widehat{x_1}; q^2)^{-1} (q\Lambda / \widehat{x_1}; q^2)^{-1} (T_1 T_2 \widehat{x_1}; q^2)^{-1} (T_3 T_4 \Lambda / \widehat{x_1}; q^2)^{-1}.$$

So the statement of the last Theorem is nothing but

$$\mathcal{H}Z = Z.$$

The identification of other parameters are:

$$QT_1^2 = a_1^4 a_4^4 = \frac{w_1^{21}}{w_1^{10}} \frac{w_1^{22}}{w_1^{11}}, \quad \tilde{q}QT_3^2 = a_0^{-4} a_5^{-4} = \frac{w_0^{10}}{w_0^{21}} \frac{w_0^{11}}{w_0^{22}},$$

$$QT_2^2 = a_1^{-4} a_4^4 = \frac{w_1^{10}}{w_1^{21}} \frac{w_1^{22}}{w_1^{11}}, \quad \tilde{q}QT_4^2 = a_0^4 a_5^{-4} = \frac{w_0^{21}}{w_0^{10}} \frac{w_0^{11}}{w_0^{22}}.$$

Thanks for listening!

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