

C. Simpson Hodge filtration on nonabelian cohomology (note by Yamazawa)

X : smooth projective variety / \mathbb{C}

M_{DR} = moduli space of (E, ∇) ,

$\left\{ \begin{array}{l} E: \text{vector bundle of rank } r \text{ on } X, \\ \nabla: E \rightarrow E \otimes_{O_X} \Omega^1_X \text{ flat holomorphic connection} \end{array} \right.$

Moduli functor.

$$M_{DR}(S) = \left\{ \begin{array}{l} (E, \nabla) \text{ on } X \times S/S, \\ \nabla: E \rightarrow E \otimes_{O_{X \times S}} \Omega^1_{X \times S/S}, \quad \nabla^2 = 0, \\ \nabla(ae) = a\nabla(e) + e \otimes da \end{array} \right\}$$

Artin stack.

$M_{DR} \rightarrow M_{DR}$ coarse moduli space

$$M_{DR} = H^1_{DR}(X, GL(r))$$

$$\begin{aligned} M_B &= H^1_B(X, GL(r)) = \{ \text{local systems of rank } r \text{ on } X \} \\ &= \text{Hom}(\pi_1(X, *), GL(r)) // \text{conj} \end{aligned}$$

$M_B \rightarrow M_B$ coarse moduli space

Riemann-Hilbert correspondence

$$M_{DR}^{\text{an}} \simeq M_B^{\text{an}}$$

$$\downarrow$$

$$M_{DR}^{\text{an}} \simeq M_B^{\text{an}}$$

Usual abelian cohomology:

$$H^1_{DR}(X, \mathbb{C}) \simeq H^1_B(X, \mathbb{C})$$

Hodge filtration. If S is a scheme and $\lambda \in \Gamma(S, \mathcal{O}_S)$

a λ -connection on $X \times S/S$ is (E, ∇) with

- E : vector bundle of rank r ,
- $\nabla : E \rightarrow E \otimes \Omega^1_{X \times S/S}$ s.t. $\nabla(ae) = a\nabla(e) + \lambda e \otimes da$,
 $(e \in \Gamma(U, E), a \in \Gamma(U, \mathcal{O}_{X \times S}))$

If $\lambda = 0$, ∇ is a Higgs field.

$$\mathcal{M}_{\lambda\text{-conn.}}(S) = \left\{ (\lambda, E, \nabla) ; \begin{array}{l} \lambda \in \mathcal{O}_S(S), \\ (E, \nabla) : \lambda\text{-connection on } X \times S/S \end{array} \right\} / \simeq$$

$$\begin{array}{ccc} \mathcal{M}_{\text{Higgs bundles}} & \hookrightarrow & \mathcal{M}_{\lambda\text{-conn.}} \hookrightarrow \mathcal{M}_{\text{DR}} \\ \downarrow & \downarrow \lambda & \downarrow \\ \{\lambda=0\} & \hookrightarrow & A^1 \hookleftarrow \{\lambda=1\} \end{array} \quad \text{This stack doesn't have a good presentation.}$$

It has an action of G_m $(t \in G_m, (\lambda, E, \nabla)) \mapsto (t\lambda, E, t\nabla)$

compatible with the usual action on A^1 .

Semistability. $H \in \text{Pic}(X)$.

A Higgs bundle (E, θ) is H -semistable

if \forall sub-Higgs sheaves $F \subset E$, $\theta(F) \subset F \otimes \Omega_X^1$, $0 < r(F) < r$,

$$\frac{c_1(F) \cdot H^{n-1}}{r(F)} \leq \frac{c_1(E) \cdot H^{n-1}}{r(E)}$$

($\begin{matrix} < \\ \text{stable} \end{matrix}$)

If $c_i(E) = 0$ in $H^{2i}(X, \mathbb{Q})$ for all i ,

then this condition is independent of the choice of H

Hitchin moduli stack

$M_H = \{ \text{semistable Higgs bundles } , c_i = 0 \text{ in } H^{2i}(X, \mathbb{Q}) \} / \sim$

$$M_{\text{Had}}(S) := \left\{ (\lambda, E, \nabla) ; \lambda \in C_S(S), (E, \nabla) : \lambda\text{-connection s.t.} \begin{array}{l} \forall s \in S \text{ with } \lambda(s) = 0, \\ (E_s, \nabla_s) \text{ is semistable with } c_i = 0. \end{array} \right\} / \sim$$

open

$M_{\lambda\text{-conn}}$

M_{Had} has good properties.

It has a coarse moduli space M_{Had} separated, quasi-projective
which parametrizes S -equivalence classes.

$$(M_H \rightarrow M_H) \hookrightarrow (M_{\text{Had}} \rightarrow M_{\text{Had}}) \hookleftarrow (M_{\text{DR}} \rightarrow M_{\text{DR}})$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow p \quad \downarrow \quad \downarrow$$

$$\{\lambda=0\} \hookrightarrow A^1 \hookleftarrow \{\lambda=1\}$$

\mathbb{G}_{m} acts on M_{Had} over the action on A^1 ,

$$\text{on } M_H \quad (t \cdot (E, \theta) := (E, t\theta))$$

$$t : p^{-1}(1) \xrightarrow{\sim} p^{-1}(t) \quad \forall t \in \mathbb{G}_{\text{m}} \subset A^1$$

Basic property : $\forall y = (\lambda, E, \nabla) \in M_{\text{Hod}}$,

$\exists!$ limit point $\lim_{t \rightarrow 0} t \cdot y \in M_H$ in the coarse moduli space.

$$\begin{array}{ccc}
 & \mathbb{G}_m & \\
 & \curvearrowright & \\
 A_0^n & \hookrightarrow & A^{n+1} \hookrightarrow A_1^n \\
 \downarrow & \downarrow z_0 & \downarrow \\
 0 \in A^1 & \rightarrow 1 & \\
 & \mathbb{A}_0^n \setminus \{0\} / \mathbb{G}_m & \uparrow \\
 & \mathbb{P}^n & \\
 & \curvearrowleft & \\
 & A_1^n \setminus \{0\} / \mathbb{G}_m &
 \end{array}$$

all points y such that $\lim_{t \rightarrow \infty} t \cdot y$ exists.

In our case, Hitchin map : $M_H \rightarrow A^n \supset \mathbb{G}_m$ (weighted)

$(E, \theta) \mapsto$ (characteristic polynomial of θ)

$$N = \{ y \in M_H \text{ s.t. } \lim_{t \rightarrow \infty} t \cdot y \text{ exists} \} = \{ y \mapsto 0 \in A^n \}$$

Hitchin
map

$$= \{ (E, \theta) ; \theta : \text{nilpotent} \}$$

"nilpotent cone"

Compactification of M_{DR}

$$(M_{\text{Hod}} \setminus N) / \mathbb{G}_m \xleftarrow[\text{open}]{} M_{\text{DR}} \quad \text{separated, proper scheme}$$

complementary divisor $D = (M_H \setminus N) / \mathbb{G}_m$.

? is it projective?

X_{DR} : a functor on Sch/\mathbb{C}

(↑
scheme of finite type.)

$X_{\text{DR}}(Y) := X(Y^{\text{red}})$, $Y^{\text{red}} \subset Y$ reduced subscheme.

an étale sheaf.

$X_{\text{DR}} = (X/\mathcal{F}_{\text{DR}})^{\text{et-sh.}}$, $\mathcal{F}_{\text{DR}} \subset X \times X$

a formal scheme

\mathcal{F}_{DR} = formal neighborhood of the diagonal

formal groupoid $(X, \mathcal{F}_{\text{DR}}) : \text{Sch} \rightarrow \text{Gpd}$.

$M_{\text{DR}} = H^1_{\text{DR}}(X, \text{GL}(n)) = H^1(X_{\text{DR}}, \text{GL}(n))$

$(E, \nabla) \longleftrightarrow E \text{ on } X + \text{descent data w.r.t. action of } \mathcal{F}_{\text{DR}}$
 $\nabla^2 = 0$

$$\begin{array}{ccc} \mathcal{F}_{\text{Dol}} & (X \times \mathbb{A}^1, \mathcal{F}_{\text{Hod}}) & \mathcal{F}_{\text{DR}} \\ \downarrow & \downarrow & \downarrow \\ 0 & \in & \mathbb{A}^1 \end{array} \Rightarrow \begin{array}{c} 1 \end{array}$$

$M_{\lambda\text{-conn.}} = \text{Hom}((X \times \mathbb{A}^1, \mathcal{F}_{\text{Hod}})/\mathbb{A}^1, \text{BGL}(n))$

This allows us to define a Hodge filtration on higher nonabelian cohomology:

Replace $\text{BGL}(n)$ by an n -stack T with enough good properties.

$$H^*(X_{\text{DR}}, T) \hookrightarrow H^*(X_{\text{Hod}}, T) = \underline{\text{Hom}}((X \times \mathbb{A}^1, \mathcal{F}_{\text{Hod}})/\mathbb{A}^1, T) \hookleftarrow H^*(X_{\text{DR}}, T)$$

$$\begin{array}{ccccccc} \downarrow & & & & & & \downarrow \\ 0 & \in & \mathbb{A}^1 & \Rightarrow & 1 & & 1 \end{array}$$

"Hodge filtration" on M_{DR} :

$$\left[M_{Hod} \supseteq \mathbb{G}_m, \quad p^*(1) \simeq M_{DR} \right]$$
$$\downarrow p$$
$$A^1$$

Y

$$\left[FY \supseteq \mathbb{G}_m, \quad p^*(1) \simeq Y \right]$$
$$\downarrow p$$
$$A^1$$

Simpson

Twistor space

X : smooth projective

$$M_{Hod}(X) = \{ (\lambda, E, \nabla); \lambda \in A^1, (E, \nabla) : \lambda\text{-connection, semistable, } c_i = 0 \}$$

$$\begin{matrix} \downarrow \lambda \\ A^1 \end{matrix}$$

Deligne gluing $M_{Hod}(X)|_{G_m} \simeq M_{DR}(X) \times G_m \xrightarrow[RH]{an} M_B(X) \times G_m$

\bar{X} : complex conjugate $X^{top} \simeq \bar{X}^{top} \xrightarrow{\exists \lambda \mapsto \lambda^{-1}}$

$$M_{Hod}(\bar{X})|_{G_m} \simeq M_{DR}(\bar{X}) \times G_m \xrightarrow[RH]{an} M_B(\bar{X}) \times G_m$$

Using this analytic isomorphism, glue together the 2 pieces $M_{Hod}(X), M_{Hod}(\bar{X})$, to get

$$\begin{array}{ccc} M_{Hod}(X) & \xleftarrow{\quad} & M_{Hod}(\bar{X}) \\ \downarrow & & \downarrow \\ A^1_0 & \hookrightarrow & \mathbb{P}^1 \hookleftarrow A^1_{\infty} \end{array} \quad \text{complex analytic space}$$

$M_{HD}(X)$ is twistor space for the hyperkähler structure on $M_H(X)$, constructed by Hitchin, Fujiki.

This construction due to Deligne

If $(E, \bar{\partial}_E, \theta, h)$ is a harmonic bundle, we get a "preferred section":

$$\begin{array}{ccc} s_{(E, \bar{\partial}_E, \theta, h)}: \mathbb{P}^1 & \rightarrow & M_{HD}(X) \\ & \searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

In the neighborhood $A_0^1 \parallel \bar{Q} + \lambda Q_E + \lambda \bar{Q}_E + \theta_E$

$$\mu = \lambda^{-1} \quad A_0^1 \ni \lambda \mapsto (E, D^\lambda) \in M_{Hod}(X)_\lambda$$

$$A_0^1 \ni \mu \mapsto (\bar{E}, \bar{D}^\mu) \in M_{Hod}(\bar{X})_\mu$$

preserved by an antipodal involution σ

$$M_{HD} \rightarrow M_{HD}$$

$$\downarrow \qquad \downarrow$$

$$\mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$x \mapsto -\bar{x}^{-1}$$

Idea: $\{\sigma\text{-invariant sections}\} \xrightarrow[\cong]{ev_\lambda} M_{Hod}(X)_\lambda$

Conj.: globally probably not true, locally OK at smooth pts.

$(\{ \text{preferred sections} \} \simeq \{ \text{harmonic bds} \} \xrightarrow[\cong]{ev_\lambda} M_{Hod}(X)_\lambda)$
 choosing h
 \cap bigger
 $\{ \sigma\text{-invariant sections} \}$

locally near a preferred section :

basic point : the normal bundle to the preferred section

$$S^k(T(M_{HD}/\mathbb{P}^1)) \simeq \underbrace{(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(1))}_{d=\dim M_{DR}} \otimes \sigma$$

$$T(\text{sections}) = H^0(\mathbb{P}^1, N) = (\mathbb{C}^2)^d \otimes \sigma$$

$$T(\sigma\text{-invariant sections}) = ((\mathbb{C}^2)^d)^\sigma = (\mathbb{R}^2)^d \xrightarrow[\cong]{ev_\lambda} \mathbb{C}^d = N_\lambda$$

\downarrow
 hyperkähler structure on M_H .

$$TM_{Hod}(X)_\lambda = N_X \underset{\text{real}}{\simeq} T(\sigma\text{-inv. sections}) \underset{\text{real}}{\simeq} N_\lambda = TM_{Hod}(X)_\lambda$$

$$TM_H(X) \quad (\lambda=0)$$

$$TM_{DR}(X) \quad (\lambda=1)$$

We get a whole \mathbb{P}^1 of complex structures on any one $TM_{H(X)}$

Calculate \Rightarrow quaternionic structure

$$I, J, K = \{0, 1, i\}$$

$$M_{HD}^{\text{reg}} \xrightarrow{C^\infty} M_H^{\text{reg}} \times \mathbb{P}^1 \quad \text{twistor space}$$

Through any point of M_{HD} , there is a unique preferred section,
locally unique σ -invariant section.

$$(g_{\mathbb{P}^1}(1))^d \leftrightarrow \begin{array}{l} \text{weight 1 twistor structure} \\ \text{gen. wt. 1 Hodge structure} \end{array}$$

The Rees construction. If V : vector bundle, F^\cdot : decreasing filtration,

$$\sum x^p F^p \subset V \otimes \mathbb{C}[[x, x^{-1}]] \quad \text{locally free subbundle over } \mathbb{C}[[x]].$$

$\text{Rees}(V, F) = \text{vector bundle} + \mathbb{G}_m\text{-action}$.

↓

$$A^1 = \text{Spec } \mathbb{C}[[x]].$$

$$\text{Rees}(V, F)_0 = \text{Gr}_{F^0}(V) \quad \text{decomposition} \leftrightarrow \mathbb{C}^* \text{-action}$$

$$\text{Rees}(V, F)_1 = V$$

$$\left(\begin{array}{c} V \supset \mathbb{G}_m \\ \downarrow \\ A^1 \end{array} \right) \rightsquigarrow F^\cdot \text{ on } V = \mathcal{V}_1 \quad \text{Gr}_{F^0}(V) = \mathcal{V}_0$$

compatible with M_{Hod} construction via the Hodge filtration

$X_{\text{DR}} = (X, \mathcal{F}_{\text{DR}})$ formal groupoid $\leftrightarrow \mathcal{D}_X$ sheaf of rings of diff. ops.

filtered by powers of the ideal

$\mathcal{F}_r \mathcal{D}_X$: operators of order $\leq r$

$$\text{Rees}(\mathcal{F}_{\text{DR}}, \mathcal{F}^*) = \mathcal{F}^*_{\text{Hod}}$$

$$R = \text{Rees}(\mathcal{D}_X, \mathcal{F})$$

G_m -invariant sections $A^1 \rightarrow M_{\text{Hod}}$

$$\begin{array}{ccc} & & \leftarrow \\ & \Downarrow & \downarrow \\ & & A_0^1 \end{array}$$

$$\xleftarrow{\text{Rees}} (V, \nabla, \mathcal{F}^*)$$

(V, ∇) : vector bdl with connection

\mathcal{F}^* : filtration s.t.

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_X^1$$

"Griffiths transversality"

$$\text{Gr}_{\mathcal{F}}(V) = \bigoplus_p \mathcal{F}^p / \mathcal{F}^{p+1}$$

$$\theta = \nabla : \mathcal{F}^p / \mathcal{F}^{p+1} \rightarrow \mathcal{F}^{p-1} / \mathcal{F}^p \otimes \Omega_X^1$$

$$\text{Rees}(V, \mathcal{F}^*, \nabla)_{\lambda=1} = (V, \nabla)$$

$$\text{Rees}(V, \mathcal{F}^*, \nabla)_{\lambda=0} = (\text{Gr}_{\mathcal{F}}(V), \theta)$$

Cond. $(\text{Gr}_{\mathcal{F}}(V), \theta)$ should be semistable as a Higgs bundle

"partial oper"

an oper is the case $\mathcal{F}^p / \mathcal{F}^{p+1}$ are line bdl

Thm. ($\dim X = 1$) $\cdot V(V, \nabla)$, there exists a partial oper str. $(V, \nabla, \mathcal{F}^*)$.

$\cdot (\text{Gr}_{\mathcal{F}}(V), \theta)$ is unique as a point in M_H .

$\cdot \mathcal{F}^*$ is unique up to translation of indices "gr-stable"

$\Leftrightarrow (\text{Gr}_{\mathcal{F}}(V), \theta)$ is a stable Higgs bundle.

Idea V : stable $\Rightarrow F$: trivial, ($E = V, \theta = 0$)

Otherwise, $H \subset V$ destabilizing subbundle

$$\rightsquigarrow F^* = (0, H, V) \quad E_1 = H \oplus \overset{\circ}{V}H$$

if semistable as a Higgs bundle $\Rightarrow E = E_1$ ok.

If not, $H_1 \subset (E_1, \theta)$ destabilizing sub-Higgs bundle

$$\rightsquigarrow \text{modify the filtration} \quad 0 \subset H_1^0 \subset H_1^1 \subset V$$

$$\rightsquigarrow E_2$$

if E_2 semistable, ok

if not,

Simpson

X : curve,

$$M_H^{(\text{reg})} \hookrightarrow G_m \quad \text{Fixed point set } P = \coprod_{\alpha} P_{\alpha}$$

$(E, \theta) \in P_{\alpha}$. Then G_m acts on E .

$$\Rightarrow E = \bigoplus_P E^P \quad \theta : E^P \rightarrow E^{P+1} \otimes \Omega_X^1$$

\uparrow
(t.e. = $t^{\pm P} e$)

These are the Higgs bundles which correspond to VHS.

$$\text{VHS. } (V, \nabla, F, h) \Rightarrow E = Gr_F(V) = \bigoplus_P \underbrace{F^P / F^{P+1}}_{E^P}$$

$$\theta = \nabla : F^P / F^{P+1} \rightarrow F^{P-1} / F^P \otimes \Omega_X^1$$

$$P = \{ \text{VHS} \} \quad P_{\alpha} \Rightarrow \begin{cases} r^P = \text{rk}(E^P), \\ d^P = \deg(E^P) \end{cases} \} \text{ combinatorial invariants of } \alpha.$$

$$ch(E^P) \in CH(X)$$

2 different types of sections of

$$M_{\text{Hod}} \subset M_{\text{HD}}$$

$$\begin{matrix} \downarrow & \downarrow \\ A^1 & \subset \mathbb{P}^1 \end{matrix}$$

- preferred sections $\mathbb{P}^1 \rightarrow M_{\text{HD}} \leftrightarrow \text{harmonic bundles}$
- G_m -invariant sections $A^1 \rightarrow M_{\text{Hod}} \leftrightarrow \text{partial oper} (V, \nabla, F)$

VHS = sections which are of both kinds.

Suppose : (V, ∇, F) is a partial oper , (V, ∇) : irreducible

$$T(M_{DR})_{(V,\nabla)} = H^1(\text{End}(V) \otimes \Omega_X, d_V) \quad F^* \text{ induces a filtration}$$

on $\text{End}(V)$ Griffiths transverse w.r.t. $\nabla_{\text{End}(V)}$ and hence on $\text{End}(V) \otimes \Omega_X$.

\rightsquigarrow filtered complex $(\text{End}(V) \otimes \Omega_X, d_V, F^*)$

\rightsquigarrow filtration on the hypercohomology $F^* H^i(\text{End}(V) \otimes \Omega_X, d_V)$
compatible with ν .

limit point construction. $y = (V, \nabla) \in M_{DR}$, $\lim_{t \rightarrow 0} t \cdot y \in P \subset M_H$

fixed by G_m .

Let $G_\alpha = \{ y = (V, \nabla) \in M_{DR} \mid \lim_{t \rightarrow 0} t \cdot y \in P_\alpha \}$

$$M_{DR} = \coprod_{\alpha \in \Pi_0(P)} G_\alpha$$

$$p \in P, \quad L_p := \{ y \in M_{DR} \mid \lim_{t \rightarrow 0} t \cdot y = p \} \quad G_\alpha = \coprod_{p \in P_\alpha} L_p.$$

G_α and L_p are locally closed subvarieties, smooth if $M_{DR} = M_{DR}^{\text{reg}}$, $M_H = M_H^{\text{reg}}$

(caution: one can have $(V, \nabla) \in M_{DR}^{\text{reg}}$ but $\lim_{t \rightarrow 0} t \cdot (V, \nabla) \notin M_H^{\text{sing}}$.)

There is $0 \in \{\alpha\}$, $\dim P_0 = \dim M_{DR}/2$.

X : compact curve, $P_0 = \{ (E, \theta=0) \text{ semistable bundles} \}$

$G_0 = \{ (V, \nabla) \text{ s.t. } V \text{ is semistable as a bundle} \}$

$$(F^0 = V, F^1 = 0)$$

Lemma. at a point $y = (V, \nabla, F)$ of M_{DR} ,

partial oper

$y \in G_\alpha$, $y \in L_p$, $\phi = (E, \theta)$ where $E = \text{Gr}_F(V)$, $\theta = "D"$

Then,

$$\begin{cases} F^0 H^1(\text{End}(V) \otimes \Omega_X^\cdot) = T(G_\alpha)_y \\ F^1 H^1(\text{End}(V) \otimes \Omega_X^\cdot) = T(L_p) \end{cases}$$

Cor. $T(L_p)$ is isotropic for the natural symplectic form.

(assuming (V, ∇) irreducible)

proof.

$$F^1 H^2(\text{End}(V) \otimes \Omega_X^\cdot) = H^2(-) = \mathbb{C}, \quad F^2 H^2(-) = 0.$$

$$v : F^1 H^1(-) \times F^1 H^1(-) \rightarrow F^2 H^2(-) = 0,$$

$$\begin{matrix} & \parallel \\ & T(L_p) \end{matrix}$$

$(E, \theta) \leftrightarrow$ a VHS

Spectral sequence for H^1 starts with

$$H^1(\underbrace{\text{Gr}_F^P(\text{End}(V) \otimes \Omega_X^\cdot)}_{\parallel}) = E_1^{P, 1-P} \Rightarrow \text{Gr}_F^P H^1.$$

$$\text{Gr}_F^P(\text{End}(E) \otimes \Omega_X^\cdot) = \text{spectral sequence for } y'.$$

$$\text{For } y' \text{ we can see that } E_1^{P, -P}, E_1^{P, 2-P} = \begin{cases} 0 & P \neq 0, 1 \\ \mathbb{C} & P = 0, 1 \end{cases}.$$

\Rightarrow degenerates for y .

Cor. X : curve. If $p \in P^{\text{reg}} \subset M_H^{\text{reg}}$ then L_p is Lagrangian in M_{DR} .

Conj. The $\{L_p\}$ forms a filtration.

Probably : Szabol's picture yesterday is this foliation for \mathbb{P}^1 -4pts, rk 2.

In general, don't know if the L_p are closed for example.

at $y = \text{a VHS } \rightarrow FPH^1$ gives a Hodge structure on $T(M_{\text{DR}})_y$ of weight 1.

This extends to a MHS on $\widehat{\mathcal{O}}_{M_{\text{DR}}, y}$.

Non compact curve.

$$X \subset \overline{X} \text{ open curve}, \quad X = \overline{X} - D$$

example : $X = \mathbb{G}_m$, $\overline{X} = \mathbb{P}^1$ bundles of rank 1

$H^1(X; \mathbb{R}) = \mathbb{R}(-1)$. Tate Hodge structure of type (1,1), weight 2.

explain the extra real parameter of the parabolic structure in this way.

$$\{ \lambda\text{-flat bundles} \} = \mathbb{C} \xrightarrow{\text{Deligne gluing}} M_{\text{HD}} = \text{line bdl.}$$

$$(O_X, d + a \frac{dz}{z}) \mapsto \begin{matrix} \text{residue at } 0 \\ = a \end{matrix} \quad \downarrow \quad \mathbb{P}^1 \quad O_{\mathbb{P}^1}(2)$$

$$\Gamma(O_{\mathbb{P}^1}(2)) = \mathbb{C}^3$$

$$\Gamma(O_{\mathbb{P}^1}(2))^{\mathbb{R}} = \mathbb{R}^3 \leftarrow \begin{matrix} \text{down} \\ \mathbb{C} \end{matrix} \quad \begin{matrix} \text{parabolic weight parameter} \\ \downarrow ev_\lambda \end{matrix}$$

Simpson Orbifold curves.

X : smooth projective curve , $D = \{p_1, \dots, p_n\} \subset X$ divisor .

$$U = X - D, \quad m_1, \dots, m_n$$

$Z = X[\frac{p_1}{m_1}, \dots, \frac{p_n}{m_n}]$ smooth DM-stack with ramification of index m_i at p_i .

↓ ramified covering at p_i .

X

$$\pi_1(Z) = \pi_1(U) / \langle y_i^{m_i} = 1 \rangle, \quad y_i : \text{loop around } p_i .$$

$\Rightarrow \rho : \pi_1(Z) \rightarrow GL(r)$ same as a map $\rho : \pi_1(U) \rightarrow GL(r)$

s.t. $C_i = \text{conjugacy class of } \rho(y_i)$, has order m_i

\Leftrightarrow semisimple with eigenvalues in $\mu_{m_i} \subset \mathbb{C}^*$.

$$M_B(Z; GL(r)) = \coprod_{(C_1, \dots, C_n)} M_B(U; C_1, \dots, C_n)$$

in almost all cases ,

\exists Galois covering $\mathbb{Z}' \xrightarrow{G} Z$ s.t. \mathbb{Z}' is smooth projective

$$\left(\begin{array}{c} \text{bundles on } Z \\ + \dots \end{array} \right) = \left(\begin{array}{c} \text{G-equivariant bundles on } \mathbb{Z}' \\ + \dots \end{array} \right)$$

$$\left(\begin{array}{c} \text{parabolic bundles on } (X, D) \\ \text{and parabolic weights at } p_i \text{ in } \frac{1}{m_i} \mathbb{Z} \\ + \dots \end{array} \right)$$

relationship with compact case

\Rightarrow we get the stratification + leaves $M_{DR}(Z) = \coprod G_\alpha = \coprod \mathcal{L}_p$.

new behavior. $X = \mathbb{P}^1$, $p_1, \dots, p_n \in X$.

$C_1, \dots, C_n \subset \mathrm{GL}(r)$ conjugacy classes of $\rho(\gamma_i)$

\rightarrow KMS-spectrum data : res(V) + parabolic weight
for each Jordan block.

Higgs ($\lambda=0$)

$$C_i = \begin{pmatrix} \cdot & e^{2\pi i \tau \alpha_j i} \\ \cdot & \cdot \end{pmatrix}$$

C_i determines a filtration type + weights.

Usually, the biggest stratum G_0 corresponds to

$$P_0 = \{(E, 0) \mid E \text{ is a semistable bundle}\}$$

$$G_0 = \{(V, \nabla) \mid V: \text{semistable}\}$$

for $X = \mathbb{P}^1$ and for some choices of parabolic weights,

$$\{\text{semistable parabolic bundles}\} = \emptyset$$

In this case the biggest stratum $G_0(C_1, \dots, C_n)$ corresponds to

$P_0(C_1, \dots, C_n) \ni$ Higgs bundles corresponding to some VHS

unitary representations with the given conjugacy classes

Q : What are the n^p and d^p of the Higgs bundles for these VHS?

Classification questions on \mathbb{P}^1

Given C_1, \dots, C_n , is $M_B(C_1, \dots, C_n) = \emptyset$?

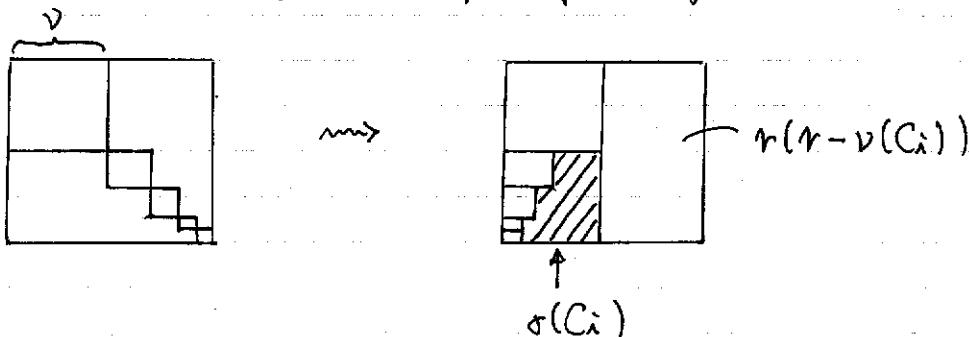
Katz : rigid case ($\dim M_B = 0$)

Kostov, Crawley-Boevey, Shaw : complete (?) answer

technique basically uses Katz's middle convolutions

$$\begin{aligned}\dim M_B(C_1, \dots, C_n) &= \sum \dim C_i - 2(r^2 - 1) \\ &= r^2 - r \sum_i v(C_i) - \sum_i s(C_i),\end{aligned}$$

where $v(C_i)$ = highest multiplicity of eigenvalues



$$\dim M_B(C_1, \dots, C_n) = 2 + r \delta(C_1, \dots, C_n) + \sum_i s_i$$

$$\text{where } \delta(C_1, \dots, C_n) = (n-2)r - \sum_i v(C_i)$$

Middle convolution

$$M_B(C_1, \dots, C_n) \rightarrow M_B(C'_1, \dots, C'_n)$$

pull back \rightarrow tensor with rank 1 \rightarrow push forward
canonical subsystem

$$r' = r + \delta(C_1, \dots, C_n)$$

Kostov "Index of rigidity zero" $\Leftrightarrow \dim M_B = 2$.

classification $\delta = 0, \tau = 0$

4 cases : $\begin{pmatrix} d & d \\ d & d \end{pmatrix} \begin{pmatrix} d & d \\ d & d \end{pmatrix} \begin{pmatrix} d & d \\ d & d \end{pmatrix} \begin{pmatrix} d & d \\ d & d \end{pmatrix}$ 4 pts

$\begin{pmatrix} d & d \\ d & d \end{pmatrix} \begin{pmatrix} d & d \\ d & d \end{pmatrix} \begin{pmatrix} d & d \\ d & d \end{pmatrix}$ 3 pts

$\begin{pmatrix} d & d \\ d & d \\ d & d \end{pmatrix} \begin{pmatrix} 2d & 2d \\ 2d & 2d \end{pmatrix} \begin{pmatrix} 2d & 2d \\ 2d & 2d \end{pmatrix}$

$\begin{pmatrix} d & d & d \\ d & d & d \\ d & d & d \end{pmatrix} \begin{pmatrix} 2d & 2d \\ 2d & 2d \\ 2d & 2d \end{pmatrix} \begin{pmatrix} 3d & 3d \\ 3d & 3d \end{pmatrix}$

Boalch 7 other wild cases.

Boalch - Etingof - Oblomkov - Rains (arXiv:math/0406480v3)

$\mu^d = 1$ prim. \Rightarrow Kostov's generic

Conj. $M_B(C_i^{(d)}) \simeq M_B(C_i^{(1)})$