Integrable structure in
melting crystal model of 5D gauge theory

joint work with Toshio Nakatsu

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1. Melting crystal model
2. Fermionic representation of partition function
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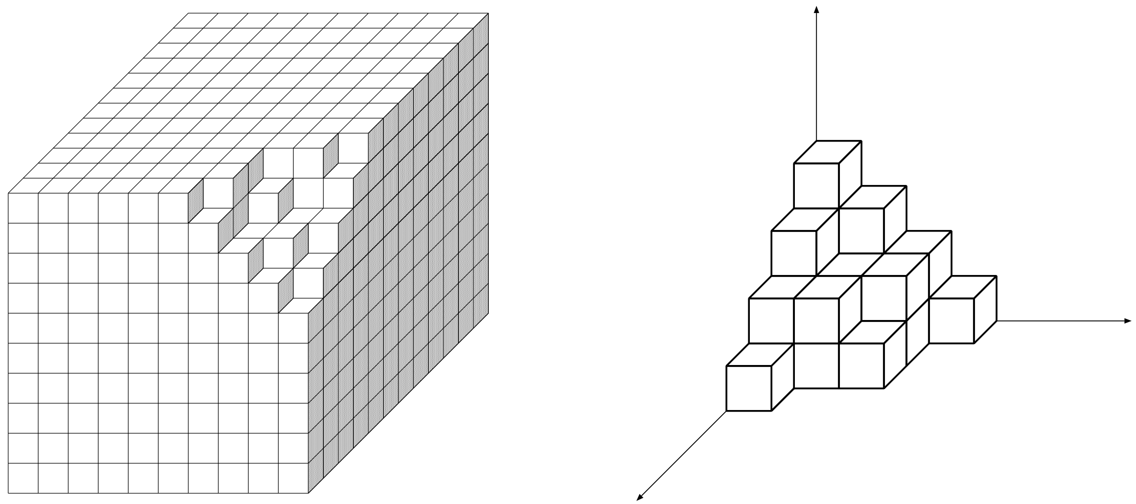
Ref: T. Nakatsu and K.T., arXiv:0710.5339 [hep-th]

1. Melting crystal model

melting crystal corner = random plane partition



Okounkov, Reshetikhin & Vafa, "Quantum Calabi-Yau and classical crystal", hep-th/0309208 ordinary partition = Young diagram

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_i \geq \lambda_{i+1}$, $\lambda_i \in \mathbf{Z}_{\geq 0}$ (length of i-th row). $|\lambda| = \sum_i \lambda_i$ (area).

plane partition = 3D Young diagram

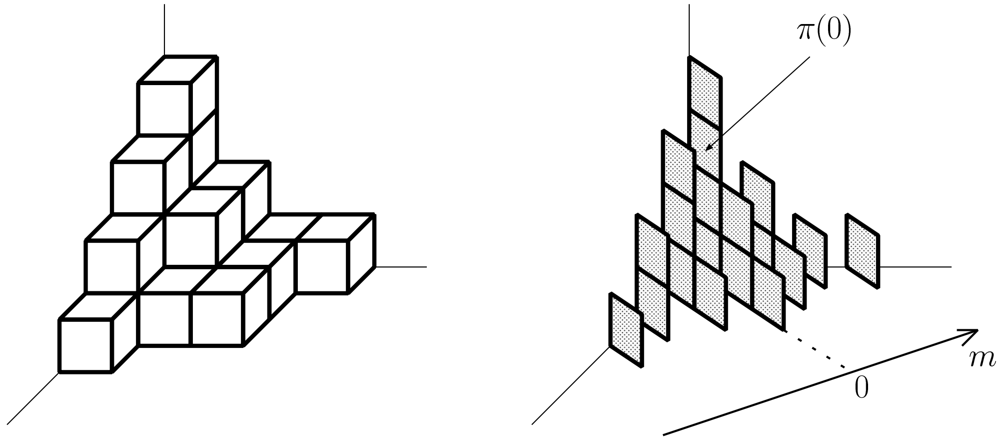
$\pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$, $\pi_{ij} \geq \pi_{i,j+1}$, $\pi_{ij} \geq \pi_{i+1,j}$, $\pi_{ij} \in \mathbf{Z}_{\geq 0}$ (height of (i,j)-th stack). $|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij}$ (volume).

partition function of random plane partition

$$Z = \sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} (1 - q^n)^{-n} \text{ (McMahon function)}, \ 0 < q < 1$$

diagonal slices of plane partition (Okounkov & Reshetikhin)

The diagonal slices $\{\pi(m)\}_{m=-\infty}^{\infty}$ of the plane partition π is a sequnce of Young diagrams that satisfy "interlacing relations"
$\cdots \preceq \pi(-2) \preceq \pi(-1) \preceq \pi(0) \succeq \pi(1) \succeq \pi(2) \succeq \cdots.$



(a) (b)

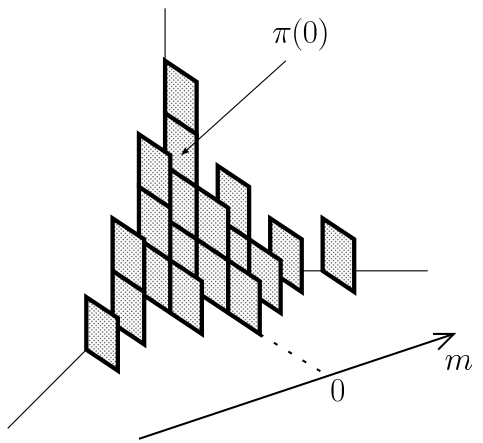
interlacing relation:

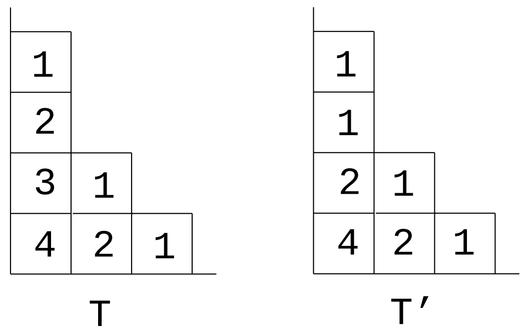
$$\lambda = (\lambda_1, \lambda_2, \ldots) \succeq \mu = (\mu_1, \mu_2, \ldots) \overset{\text{def}}{\Longleftrightarrow} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

plane partition $\pi \mapsto$ pair (T, T') of semi-standard tableaux

The plane partition π determines a pair (T, T') of semi-standard tableaux of shape $\lambda = \pi(0)$ by putting "$m+1$" in boxes of the skew diagram $\pi(\pm m)/\pi(\pm(m+1))$.

T: $\lambda = \pi(0) \succeq \pi(-1) \succeq \pi(-2) \succeq \cdots$
T': $\lambda = \pi(0) \succeq \pi(1) \succeq \pi(2) \succeq \cdots$





partition function as sum over semi-standard tableaux

By the mapping $\pi \mapsto (T, T')$, the partition function $Z = \sum_{\pi} q^{|\pi|}$ can be converted to a sum over T, T' and their shape λ:

$$Z = \sum_{\lambda} \sum_{T,T':\text{shape}\,\lambda} q^T q^{T'}$$

The weights are determined by entries of the tableaux:

$$q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|},$$

$$q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}$$

partition function in terms of Schur functions

The partial sums over the semi-standard tableaux T, T' give a special value of the Shur function:

$$\sum_{T:\text{shape}\,\lambda} q^{T} = \sum_{T':\text{shape}\,\lambda} q^{T'} = s_\lambda(q^\rho), \quad \rho = (\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots)$$

The partition function can be thus rewritten as

$$Z = \sum_{\lambda} s_\lambda(q^\rho)^2$$

Remark: Hook formula for $s_\lambda(q^\rho)$

$$s_\lambda(q^\rho) = q^{n(\lambda)+|\lambda|/2} \prod_{(i,j)\in\lambda} (1 - q^{h(i,j)})^{-1}, \quad n(\lambda) = \sum_{i=1}^{\infty} (i-1)\lambda_i$$

deformation by potential $\Phi(t, \lambda, p)$

We consider a deformed model

$$Z_p(t) = \sum_\lambda s_\lambda(q^\rho)^2 e^{\Phi(t,\lambda,p)}, \quad \Phi(t, \lambda, p) = \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, p)$$

with potentials

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} q^{k(p+\lambda_i-i+1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}$$

The right hand side of this definition of $\Phi_k(\lambda, p)$ is understood to be a finite sum (hence a rational function of q) by cancellation of terms between the two sums:

$$\Phi_k(\lambda, p) = \sum_{i=1}^{\infty} (q^{k(p+\lambda_i-i+1)} - q^{k(p-i+1)}) + q^k \frac{1-q^{pk}}{1-q^k}$$

melting crysital model and 5D SUSY gauge theory

Melting crystal model with external potential:

$$Z_p(t) = \sum_{\pi} q^{|\pi|} e^{\Phi(t,\pi(0),p)} = \sum_{\lambda} s_\lambda(q^\rho)^2 q^{\Phi(t,\lambda,p)}$$

5D $\mathcal{N} = 1$ SUSY U(1) gauge theory:

$$Z_p(t) = \sum_{\pi} q^{|\pi|} Q^{\pi(0)} e^{\Phi(t,\pi(0),p)} = \sum_{\lambda} s_\lambda(q^\rho)^2 Q^{|\lambda|} q^{\Phi(t,\lambda,p)},$$

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2$$

(5D analogue of Nekrasov's 4D instanton sum)

Goal: Show that 1D Toda hierarchy is a common integrable structure in these models.

2. Fermionic representation of partition function

complex fermion system

$\psi(z) = \sum_{m=-\infty}^{\infty} \psi_m z^{-m-1}$, $\psi^*(z) = \sum_{m=-\infty}^{\infty} \psi_m^* z^{-m}$ with anti-commutation relations

$$\{\psi_m, \psi_n^*\} = \delta_{m+n,0}, \quad \{\psi_m, \psi_n\} = \{\psi_m^*, \psi_n^*\} = 0$$

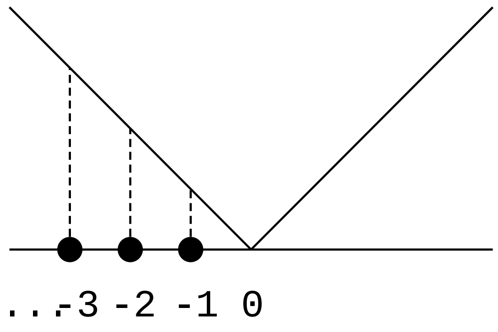
Ground state (Fermi sea) $|p\rangle$ in charge p sector

$$\psi_m|p\rangle = 0 \quad \text{for } m \geq -p, \quad \psi_m^*|p\rangle = 0 \quad \text{for } m \geq p+1$$

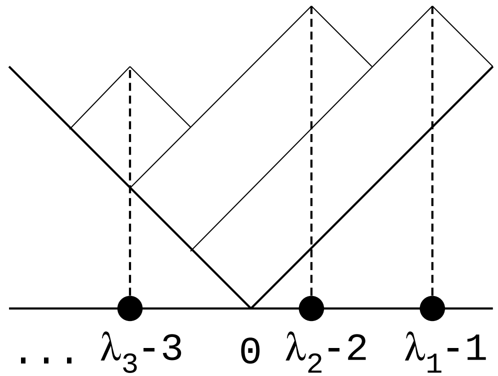
Fock space spannded by states labelled by partitions (or Young diagrams)

$$F = \bigoplus_{p=-\infty}^{\infty} F_p, \quad F_p = \bigoplus_{\lambda} \mathbf{C}|\lambda; p\rangle$$

States labelled by Young diagrams (charge 0 sector)



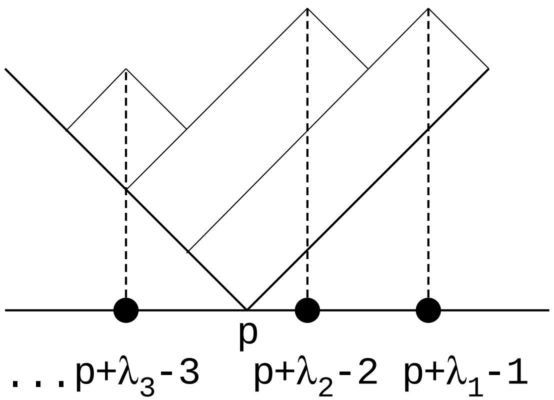
$$\emptyset = (0, 0, \ldots), \text{ charge } 0 \mapsto |\emptyset; 0\rangle$$



$$\lambda = (\lambda_1, \lambda_2, \ldots), \text{ charge } 0 \mapsto |\lambda; 0\rangle$$

$$\lambda = (\lambda_1, \lambda_2, \ldots) \mapsto \{\lambda_i - i\}_{i=1}^{\infty} \subset \mathbf{Z} \text{ (Maya diagram)}$$

States labelled by Young diagrams (charge p sector)



$$\lambda = (\lambda_1, \lambda_2, \ldots), \text{ charge } p \mapsto |\lambda; p\rangle$$

$(\lambda, p) \mapsto \{p + \lambda_i - i\}_{i=1}^{\infty} \subset \mathbf{Z}$ (Maya diagram of charge p)

If $\lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots)$,

$$|\lambda; p\rangle = \psi_{-(p+\lambda_1-1)-1} \cdots \psi_{-(p+\lambda_n-n)-1} \psi^*_{-(p-n)+1} \cdots \psi^*_{-(p-1)+1} |p\rangle$$

U(1) current and fermionic representation of tau function

$J(z) = {:}\psi(z)\psi^*(z){:} = \sum_{k=-\infty}^{\infty} J_m z^{-m-1}, \ J_m = \sum_{n=-\infty}^{\infty} {:}\psi_{m-n}\psi_n^*{:}$ with

commutation relations

$$[J_m, J_n] = m\delta_{m+n,0} \quad \text{(Heisenberg algebra)}$$

J_m's play the role of "Hamiltonians" in the usual fermionic formula of tau functions of the KP and (2D) Toda hierarchies:

$$\tau_p(t,\bar{t}) = \langle p| \exp(\sum_{m=1}^{\infty} t_m J_m) g \exp(-\sum_{m=1}^{\infty} \bar{t}_m J_{-m})|p\rangle, \ g \in \mathrm{GL}(\infty)$$

Hamiltonians for fermionic representation of $Z_p(t)$

$$H_k = \sum_{n=-\infty}^{\infty} q^{kn} {:}\psi_{-n}\psi_n^*{:}$$

The states $|\lambda; p\rangle$ are eigenvectors of these "Hamiltonians" and the potential functions $\Phi_k(\lambda, p)$ are their eigenvalues:

$$H_k|\lambda; p\rangle = \Phi_k(\lambda, p)|\lambda; p\rangle$$

ferminonic representation of $Z_p(t)$

$$Z_p(t) = \langle p|G_+ e^{H(t)} G_-|p\rangle$$

where

$$H(t) = \sum_{k=1}^{\infty} t_k H_k, \quad G_\pm = \exp\Big(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k}\Big)$$

$G_\pm$ generate random plane partition (Okounkov & Reshetikhin)

$G_\pm$ are a product of vertex operators $\Gamma_\pm(m)$:

$$G_+ = \prod_{m=-\infty}^{-1} \Gamma_+(m), \quad G_- = \prod_{m=0}^{\infty} \Gamma_-(m),$$
$$\Gamma_\pm(m) = \exp\Big(\sum_{k=1}^{\infty} \frac{1}{k} q^{\mp k(m+1/2)} J_{\pm k}\Big)$$

They generate a "half" of random plane partition π:

$$\langle p|G_+ = \sum_\lambda \sum_{T:\mathrm{shape}\,\lambda} q^T \langle \lambda; p| = \sum_\lambda s_\lambda(q^\rho)\langle \lambda; p|,$$
$$G_-|p\rangle = \sum_\lambda \sum_{T:\mathrm{shape}\,\lambda} q^T |\lambda; p\rangle = \sum_\lambda s_\lambda(q^\rho)|\lambda; p\rangle$$

Consequently, $\langle p|G_+ e^{H(t)} G_-|p\rangle = \sum_\lambda s_\lambda(q^\rho)^2 e^{\Phi(t,\lambda,p)} = Z_p(t)$.

3. Quantum torus Lie algebra

basis $V_m^{(k)}$ $(k = 0, 1, \dots,\ m \in \mathbf{Z})$

$$\begin{aligned} V_m^{(k)} &= q^{-km/2} \sum_{n=-\infty}^{\infty} q^{kn} {:}\psi_{m-n}\psi_n^*{:} \\ &= q^{k/2} \oint \frac{dz}{2\pi i} z^m {:}\psi(q^{k/2}z)\psi^*(q^{-k/2}z){:} \end{aligned}$$

Remark: $J_m = V_m^{(0)}$, $H_k = V_0^{(k)}$. $V_m^{(k)}$ coincides with Okounkov and Pandharipande's operator $\mathcal{E}_m(z)$ specialized to $z = q^k$.

commutation relations

$$[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})(V_{m+n}^{(k+l)} - \delta_{m+n,0}\frac{q^{k+l}}{1-q^{k+l}})$$

Remark: This is a (central extension of) q-deformation of the Poisson algebra of functions on a 2-torus.

adjoint action by $G_\pm$ (1)

Fermion fields $\psi(z), \psi^*(z)$ transform as

$$\begin{aligned}
G_+\psi(z){G_+}^{-1} &= (q^{1/2}z;q)_\infty^{-1}\psi(z), \\
G_+\psi^*(z){G_+}^{-1} &= (q^{1/2}z;q)_\infty\psi^*(z), \\
G_-\psi(z){G_-}^{-1} &= (q^{1/2}z^{-1};q)_\infty\psi(z), \\
G_-\psi^*(z){G_-}^{-1} &= (q^{1/2}z^{-1};q)_\infty^{-1}\psi^*(z)
\end{aligned}$$

where $(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n)$.

adjoint action by $G_\pm$ (2)

The forgoing formulae for fermion fields imply that the fermion bilinear $\psi^*(q^{-k/2}z)\psi(q^{k/2}z)$ transforms as

$$
\begin{aligned}
& G_+\psi^*(q^{-k/2}z)\psi(q^{k/2}z){G_+}^{-1} \\
= \; & \frac{(q^{1/2}\cdot q^{-k/2}z;q)_\infty}{(q^{1/2}\cdot q^{k/2}z;q)_\infty}\psi^*(q^{-k/2}z)\psi(q^{k/2}z) \\
= \; & \prod_{m=1}^{k}(1-zq^{(k+1)/2-m})\psi^*(q^{-k/2}z)\psi(q^{k/2}z)
\end{aligned}
$$

A similar transformation law holds for the adjoint action by G_- as well.

shift symmetry among $V_m^{(k)}$'s

From the foregoing formulae, one can deduce the following symmetry among the basis of the quantum torus Lie algebra:

$$G_-G_+\left(V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k}\right)(G_-G_+)^{-1} = (-1)^k\left(V_{m+k}^{(k)} - \delta_{m+k,0}\frac{q^k}{1-q^k}\right)$$

In particular,

$$G_-G_+\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)(G_-G_+)^{-1} = (-1)^k V_k^{(k)},$$

$$(G_-G_+)^{-1}\left(V_0^{(k)} - \frac{q^k}{1-q^k}\right)G_-G_+ = (-1)^k V_{-k}^{(k)}$$

This is a key to identification of the integrable structure.

4. Integrable structure

rewriting partition function of melting crystal model (1)

$$Z_p(t) = \langle p|G_+ e^{H(t)} G_-|p\rangle$$

Split $G_+ e^{H(t)} G_-$ into several pieces as

$$\begin{aligned} G_+ e^{H(t)} G_- &= G_+ e^{H(t)/2} e^{H(t)/2} G_- \\ &= G_+ e^{H(t)/2} {G_+}^{-1} \cdot G_+ G_- \cdot {G_-}^{-1} e^{H(t)/2} G_- \end{aligned}$$

and use the formulae (a special case of shift symmetry)

$$G_- G_+ \Big(H_k - \frac{q^k}{1-q^k} \Big) (G_- G_+)^{-1} = (-1)^k V_k^{(k)},$$

$$(G_- G_+)^{-1} \Big(H_k - \frac{q^k}{1-q^k} \Big) G_- G_+ = (-1)^k V_{-k}^{(k)}$$

rewriting partition function of melting crystal model (2)

The forgoing formulae imply that

$$\begin{aligned} G_+\Big(H_k - \frac{q^k}{1-q^k}\Big){G_+}^{-1} &= (-1)^k {G_-}^{-1} V_k^{(k)} G_-, \\ {G_-}^{-1}\Big(H_k - \frac{q^k}{1-q^k}\Big)G_- &= (-1)^k G_+ V_{-k}^{(k)} {G_+}^{-1} \end{aligned}$$

$V_{\pm k}^{(k)}$ on the right hand side can be transformed to $J_{\pm k}$ as

$$q^{W/2} V_k^{(k)} q^{-W/2} = V_k^{(0)} = J_k, \quad q^{-W/2} V_{-k}^{(k)} q^{W/2} = V_{-k}^{(0)} = J_{-k}$$

where W is a special element of W_∞ algebra:

$$W = W_0^{(3)} = \sum_{n=-\infty}^{\infty} n^2 {:}\psi_{-n}\psi_n^*{:}$$

rewriting partition function of melting crystal model (3)

Thus we have the relation

$$
\begin{aligned}
G_+\Bigl(H_k - \frac{q^k}{1-q^k}\Bigr){G_+}^{-1} &= (-1)^k {G_-}^{-1} q^{-W/2} J_k q^{W/2} G_-, \\
{G_-}^{-1}\Bigl(H_k - \frac{q^k}{1-q^k}\Bigr)G_- &= (-1)^k G_+ q^{W/2} J_{-k} q^{-W/2} {G_+}^{-1}
\end{aligned}
$$

hence

$$
\begin{aligned}
& G_+ e^{H(t)/2} {G_+}^{-1} \\
= \; & \exp\Bigl(\sum_{k=1}^{\infty} \frac{t_k q^k}{2(1-q^k)}\Bigr) {G_-}^{-1} q^{-W/2} \exp\Bigl(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k\Bigr) q^{W/2} G_-,
\end{aligned}
$$

and a similar expression for ${G_-}^{-1} e^{H(t)/2} G_-$.

rewriting partition function of melting crystal model (4)

We can thus eventually rewrite $G_+ e^{H(t)} G_-$ as

$$G_+ e^{H(t)} G_- = \exp\Big(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k}\Big) {G_-}^{-1} q^{-W/2} \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k\Big) \times \\ \times\, g \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_{-k}\Big) q^{-W/2} {G_+}^{-1}$$

where

$$g = q^{W/2} (G_- G_+)^2 q^{W/2} \in \mathrm{GL}(\infty)$$

rewriting partition function of melting crystal model (5)

Since $\langle p|{G_-}^{-1}q^{-W/2} = q^{-p(p+1)(2p+1)/12}\langle p|$ and $q^{-W/2}{G_+}^{-1}|p\rangle = q^{-p(p+1)(2p+1)/12}|p\rangle$, the partition function $Z_p(t)$ can be expressed as

$$\begin{aligned} Z_p(t) &= \exp\Bigl(\sum_{k=1}^{\infty}\frac{t_k q^k}{1-q^k}\Bigr)q^{-p(p+1)(2p+1)/6}\times \\ &\qquad \times \langle p|\exp\Bigl(\sum_{k=1}^{\infty}\frac{(-1)^k t_k}{2}J_k\Bigr)g\exp\Bigl(\sum_{k=1}^{\infty}\frac{(-1)^k t_k}{2}J_{-k}\Bigr)|p\rangle \end{aligned}$$

The last piece $\langle p|\cdots|p\rangle$ may be interpreted as a special value of the tau function

$$\tau_p(t,\bar{t}) = \langle p|\exp(\sum_{k=1}^{\infty}t_k J_k)g\exp(-\sum_{k=1}^{\infty}\bar{t}_k J_{-k})|p\rangle$$

of 2D Toda hierarchy. However, this is not the end of the story.

identities of expectation values

Actually, we can start from different splitting of $G_+ e^{H(t)} G_-$ as well:

$$G_+ e^{H(t)} G_- = G_+ e^{H(t)} {G_+}^{-1} \cdot G_+ G_- = G_+ G_- \cdot {G_-}^{-1} e^{H(t)} G_-$$

This leads to apparently different expressions of $Z_p(t)$, which imply that the following identities hold:

$$\begin{aligned}
& \langle p| \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_k\Big) g \exp\Big(\sum_{k=1}^{\infty} \frac{(-1)^k t_k}{2} J_{-k}\Big)|p\rangle \\
= \ & \langle p| \exp\Big(\sum_{k=1}^{\infty} (-1)^k t_k J_k\Big) g |p\rangle \\
= \ & \langle p| g \exp\Big(\sum_{k=1}^{\infty} (-1)^k t_k J_{-k}\Big)|p\rangle
\end{aligned}$$

What do they mean?

$g = q^{W/2}(G_-G_+)^2 q^{W/2}$ determines solution of 1D Toda hierarchy

The foregoing identities can be directly derived from the relations

$$J_k g = g J_{-k},\ k = 1, 2, 3, \ldots$$

(a consequence of the shift symmetry of $V_m^{(k)}$'s). From these relations one can derive the identities

$$\tau_p(t, \bar{t}) = \tau_p(t - \bar{t}, 0) = \tau_p(0, \bar{t} - t)$$

for the tau function $\tau_p(t, \bar{t})$ of 2D Toda hierarchy, which thereby reduces to a tau function of 1D Toda hierarchy. Thus 1D Toda hierarchy turns out to be an underlying integrable structure of the partition function $Z_p(t)$ of the melting crystal model.

integrable structure in 5D SUSY U(1) gauge theory

$Z_p(t)$ has a fermionic representation of the form

$$Z_p(t) = \langle p|G_+ Q^{L_0} e^{H(t)} G_- |p\rangle$$

where $L_0 = \sum_{n=-\infty}^{\infty} n{:}\psi_{-n}\psi_n^*{:}$ (element of Virasoro algebra). The foregoing calculations can be repeated for this case as well and lead to a similar conclusion. The counterpart of g is given by

$$g = q^{W/2} G_- G_+ Q^{L_0} G_- G_+ q^{W/2}$$

and satisfies the relation

$$J_k g = g J_{-k},\ k = 1, 2, 3, \ldots$$

Thus a relevant integrable structure is again 1D Toda hierarchy.

Concluding remarks

4D limit ($R \to 0$) (cf. Marshakov and Nekrasov's work on 4D case)
Not straightforward

relation to topological strings
1. Another interpretation of $\langle p|G_+ Q^{L_0} e^{H(t)} G_- |p\rangle$ ($q = e^{-g_{\mathrm{st}}}$, $Q = e^{-a}$) as A-model amplitude on $\mathcal{O} \oplus \mathcal{O}(-2) \to \mathbf{CP^1}$
2. Generating function of $W_{\lambda\mu} \sim c_{\lambda\mu\bullet}$ as solution of 2D Toda hierarchy with $g = q^{W/2} G_+ G_- q^{W/2}$ (Zhou)

thermodynamic limit (rescaling t_k's and letting $\hbar \to 0$ in $q = e^{-R\hbar}$)
Dispersionless Toda hierarchy? (work in progress)

more relations satisfied by g Constraints with quantum/classical torus algebraic structure? (work in progress)