

UCSD  
Mathematics Department

# The Tropical Vertex

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## Directory

- [Table of Contents](#)
- [Begin Article](#)

**Goal.** Explain both sides (the A and B-model sides) of a phenomenon which lies at the heart of mirror symmetry.

## 1. The Tropical Vertex Group (B-model)

Fix the following data:

$$M = \mathbb{Z}^2, \quad N = \operatorname{Hom}(M, \mathbb{Z}),$$

$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$$

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We will sometimes write

$$\mathbb{k}[M] = \mathbb{k}[x^{\pm 1}, y^{\pm 1}],$$

so an element

$$z^m \in \mathbb{k}[M]$$

can be written as

$$x^a y^b$$

if

$$m = (a, b).$$

**Definition.** The tropical vertex group  $H(R)$  is the subgroup of  $\text{Aut}(\mathbb{k}[M] \otimes_{\mathbb{k}} R)$  generated by automorphisms of the form

$$z^m \mapsto z^m f^{\langle n_0, m \rangle}$$

where

- $n_0 \in N$
- $f \in \mathbb{k}[z^{m_0}] \otimes_{\mathbb{k}} R \subseteq \mathbb{k}[M] \otimes_{\mathbb{k}} R$  for some **non-zero**  $m_0 \in M$ .
- $f - 1 \in z^{m_0} \mathfrak{m}$ .
- $\langle n_0, m_0 \rangle = 0$ .

**Remarks.**

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- Elements of  $H(R)$  are symplectomorphisms, preserving the symplectic form

$$\Omega = \frac{dx}{x} \wedge \frac{dy}{y}$$

- $z^{m_0}$  is left invariant by the automorphism

$$z^m \mapsto z^m f^{\langle n_0, m \rangle}$$

**Typical example.** With  $R = \mathbb{k}[[t]]$ ,

$$x \mapsto x$$

$$y \mapsto y(1 + tx)$$

is a typical example of one of the generators of  $H(R)$ . Here

$$m_0 = (1, 0)$$

$$n_0 = (0, 1)$$

$$f = 1 + tx$$

## 2. Scattering diagrams

**Definition.** A *ray* is a pair  $(\mathfrak{d}, f_{\mathfrak{d}})$  where  $\mathfrak{d} \subseteq M_{\mathbb{R}}$  is given by  $\mathfrak{d} = m'_0 - \mathbb{R}_{\geq 0} m_0$  for some  $m'_0 \in M_{\mathbb{R}}$  and **non-zero**  $m_0 \in M$ , and

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**Definition.** A *line* is a pair  $(\mathfrak{d}, f_{\mathfrak{d}})$  where  $\mathfrak{d} \subseteq M_{\mathbb{R}}$  is given by  $\mathfrak{d} = m'_0 - \mathbb{R} m_0$  for some  $m'_0 \in M_{\mathbb{R}}$  and **non-zero**  $m_0 \in M$ , and

$$f_{\mathfrak{d}} \in \mathbb{k}[z^{m_0}] \otimes_{\mathbb{k}} R$$

satisfies

$$f_{\mathfrak{d}} - 1 \in z^{m_0} \mathfrak{m}.$$

**Definition.** A *scattering diagram*  $\mathfrak{D}$  is a set of lines and rays such that for any  $n > 0$ , there are only a finite number of elements  $(\mathfrak{d}, f_{\mathfrak{d}})$  with

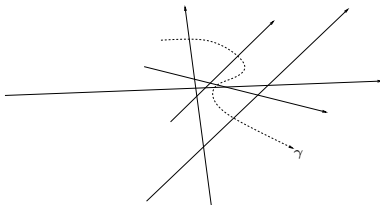
$$f_{\mathfrak{d}} \not\equiv 1 \pmod{\mathfrak{m}^n}.$$

Consider any path

$$\gamma : [0, 1] \rightarrow M_{\mathbb{R}}$$

which

- is transversal to every element of  $\mathfrak{D}$  it intersects;
- does not pass through the endpoint of any ray or the intersection of any two elements;
- only passes through any given ray a finite number of times.



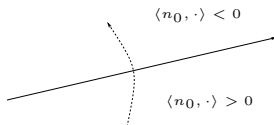
To such a path, we can associate a **path-ordered product** of automorphisms, as follows.

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First, when  $\gamma$  crosses an element  $(\mathfrak{d}, f_{\mathfrak{d}})$ , we obtain an element of  $H(R)$  given by

$$z^m \mapsto z^m f_{\mathfrak{d}}^{\langle m, n_0 \rangle},$$

where  $n_0 \in N$  is primitive with  $\langle n_0, m_0 \rangle = 0$  chosen with the following sign convention:

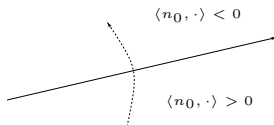


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This defines an element  $\theta_{\gamma, \mathfrak{d}} \in H(R)$ .

The path-ordered product is then defined by

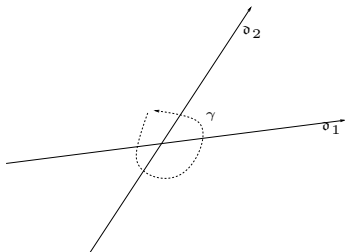
$$\theta_{\mathfrak{D},\gamma} = \prod \theta_{\mathfrak{d},\gamma},$$

where the product runs over all  $\mathfrak{d}$  crossed by  $\gamma$ , in the order traversed by  $\gamma$ .

**Example: Commutators**

$$\mathfrak{D} = \{(\mathfrak{d}_1, f_1), (\mathfrak{d}_2, f_2)\},$$

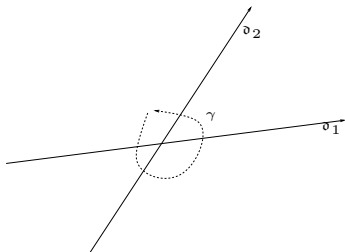
where  $\mathfrak{d}_1, \mathfrak{d}_2$  are lines through the origin.



**Example: Commutators I**

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Then

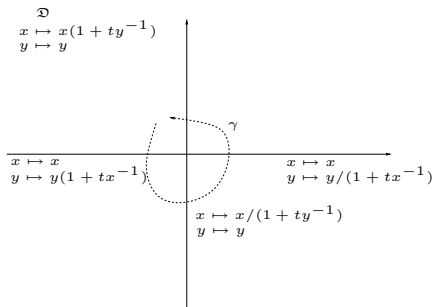
$$\theta_{\mathfrak{D}, \gamma} = \theta_2^{-1} \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1,$$

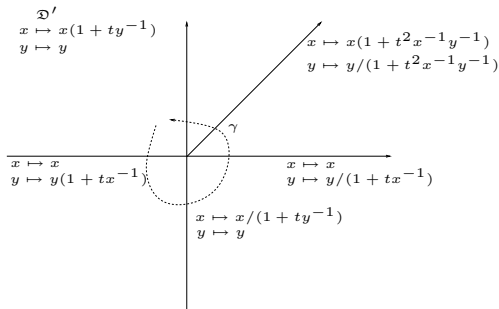
where  $\theta_1$  and  $\theta_2$  are the elements of  $H(R)$  associated to the first two crossings.

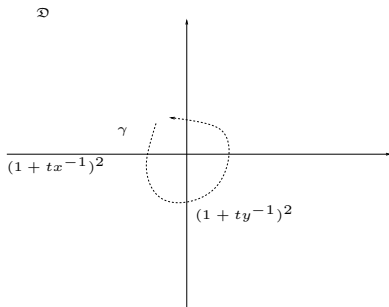
**Kontsevich-Soibelman Lemma.** Given a scattering diagram  $\mathfrak{D}$ , there is a scattering diagram  $\mathfrak{D}'$  containing  $\mathfrak{D}$  such that  $\mathfrak{D}' \setminus \mathfrak{D}$  consists only of rays, and

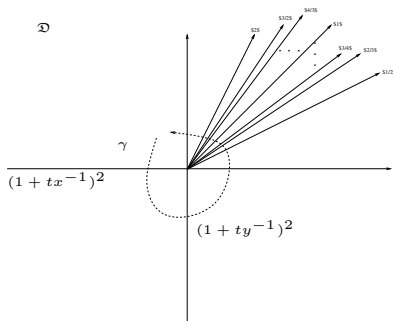
$$\theta_{\mathfrak{D}', \gamma} = id$$

for every closed loop  $\gamma$  for which  $\theta_{\mathfrak{D}', \gamma}$  is defined.

**Example: Commutators II**

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**Example: Commutators III**

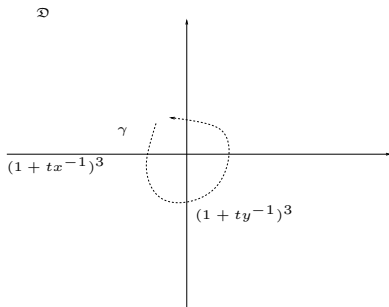
**Example: Commutators III**

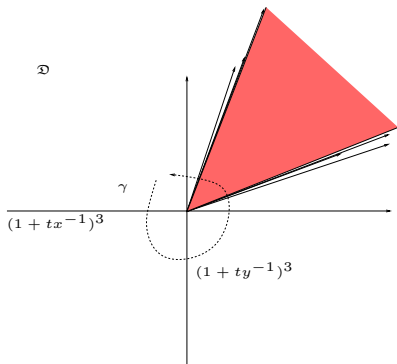
Lines of slope  $(n+1)/n$ ,  $n \geq 1$ :  $(1 + t^{2n+1}x^{-n}y^{-n-1})^2$

Lines of slope  $n/(n+1)$ ,  $n \geq 1$ :  $(1 + t^{2n+1}x^{-n-1}y^{-n})^2$

Line of slope 1:

$$(1 - t^2x^{-1}y^{-1})^{-4} = \frac{(1 + t^2x^{-1}y^{-1})^4}{(1 - t^4x^{-2}y^{-2})^{2 \cdot 2}}.$$

**Example: Commutators IV**

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Have rays of slope  $3, 8/3, 21/8, \dots$  converging to  $(3 + \sqrt{5})/2$ .

Have rays of slope  $1/3, 3/8, 8/21, \dots$  converging to  $(3 - \sqrt{5})/2$ .

Have rays of **all rational slopes** between  $(3 - \sqrt{5})/2$  and  $(3 + \sqrt{5})/2$ .

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For example, the function attached to the line of slope 1 is

$$\left( \sum_{n=0}^{\infty} \frac{1}{3n+1} \binom{4n}{n} t^{2n} x^{-n} y^{-n} \right)^9$$

$$= \frac{(1 + t^2 x^{-1} y^{-1})^9 \cdot (1 + t^6 x^{-3} y^{-3})^{3 \cdot 54} \dots}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 18} \cdot (1 - t^8 x^{-4} y^{-4})^{4 \cdot 252} \dots}$$

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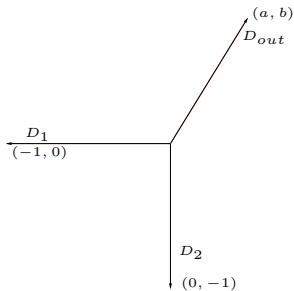
$$= \frac{(1 + t^2 x^{-1} y^{-1})^9 \cdot (1 + t^6 x^{-3} y^{-3})^{3 \cdot 54} \dots}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 18} \cdot (1 - t^8 x^{-4} y^{-4})^{4 \cdot 252} \dots}$$

**Remark.** This description is based on computer calculations and may not yet have been verified.

### 3. The tropical vertex, A-model

[Work in progress, joint with Bernd Siebert.]

Consider a weighted projective space  $X$  given by the fan:



The three labelled rays correspond to three toric divisors,  $D_1$ ,  $D_2$ , and  $D_{out}$ .

Pick two integers  $d_1, d_2 > 0$  and general sets of points

$$\begin{aligned} S_1 &\subseteq D_1, \\ S_2 &\subseteq D_2 \end{aligned}$$

with

$$\#S_1 = d_1, \#S_2 = d_2,$$

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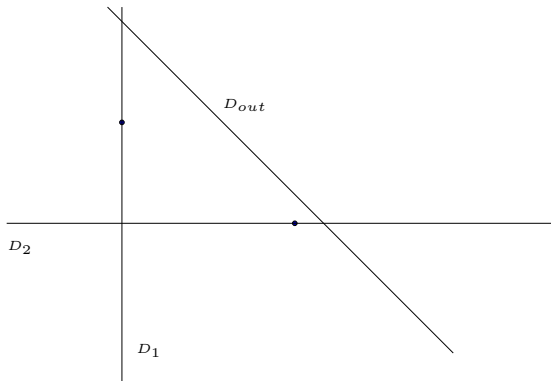
**Definition.** Let  $N_d$  be the number of maps  $\varphi : \mathbb{P}^1 \rightarrow X$  (up to reparametrization) satisfying the following properties:

1. Whenever  $\varphi(p) \in D_i$ ,  $i = 1, 2$ , then  $\varphi(p) \in S_i$  and  $\varphi$  is transversal to  $D_i$  at  $\varphi(p)$ .
2. There is a unique  $q \in \mathbb{P}^1$  such that  $\varphi(q) \in D_{out}$ .
3. The intersection multiplicity of  $\varphi(\mathbb{P}^1)$  with  $D_{out}$  at  $\varphi(q)$  is  $d$ .

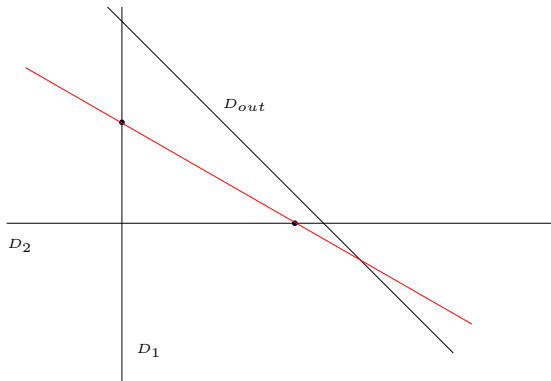
**Remark.** This definition is rather vague at the moment, and needs a more precise formulation. In particular, the main question is how to count multiple covers.

**Examples.**  $d_1 = d_2 = 1$ ,  $(a, b) = (1, 1)$ .

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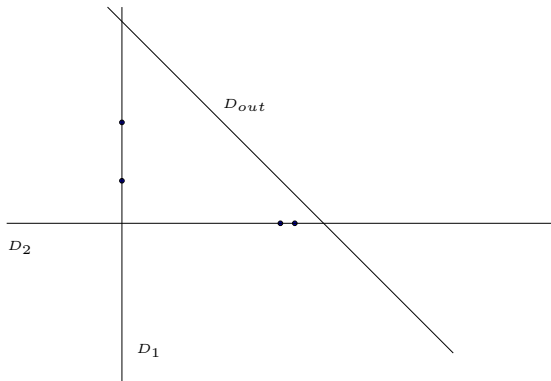


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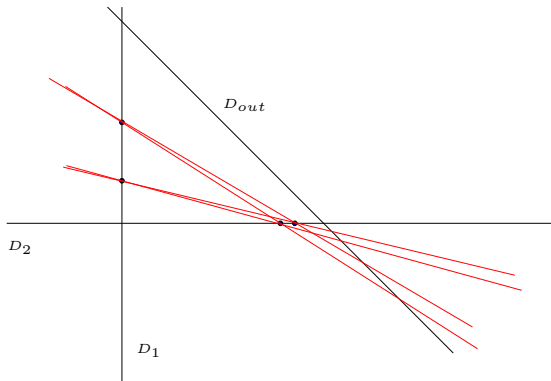


$$N_1 = 1, N_d = 0, d \geq 2.$$

**Examples.**  $d_1 = d_2 = 2$ ,  $(a, b) = (1, 1)$ .

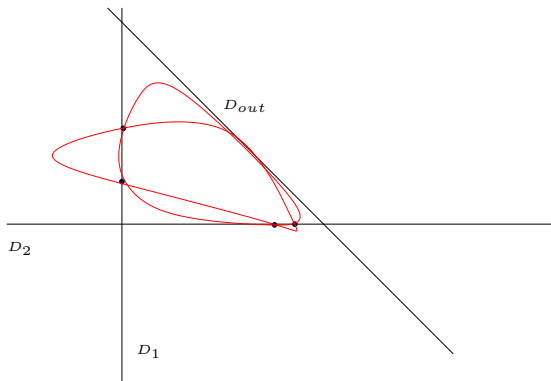


**Examples.**  $d_1 = d_2 = 2$ ,  $(a, b) = (1, 1)$ .



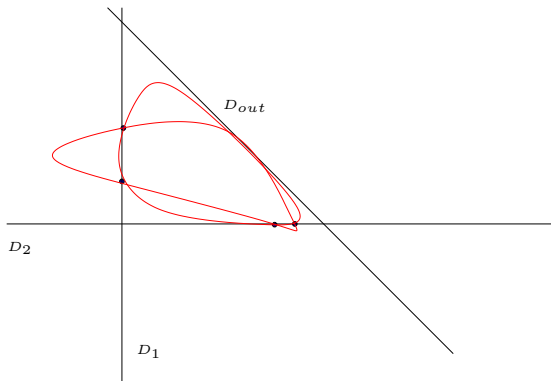
$$N_1 = 4$$

**Examples.**  $d_1 = d_2 = 2$ ,  $(a, b) = (1, 1)$ .



$$N_1 = 4 \quad N_2 = 2, \quad N_d = 0, \quad d \geq 3$$

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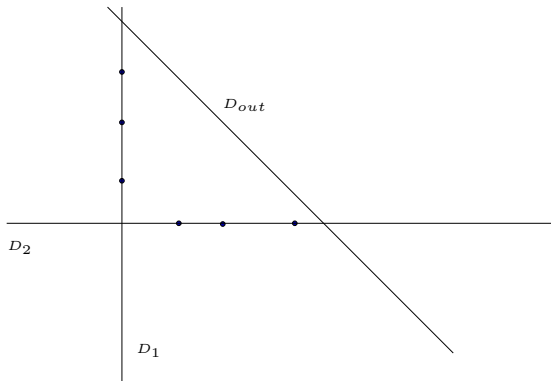


$$N_1 = 4 \quad N_2 = 2, \quad N_d = 0, \quad d \geq 3$$

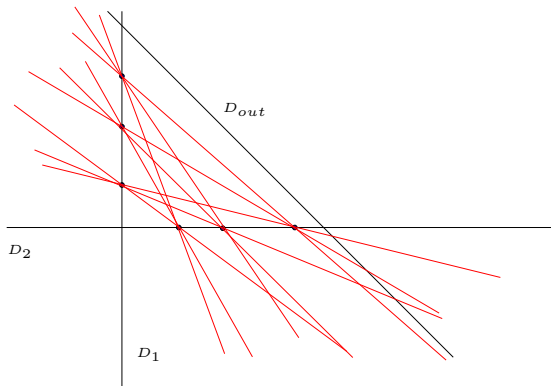
Compare with the function attached to the ray of slope 1:

$$\frac{(1 + t^2 x^{-1} y^{-1})^4}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 2}}$$

**Examples.**  $d_1 = d_2 = 3$ ,  $(a, b) = (1, 1)$ .

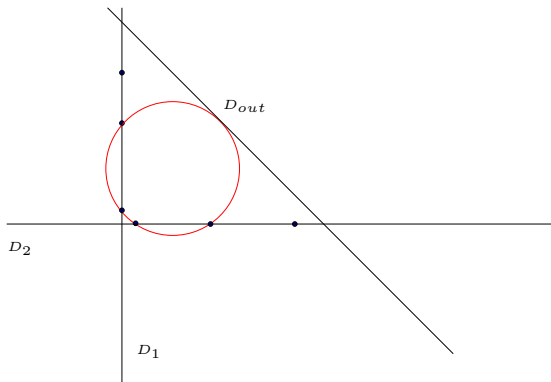


**Examples.**  $d_1 = d_2 = 3$ ,  $(a, b) = (1, 1)$ .



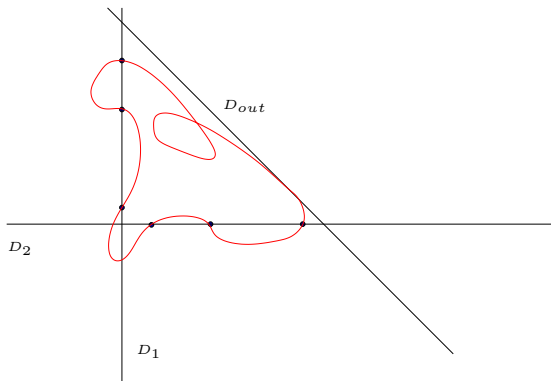
$$N_1 = 9$$

**Examples.**  $d_1 = d_2 = 3$ ,  $(a, b) = (1, 1)$ .



$$N_1 = 9, N_2 = 3 \times 3 \times 2 = 18$$

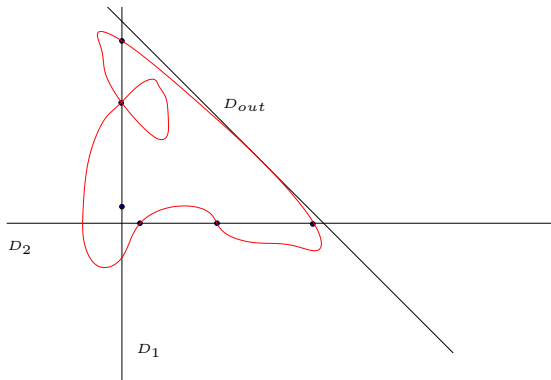
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18 such cubics.

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$$N_1 = 9, N_2 = 3 \times 3 \times 2 = 18, N_3 = 18 + 36 = 54, \dots$$

$$2 \times 3 \times 2 \times 3 = 36 \text{ such cubics}$$

**Summary.**  $d_1 = d_2 = 3$ ,  $(a, b) = (1, 1)$ .

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Compare with

$$\frac{(1 + t^2 x^{-1} y^{-1})^9 \cdot (1 + t^6 x^{-3} y^{-3})^{3 \cdot 54} \dots}{(1 - t^4 x^{-2} y^{-2})^{2 \cdot 18} \cdot (1 - t^8 x^{-4} y^{-4})^{4 \cdot 252} \dots}$$

**Conjecture.** Let  $\mathfrak{D}$  be the scattering diagram consisting of two lines with attached functions  $(1 + tx^{-1})^{d_1}$  and  $(1 + ty^{-1})^{d_2}$  and let  $\mathfrak{D}'$  be the scattering diagram obtained from the Kontsevich-Soibelman Lemma. Then the function attached to the ray generated by a primitive vector  $(a, b)$  is

$$\prod_{d=1}^{\infty} (1 + (-1)^{d+1} t^{d(a+b)} x^{-da} y^{-db})^{(-1)^{d+1} d \cdot N_d}.$$

**Remark.** We can prove this conjecture modulo the correct definition of the  $N_d$ 's. We use tropical techniques to prove it.

## 4. The $B$ -model for the mirror of $\mathbb{P}^2$

**Recall.** The mirror of  $\mathbb{P}^2$  is a Landau-Ginzburg model, which can be represented as

$$(\mathbb{C}^\times)^2 = \operatorname{Spec} \mathbb{C}[x, y, z]/(xyz - q)$$

where  $q$  is a non-zero parameter, and the Landau-Ginzburg potential is

$$W := x + y + z.$$

The **small** quantum cohomology ring is the Jacobian ring of  $W$ ,

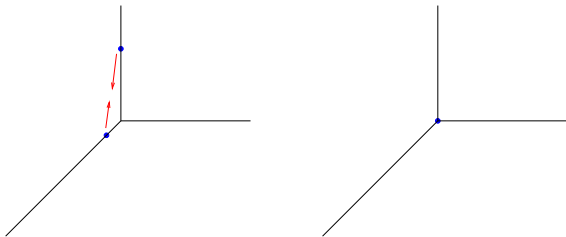
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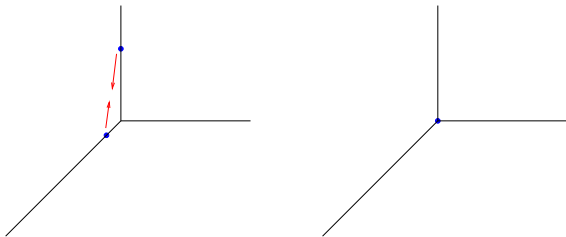
$$\mathbb{C}[x, y, z]/(x - y, x - z, xyz - q) \cong \mathbb{C}[H]/(H^3 - q).$$

Small quantum cohomology of  $\mathbb{P}^2$  just captures the fact that there is one line through two points.

Tropically, we can imagine these two points coming together along a line of generic slope, hence forcing the vertex of the line to lie at a given point in  $\mathbb{R}^2$ :



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The Jacobian ring essentially just reads off the existence of one tropical curve with one ray in each of the three directions corresponding to  $x$ ,  $y$  and  $z$ . (Chan and Leung).

For big quantum cohomology, need to put a Frobenius manifold structure on the universal unfolding of  $W$ .

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Here we work on  $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xyz - 1)$ ,

$$W = x + y + z,$$

and the universal unfolding is parametrized by  $(t_0, t_1, t_2)$ :

$$t_0 + (1 + t_1)W + t_2W^2$$

We want a Frobenius manifold structure on  $\mathcal{M} = \operatorname{Spec} \mathbb{C}[[t_0, t_1, t_2]]$ .  
(K. Saito, Barannikov, Sabbah, Hertling)

One key point is get flat (canonical) coordinates  $(y_0, y_1, y_2)$  on  $\mathcal{M}$ .

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e.g. to fourth order,

$$\begin{aligned}y_0 &= t_0 - \frac{27}{2}t_2^2 + \frac{27}{2}t_1t_2^2 - \frac{27}{2}t_1^2t_2^2 + \cdots \\y_1 &= 3t_1 - \frac{3}{2}t_1^2 + t_1^3 - \frac{135}{2}t_2^3 - \frac{3}{4}t_1^4 + \frac{405}{2}t_1t_2^3 + \cdots \\y_2 &= 9t_2(1 - 2t_1 + 3t_1^2 - 4t_1^3 + \frac{81}{4}t_2^3 + \cdots)\end{aligned}$$

(Conjectural) method for producing a canonical unfolding to order  $3d$ :

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We will build a scattering diagram over the ring

$$R = \frac{\mathbb{C}[g_1, \dots, g_d, r_1, \dots, r_d, b_1, \dots, b_d]}{(g_1^2, \dots, g_d^2, r_1^2, \dots, r_d^2, b_1^2, \dots, b_d^2)}$$

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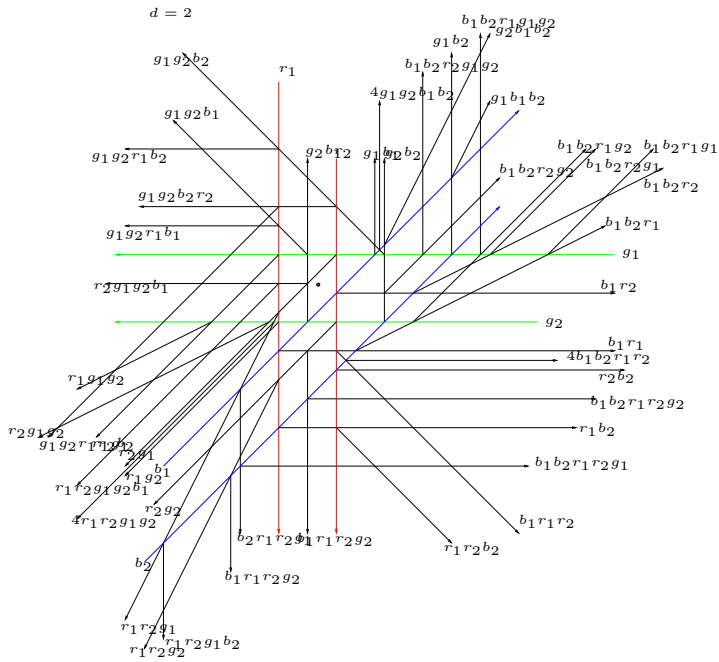
We begin with  $\mathfrak{D}$  consisting of  $3d$  lines, with attached functions

$$\begin{aligned} 1 + g_1x, & \quad \dots, \quad 1 + g_dx \\ 1 + r_1y, & \quad \dots, \quad 1 + r_dy \\ 1 + b_1z, & \quad \dots, \quad 1 + b_dz \end{aligned}$$

(with  $xyz = q$ .)

We have three groups of  $d$  parallel lines.

The Kontsevich-Soibelman Lemma then gives a new scattering diagram  $\mathfrak{D}'$ .



We now construct  $W$  by picking a point, and considering all ways of “transporting” the monomials  $x$ ,  $y$  and  $z$  to arrive at a chosen base-point.

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We “transport” a monomial  $cz^{(a,b)}$  along a straight line whose tangent direction is  $-(a, b)$ .

We now construct  $W$  by picking a point, and considering all ways of “transporting” the monomials  $x$ ,  $y$  and  $z$  to arrive at a chosen base-point.

We “transport” a monomial  $cz^{(a,b)}$  along a straight line whose tangent direction is  $-(a, b)$ .

When a monomial crosses a line or ray of the scattering diagram, we apply the corresponding automorphism to the monomial, and then are allowed to replace the monomial with any monomial which appears in the resulting expression.



$W$  is now a sum of a new parameter  $y_0$  and all the monomials appearing in this way.

$$\begin{aligned}
W = & y_0 + \\
& x + x^2 z b_2 g_2 + 2 b_2 g_2 x^2 z^2 b_1 + x z b_1 + b_2 z x + x z^2 b_1 b_2 \\
& + b_1 b_2 r_1 z^2 x y + r_1 x^2 z^2 y b_1 b_2 g_2 \\
& + r_2 x y + r_2 x^2 y z b_2 g_2 + g_2 b_1 x^2 z^2 y b_2 r_2 + y + x y g_1 + b_2 g_1 g_2 x^2 z y \\
& + b_1 b_2 g_1 g_2 z^2 x^2 y + r_2 g_1 x y^2 + z g_2 x^2 r_2 y^2 b_2 g_1 \\
& + z + g_2 x z + g_2 b_1 x z^2 + z r_1 y + r_1 z^2 x y b_1 g_2 + r_1 r_2 g_1 x y^2 z
\end{aligned}$$

and

$$x y z = e^{y_1}.$$

This deformation of  $x + y + z$  over

$$\mathrm{Spec} \mathbb{C}[[y_0, y_1, g_1, \dots, g_d, r_1, \dots, r_d, b_1, \dots, b_d]]/(g_i^2, r_i^2, b_i^2),$$

and hence gives a map to the universal unfolding (in canonical coordinates)

$$\begin{aligned} & \mathrm{Spec} \mathbb{C}[[y_0, y_1, g_1, \dots, g_d, r_1, \dots, r_d, b_1, \dots, b_d]]/(g_i^2, r_i^2, b_i^2) \\ \rightarrow & \mathrm{Spec} \mathbb{C}[[y_0, y_1, y_2]]. \end{aligned}$$

**Conjecture.** This map is given by

$$y_0 \mapsto y_0$$

$$y_1 \mapsto y_1$$

$$y_2 \mapsto \sum_{i=1}^d g_i + r_i + b_i$$

Checked “by hand” in the  $d = 1, 2$  cases.

Why does  $W$  contain the information we want?

Why does  $W$  contain the information we want?

Intution: Whenever three monomials in  $W$  multiply to give a term of the form

$$f(g_i, r_i, b_i)(xyz)^d,$$

we obtain a tropical curve in tropical  $\mathbb{P}^2$  of degree  $d+1$ , which passes at infinity passes through  $3d$  points, and through “two points” at the chosen base-point.

