

Quivers, Superpotentials,
Dimer Models

Jan Stienstra

Two-variable hypergeometric systems
and dessins d'enfants

reference

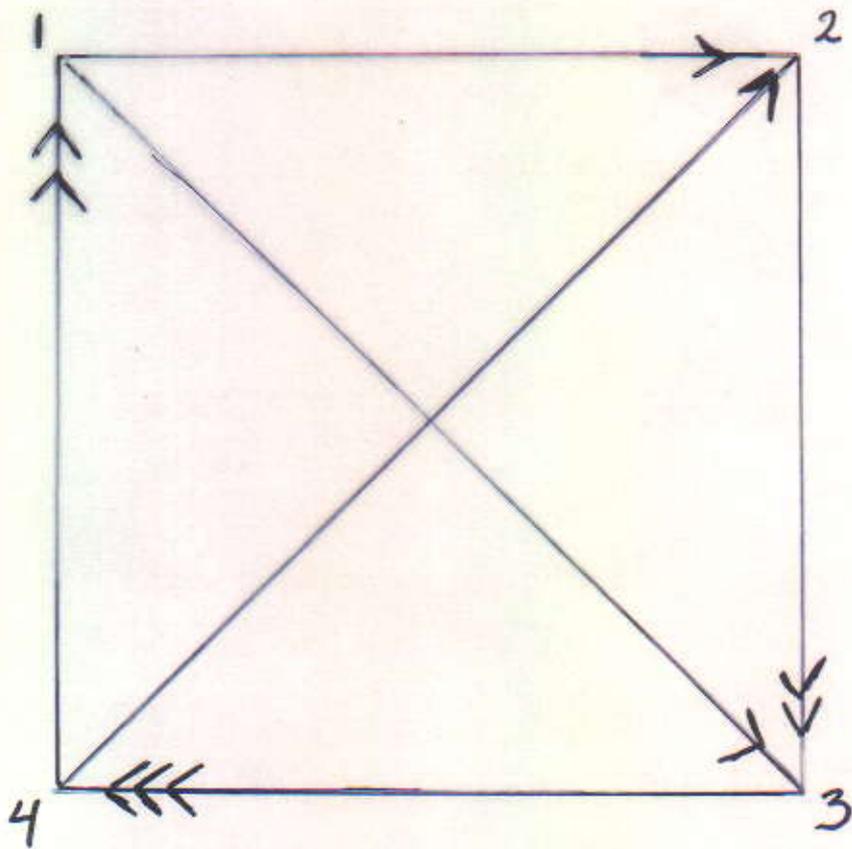
Hypergeometric Systems in two Variables,
Quivers, Dimers and Dessins d'Enfant.

arXiv: 0711.0464

Quiver = directed graph

vertices = nodes

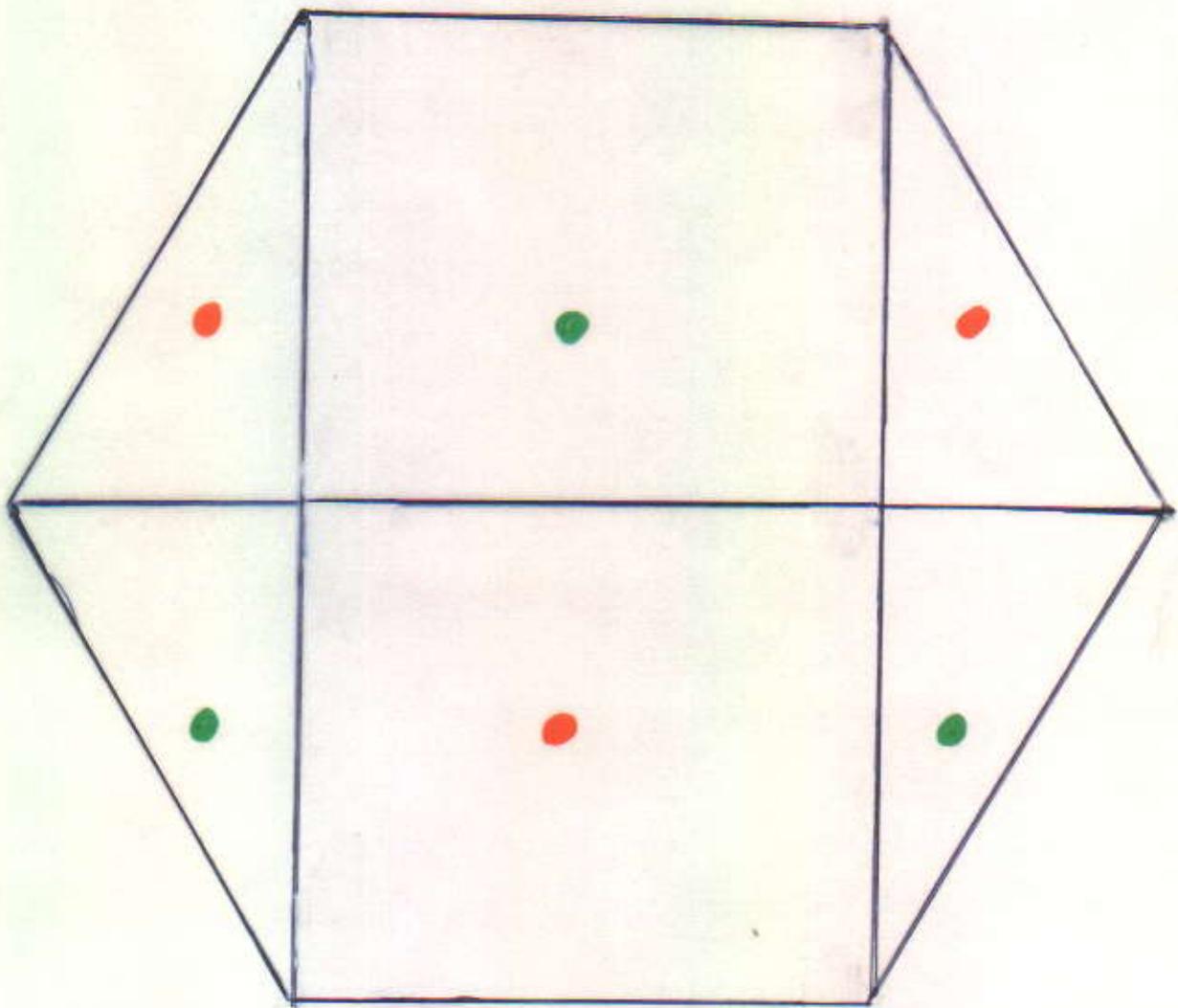
arrows = oriented edges



3^A

Question (geometric formulation)

Embed Quiver into oriented
surface without boundary



opposite edges of hexagon identified
→ quiver embedded in torus

Quiver embedded in oriented surface
without boundary
=
dessin d'enfants

Belyi's theorem

dessin d'enfants:

surface = complex points of an algebraic
curve C defined over a
number field

+ morphism $\varphi: C \rightarrow \mathbb{P}^1$
ramified only over $0, 1, \infty$

s.t. quiver = φ^{-1} (unit circle)

Superpotential from embedding

is list of oriented boundaries of 2-cells
written as a polynomial in
non-commuting variables

$$\begin{aligned}
 &+ X_1 X_4 X_7 X_5 + X_2 X_9 X_6 + X_3 X_8 X_{10} \\
 &- X_1 X_9 X_7 X_{10} - X_3 X_4 X_6 - X_2 X_5 X_8
 \end{aligned}$$

Dimer configuration

(or perfect matching)

is a subset P of the set of
arrows such that for every 2-cell
exactly one edge in the boundary
belongs to P

Bi-adjacency matrix

Every edge connects two vertices and separates two 2-cells (one red cell, one green cell)

Since the number of red cells
 = // green //

one can conveniently represent all information of the embedded quiver as a matrix:

rows \longleftrightarrow red cells

columns \longleftrightarrow green cells

edge ε connecting vertices a, b
 separating cells i , j

matrix entry $X_{\varepsilon} U_a U_b$

in row i , column j

$$\begin{bmatrix} X_6 u_1 u_3 & X_3 u_3 u_4 & X_4 u_1 u_4 \\ X_2 u_3 u_4 & X_8 u_2 u_4 & X_5 u_2 u_3 \\ X_9 u_1 u_4 & X_{10} u_2 u_3 & X_1 u_3 u_4 + X_7 u_1 u_2 \end{bmatrix}$$

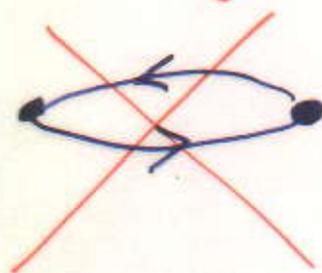
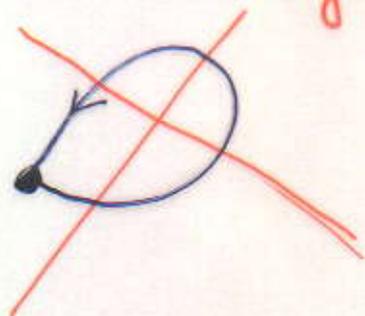
Theorem (\approx Kasteleyn)
 dimer configurations
 \longleftrightarrow

(monomial) terms in

the determinant of
 the bi-adjacency matrix

Assume Quiver satisfies:

- connected
- no oriented cycles of length ≤ 2



- in every node:
 $\# \text{ incoming arrows} = \# \text{ outgoing arrows}$
- anti-symmetric adjacency matrix
 has rank 2

rows and columns of \quad correspond
 with vertices of quiver

$$(i, j)\text{-entry} = \# \text{ arrows } i \rightarrow j \\ - \# \text{ arrows } j \rightarrow i$$

For such a quiver
the anti-symmetric adjacency matrix
is of the form

$$B^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B$$

where B is a $2 \times N$ -matrix
with columns $b_1, \dots, b_N \in \mathbb{Z}^2$

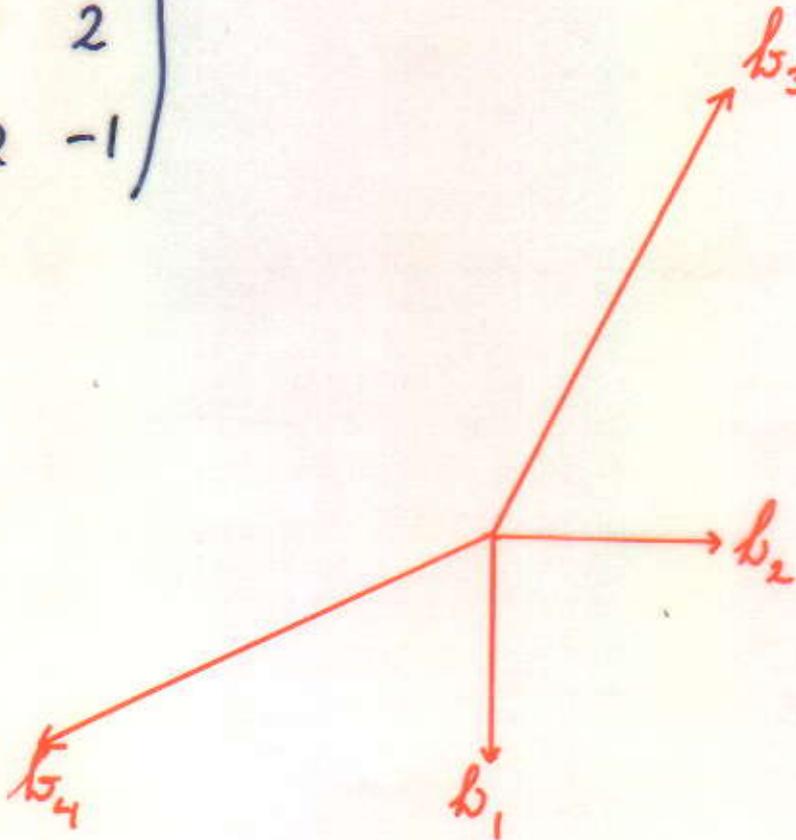
such that

- $b_i \neq 0 \quad \forall i$
- $b_1 + \dots + b_N = 0$
- b_1, \dots, b_N span \mathbb{Z}^2 (over \mathbb{Z})

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$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 2 & -1 \\ -1 & -2 & 0 & 3 \\ 2 & 1 & -3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 2 & -1 \end{pmatrix}$$



Gelfand - Kapranov - Zelevinsky's

A-hypergeometric systems

$$A = \{\alpha_1, \dots, \alpha_N\} \subset \mathbb{Z}^{k+1}$$

such that

- A generates \mathbb{Z}^{k+1}
- \exists linear $h: \mathbb{Z}^{k+1} \rightarrow \mathbb{Z}$
s.t. $h(\alpha_j) = 1 \quad \forall j$

$$\mathbb{L} = \{(l_1, \dots, l_N) \in \mathbb{Z}^N \mid l_1 \alpha_1 + \dots + l_N \alpha_N = 0\}$$

our present interest: rank $\mathbb{L} = 2$

\mathbb{Z} -basis for $\mathbb{L} \iff$

$$\mathbb{L} = \mathbb{Z}^2 B$$

with B $2 \times N$ -matrix over \mathbb{Z}
columns of B : b_1, \dots, b_N

Secondary fan

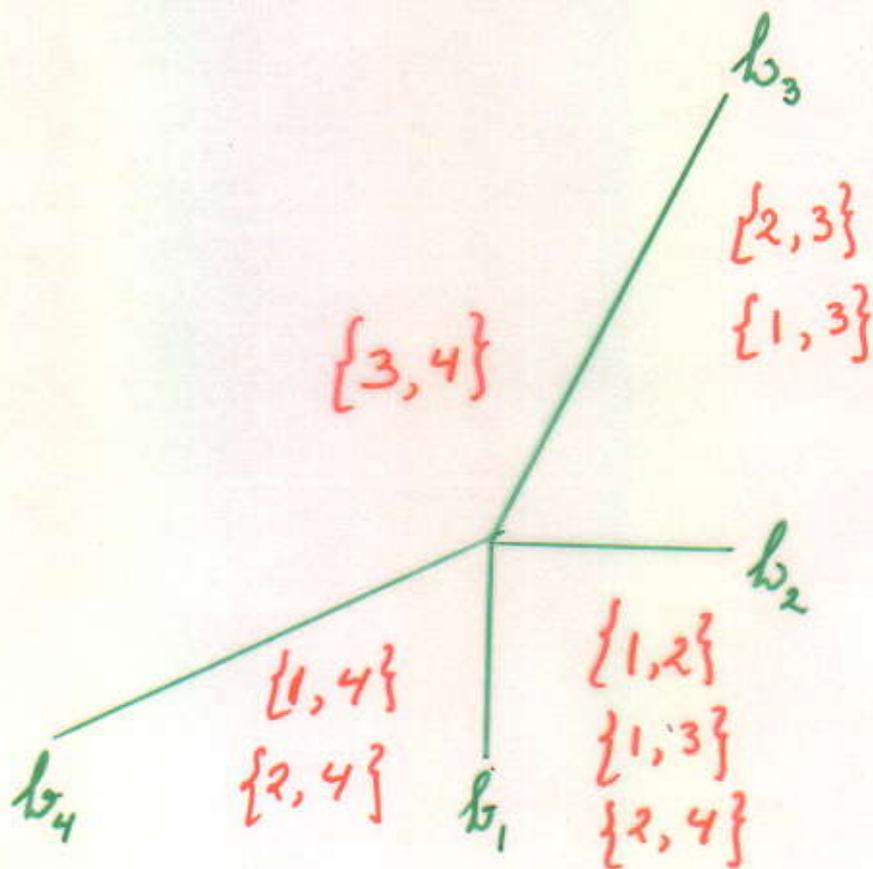
 in \mathbb{R}^2

1-dim. cones of sec. fan $\mathbb{R}_{\geq 0} b_i$

$$i = 1, \dots, N$$

for every 2-dim. cone of sec. fan C

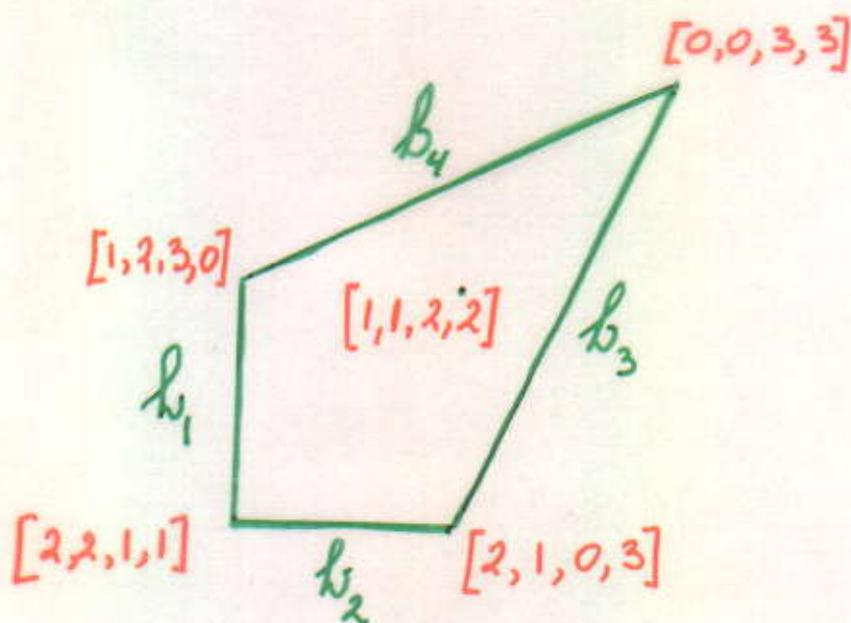
set $L_C = \{(i, j) \mid C = \mathbb{R}_{\geq 0} b_i + \mathbb{R}_{\geq 0} b_j\}$



Theorem: $\forall C$ arrows with endpoints $\{(i, j) \in L_C\}$ form a dimer configuration

Secondary polygon

= convex polygon obtained by putting b_1, \dots, b_N in appropriate order head to tail

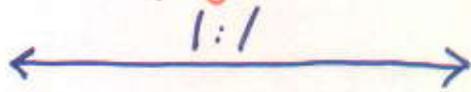


Theorem:

secondary polygon

= Newton polygon (w.r.t u_1, \dots, u_N)
of determinant
of bi-adjacency matrix

dimer configurations



monomials in det. bi-adj. matrix



lattice points in secondary polygon

for every vertex of secondary polygon
unique dimer configuration

if one omits from the quiver
the arrows in a dimer configuration
on the boundary of secondary
polygon,
then the resulting quiver has
no oriented cycles

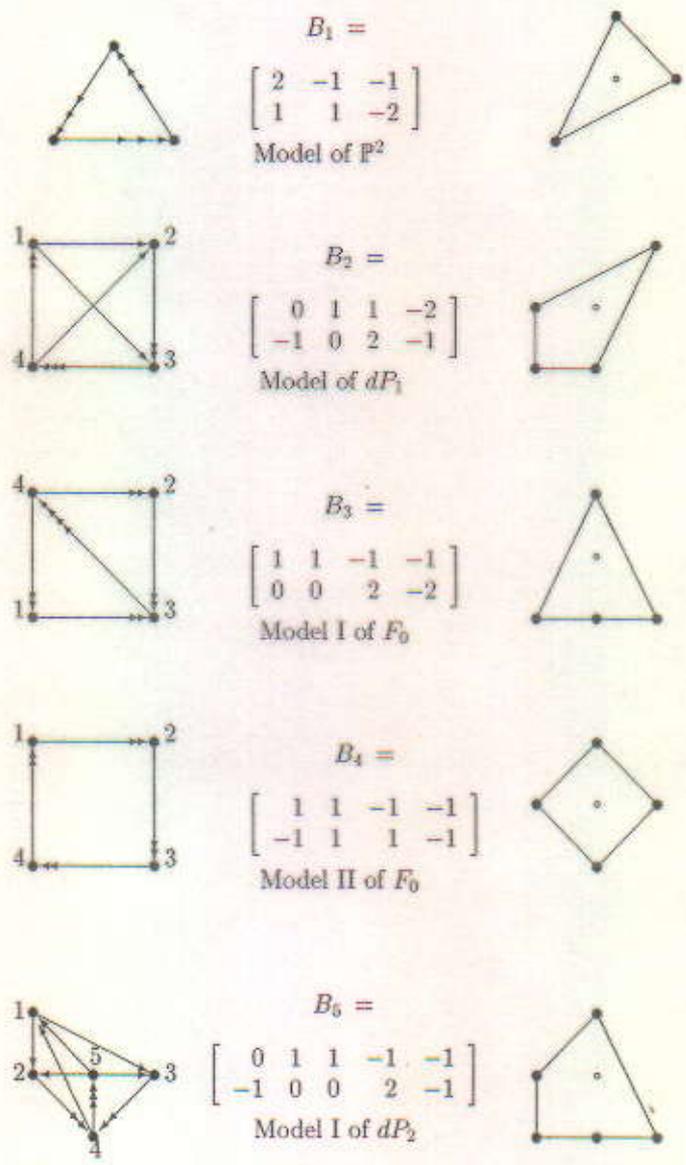


Figure 3: Quivers from [3] Figures 10, 11, 4, 12 and corresponding polygons.

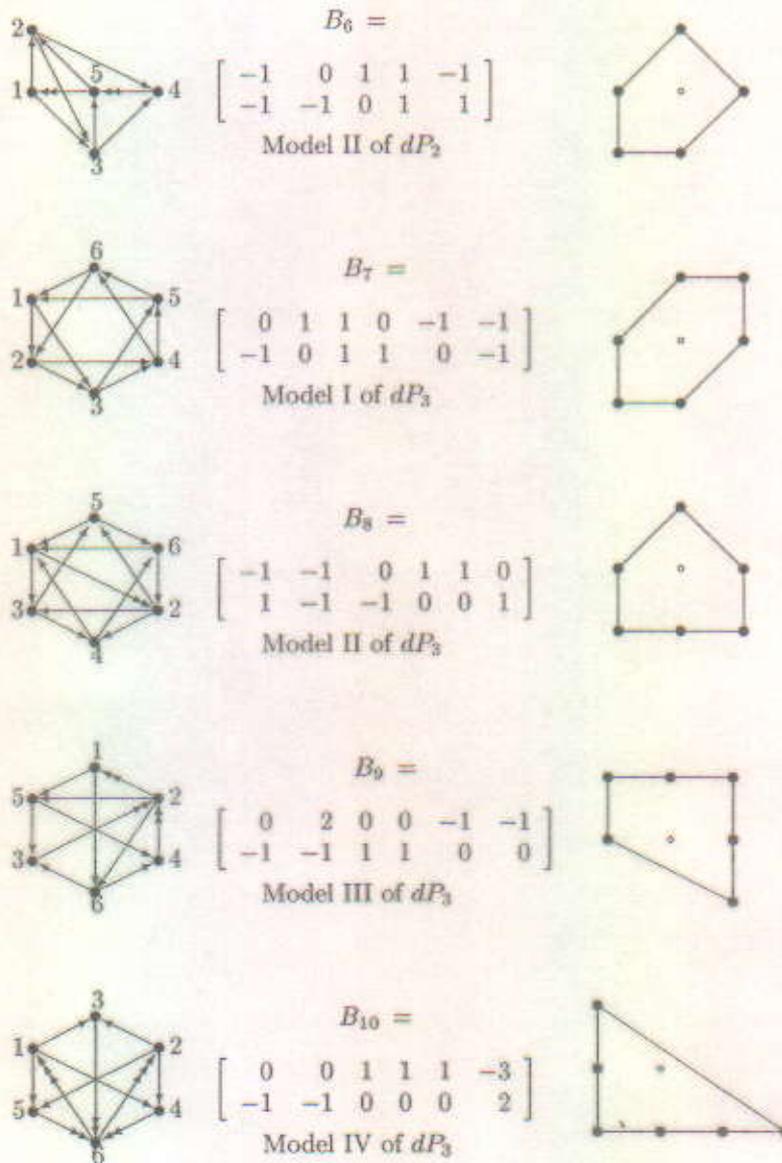


Figure 4: Quivers from [3] Figures 12, 9 and corresponding polygons.

Observation about

Principal A-determinant

recall bi-adjacency matrix

rows \leftrightarrow red cells

columns \leftrightarrow green cells

edge ε connecting vertices a, b
separating cells i , j

i, j matrix entry $X_{\varepsilon} u_a u_b$

substitute: $X_{\varepsilon} = \#$ edges with same
endpoints as ε

let

$D(u_1, \dots, u_N) =$

det bi-adj. matrix after substitution

Then (in examples)

$$(u_1 \cdots u_N)^{\# \text{red cells}} \mathbb{D}(u_1^{-1}, \dots, u_N^{-1})$$

equals

principal A -determinant

↑
describes singularities of GKZ system

Example

$$B = \begin{bmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} u_1 u_3 & 3 u_3 u_4 & 2 u_1 u_4 \\ 3 u_3 u_4 & u_2 u_4 & 2 u_2 u_3 \\ 2 u_1 u_4 & 2 u_2 u_3 & 3 u_3 u_4 + u_1 u_2 \end{bmatrix}$$

=

$$27 u_3^3 u_4^3 + 4 u_1 u_2^2 u_3^3 + 4 u_1^2 u_2 u_4^3 \\ - 18 u_1 u_2 u_3^2 u_4^2 - u_1^2 u_2^2 u_3 u_4$$

discriminant of $(u_1 + u_3 T + u_4 T^2 + u_2 T^3)$

=

$$27 u_1^2 u_2^2 + 4 u_1 u_4^3 + 4 u_2 u_3^3 \\ - 18 u_1 u_2 u_3 u_4 - u_3^2 u_4^2$$

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Algorithm for embedding Quiver into oriented surface w.o. boundary

step 1. Quiver \rightsquigarrow $2 \times N$ -matrix B

step 2. (geometric) formulation
pick $\gamma \in \mathbb{R}^N$ s.t. $*$
look at 2-plane

$$\gamma + \mathbb{R}^2 B \subset \mathbb{R}^N$$

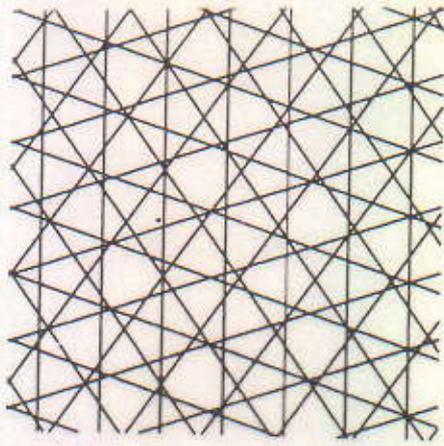
write $\mathcal{H}_{i,k} =$ hyperplane in \mathbb{R}^N
with equation $x_i = k$

Standard N -grid on \mathbb{R}^N

- consists of hyperplanes $\mathcal{H}_{i,k}$ with $1 \leq i \leq N$, $k \in \mathbb{Z}$.

- cuts out on $\gamma + \mathbb{R}^2 B$
an N -grid of lines

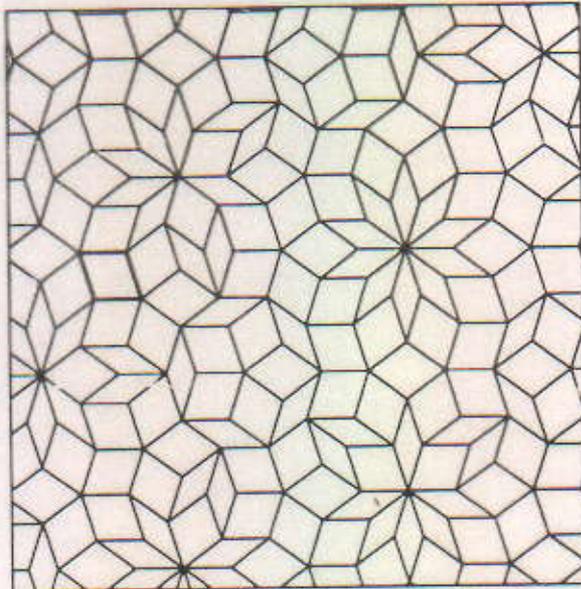
- * every point of $\gamma + \mathbb{R}^2 B$ lies on at most 2 grid lines



Figuur 25: Gedeelte van een vijftralie.

\approx de Bruijn's construction
of Penrose tilings

3.1.7. Voorbeeld. Figuur 26 toont een gedeelte van de betegeling behorende bij de



Figuur 26: Gedeelte van de betegeling behorende bij $\gamma = (0, 0.91, -0.29, -0.79, -0.02)$.

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Convert cell structure on plane
to dual rhombus tiling

Step 2 (algebraic formulation)

pick $\gamma \in \mathbb{R}^N$ (s.t. $*$)

look at map

$$F: \mathbb{R}^2 \longrightarrow \mathbb{Z}^N$$

$F(x) =$ componentwise floor of $(\gamma + xB)$

set $\underline{S^0} = F(\mathbb{R}^2) \subset \mathbb{Z}^N \subset \mathbb{R}^N$

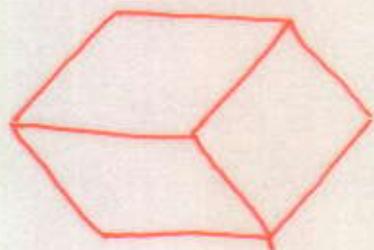
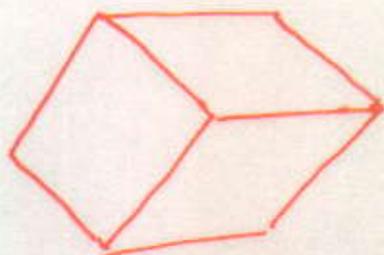
Remark

set $S =$ union of unit squares
in \mathbb{R}^N with vertices in S^0

there is a linear map $\mathbb{R}^N \rightarrow$ plane
which maps the surface S
with its tiling by unit squares
"isomorphically" onto above rhombus
tiling of the plane

Step 3

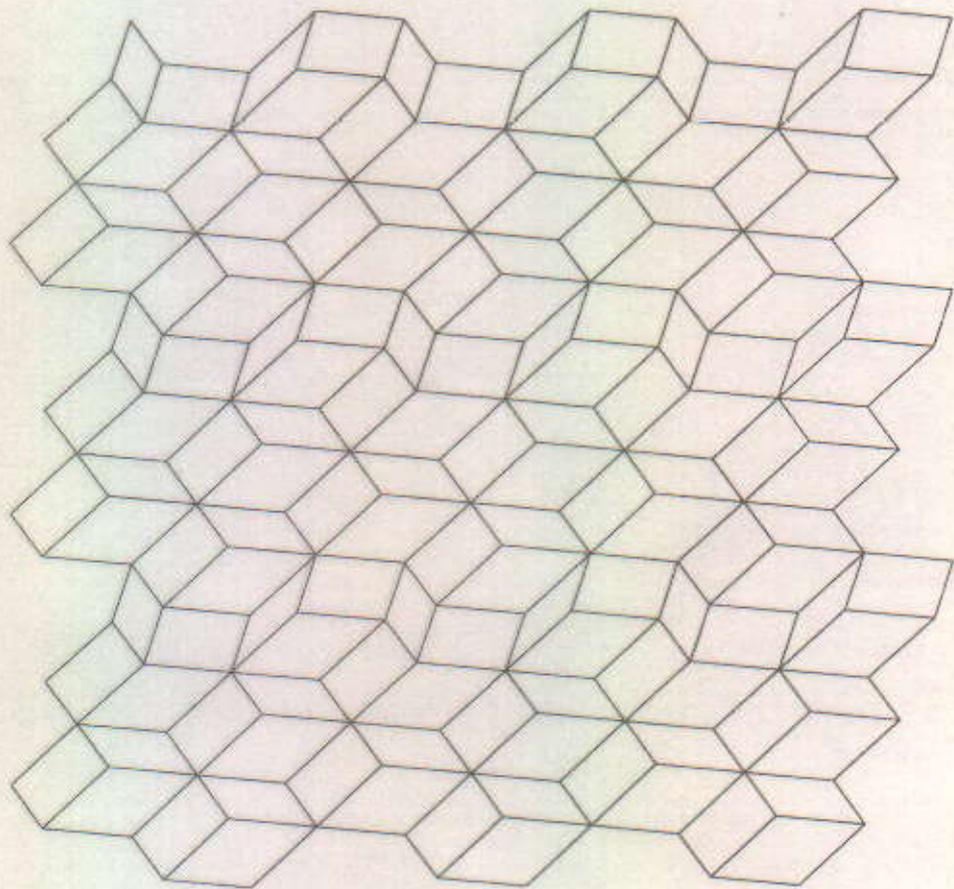
elementary move on rhombus tiling



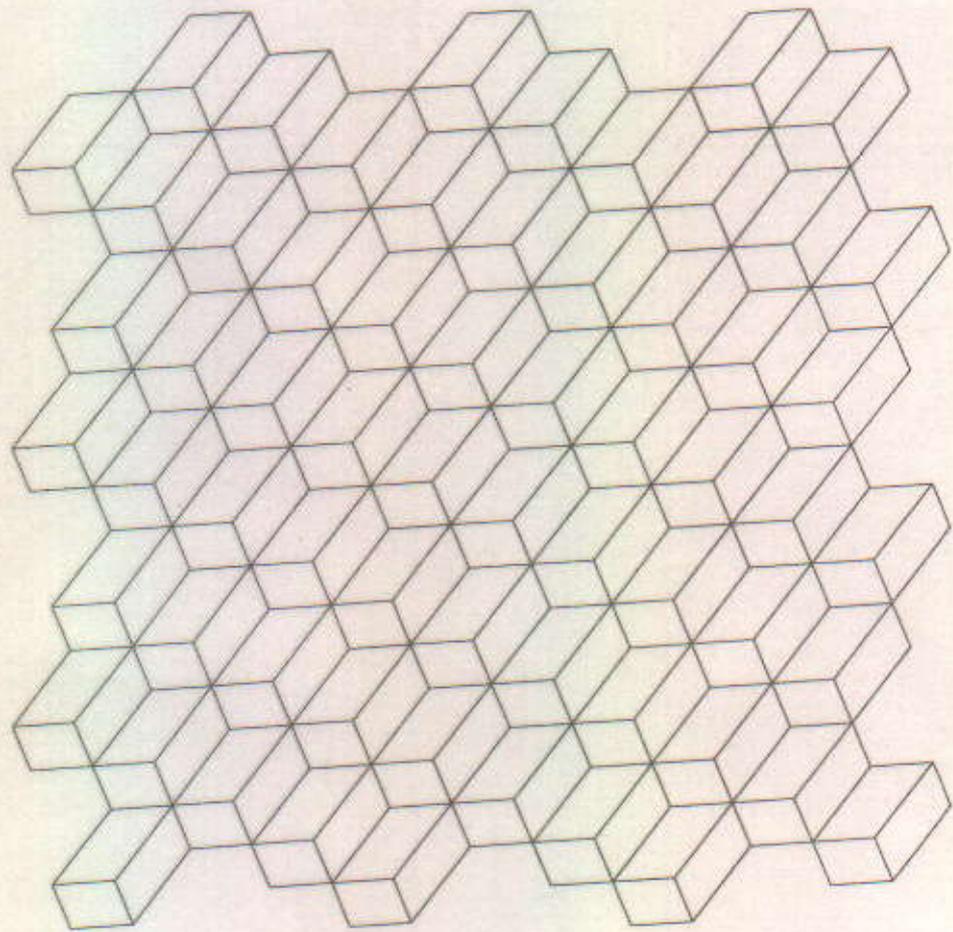
by translations (no rotations!)
of individual tiles.

Generate complete list of all rhombus tilings of the plane which are obtained from initial tiling by repeated elementary moves (preserving periodicity of tiling)

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Structure of crossing ribbons in rhombus tiling



is equivalent to input quiver
 lines mod. periodicity \longleftrightarrow vertices
 intersection points \longleftrightarrow arrows
 orientation of lines \longleftrightarrow orientation of arrows
 is preserved by elementary moves

Step 4

Each rhombus tiling is image of surface in \mathbb{R}^N which is a union of unit squares with vertices in \mathbb{Z}^N and is completely determined by this set of vertices $S^\circ \subset \mathbb{Z}^N$

$$\text{Let } \varphi: \mathbb{Z}^N \rightarrow \mathbb{Z}$$

$$\varphi(z_1, \dots, z_N) = z_1 + z_2 + \dots + z_N$$

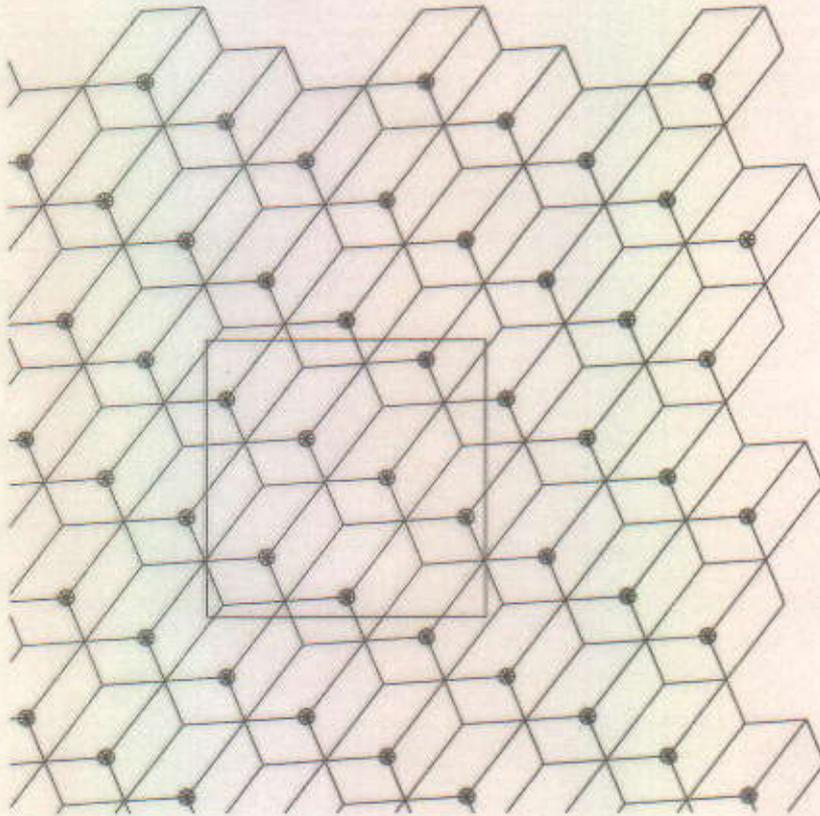
Select from the list of rhombus tilings those tilings for which φ takes only 3 values on the set of vertices S°

In selected tiling mark vertices

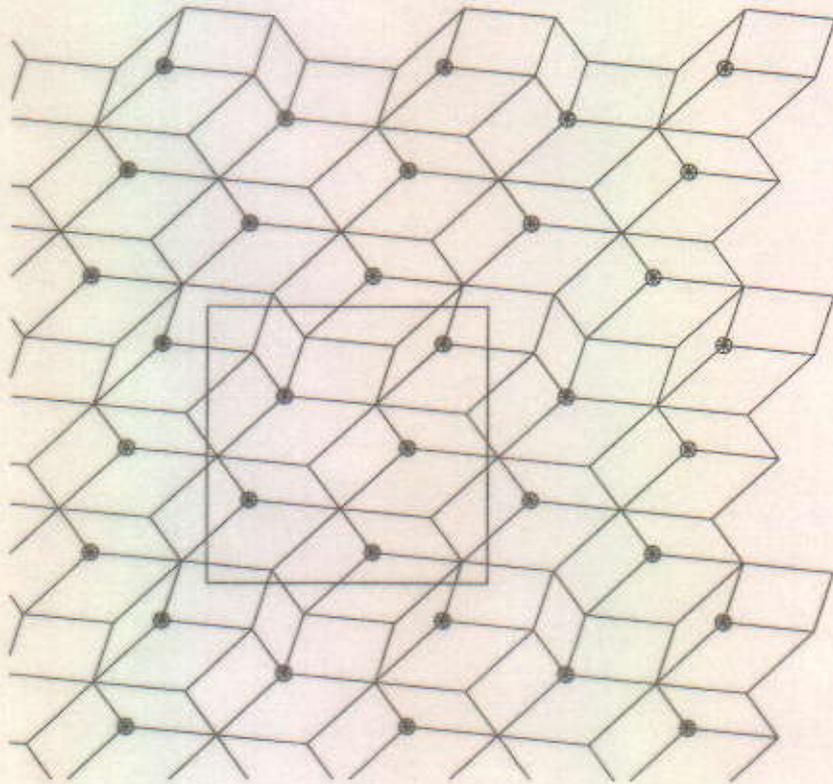
where φ is maximal as \bullet

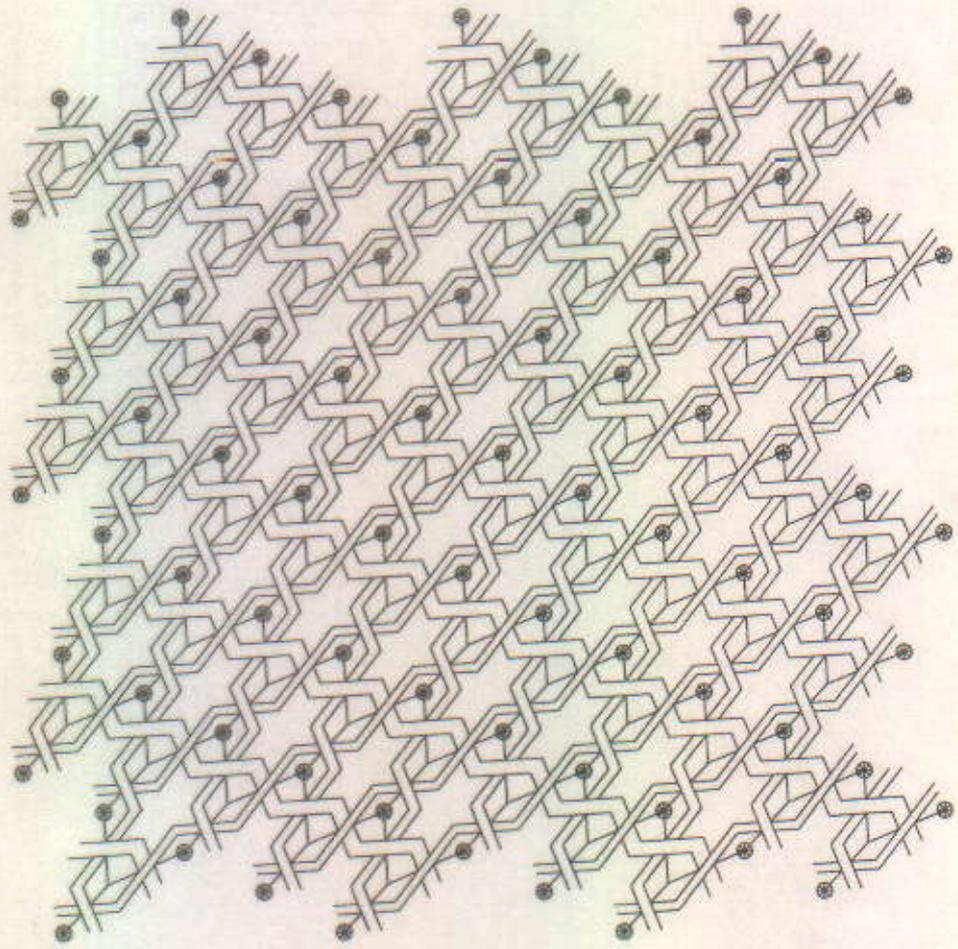
————//———— minimal \circ

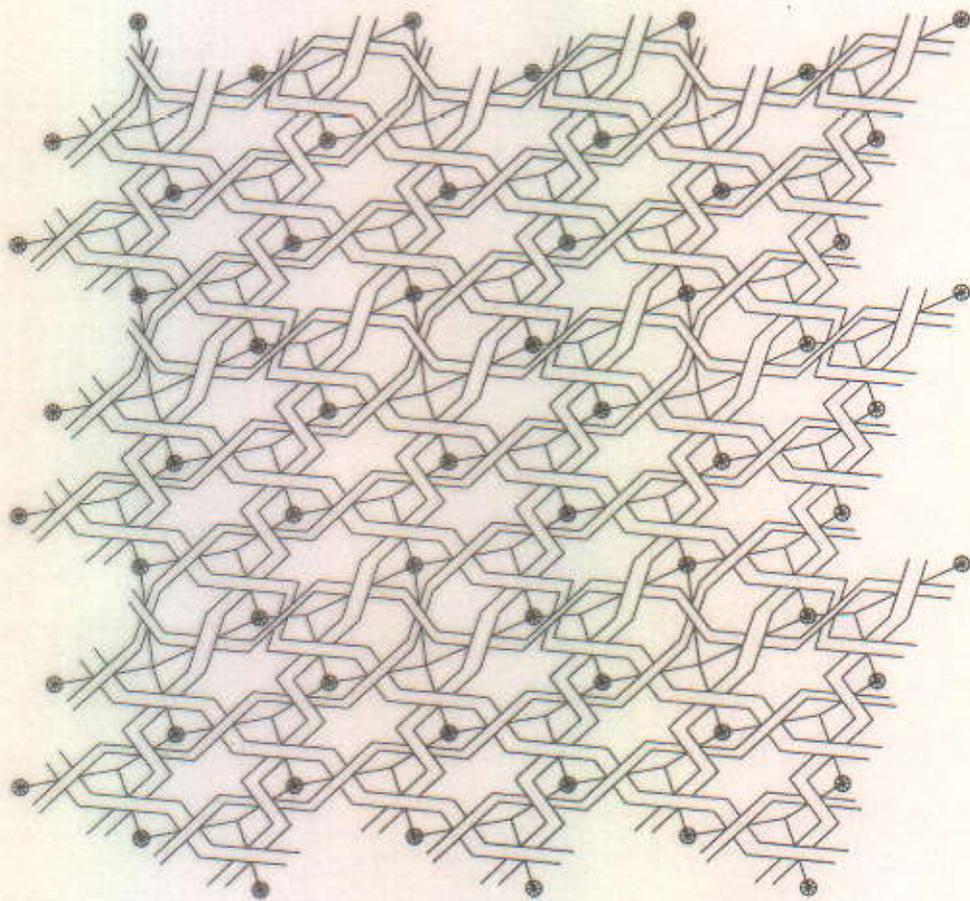
B is $\begin{matrix} 3 & -2 & -1 \\ -1 & 3 & -2 \end{matrix}$



B is $\begin{matrix} 1-2 & 0 & 1 \\ 2 & 1-1 & -2 \end{matrix}$







Final step: how to get a bi-adjacency matrix from a selected rhombus tiling:

- number black vertices $1, 2, \dots, M$
- — // — white — // — $1, 2, \dots, M$
- recall ribbons $\leftarrow \begin{smallmatrix} | \\ | \\ | \end{smallmatrix} \rightarrow$ grid directions
 $\leftarrow \begin{smallmatrix} | \\ | \\ | \end{smallmatrix} \rightarrow$ quiver nodes
already numbered $1, 2, \dots, N$
- number rhombi $1, 2, \dots, K$
recall rhombi $\leftarrow \begin{smallmatrix} | \\ | \\ | \end{smallmatrix} \rightarrow$ quiver edges
- note:
 - every rhombus has 1 white and 1 black vertex
 - every rhombus lies on exactly 2 ribbons

Bi-adjacency matrix

rows \longleftrightarrow black vertices
columns \longleftrightarrow white vertices

rhombus ε on ribbons a, b
with black vertex i
-||- white -||- j



matrix entry $X_\varepsilon \llbracket a \llbracket b$
in row i , column j

