2008

代数幾何学シンポジウム

記録

平成 20 年度科学研究費補助金　基盤研究 (S)
(課題番号 19104001，代表　桂 利行)
平成 20 年度科学研究費補助金　基盤研究 (S)
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平成 20 年度科学研究費補助金　若手研究 (A)
(課題番号 20684003，代表　高橋 篤史)

於 兵庫県立城崎大会議館
(2008 年 10 月 20 日～10 月 24 日)

永田先生を偲んで

永田雅宜先生は本年8月27日に逝去されました。永田先生に縁のあるもの200名余りが集い、11月23日にお別れ会を開かせていただきました。

本年度の城崎シンポジウム主催者の御厚意により、以下に「お別れ会」の案内状の主要部分と、奥様の永田千種様に「城崎シンポジウム」と「お別れ会」に御用意いただいたものを転載いたします。永田千種様は名古屋大学理学部数学科で永田先生の 1 年後輩であり、その頃の名古屋大学数学教室の教員・学生と日本の数学研究者の熱い思いを読み取っていただけると思います。

丸山正樹

「お別れ会」案内状の主要部分

永田先生は可換環論と代数幾何学のリーダーとして数々の業績を挙げ、世界を驚かせてきました。今日 Nagata ring と呼ばれる擬幾何環についての精力的な研究は「ネター環の密林の中で安全に探索できる領域を見つけ出し」、また「幾多の病理学的例により、その領域の境界を指し示し」(L. Illusie) ました。局所環のヘンゼル化理論の創始と完成は殆ど全て永田先生によるものであり、後の étale 射、代数空間理論の基礎になるものです。正則局所環が素元分解環であることの決着に永田先生が中心的役割を果たされたこともよく知られています。

デデキント環上の代数幾何学についての一連の業績は、後のスキーム理論など代数幾何学の基礎付けに計り知れない影響を与えました。全ての代数多様体は完備代数多様体の開集合であるという、代数幾何学になくてはならない定理は、射影的でない非特異 3 次元完備代数多様体と正規完備代数曲面の例の構成と相俟って、完備代数多様体の位置づけを明確にしたものです。

可換環論と代数幾何学を深く洞察した永田先生の研究は、不変式論の世界を塗り替えてしまったといっても過言ではないでしょう。代数群の多項式環への有理作用について、その不変式全体は有限生成代数になるであろうという Hilbert の第 14 問題に対する反例は、代数幾何学と不変式の専門家を震撼させるとともに、この分野が想像以上に複雑かつ豊かなものであることを示しました。正標数の場合でも簡約可能群が不変式の有限性についての十分条件であることを導く Mumford 予想の重要性を早くから指摘して、この予想の肯定的解決

が主張する不変式論における結果とその代数幾何学への応用を明らかにしました。

学術コミュニティにおいては日本数学会の理事、学術会議会員、国際数学連合の副会長などの要職を務められ、数学の発展だけでなく、学界における数学の位置づけを向上させることに意を尽くされたこともよく知られています。

永田先生は京都大学で理学部選出評議員として大学の管理・運営に多大の貢献をされ、「入学者選抜方法研究委員会」で大学入学試験について先生の高い見識を発揮されました。入学試験における数学者の役割の重要性を早くから主張したのも永田先生です。京都大学退職後は岡山理科大学で学生の指導に当たり、その後は兵庫県八千代町で子供たちに数の面白さ、自分で考えることの重要性などを伝え、ご自身の教育についての夢を追われたと伺っています。

皆様へ　　　　　妻　永田千種より

本日はご参加頂きまして有難うございました。故人共々心から御礼申し上げます。雅宜と共に歩みました道を振り返りつつ、私的な立場から、少し記させて頂きます。

昭和23年春、名古屋の本山から唐山、東山に続くなだらかな丘陵はまだ自然の姿のままで、とき色の山つつじが咲き誇っていました。

そこに6棟の粗末な兵舎のような低い建物がぽつりぽつりと並んでいるだけでした。これが、戦争で沈滞した研究の焔をいち早く再び輝かせた、名古屋大学理学部でした。建物は貧しくても、住人は若いエネルギーの充ち溢れた集団でした。

一番奥に中山正先生の部屋がありました。中山先生と角谷静夫先生が、戦争で疲弊し、閉ざされていた日本の研究者たちの世界に真っ先に外国の新しい息吹を紹介されたのでした。

中山先生の他に黒田成勝先生（整数論。高木貞治先生の思想も人間味も、そのまま受け継がれた方）、能代清先生（関数論。謹厳だが、学生を親身にみて下さった方）、吉田耕作先生（関数解析。広範囲の研究をバリバリなさるエネルギッシュな方）、小野勝次先生（数学基礎論。陸上競技の分野でも有名人でいらした方）の何れも若々しい5人の教授。先生方と学生を結びつける役割を果たされたのが、若き日の伊藤清先生。詳細で解り易い講義を早口で進められました。静間先生や東屋先生の独特な語り口と光る個性にも心を和ませられました。

そして、年はもっと若いけれど、新しい研究に果敢に取り組んで、成果を挙げられる研究者たちが、キラ星のように、たむろしておられました。今では考えられないような粗末な建物。物質的には恵まれない環境の中に、人間的には、研究者にとっては、実に恵まれた場がありました。

同じ建物に学生のたまり場の一室もありました。戦争直後のこととて、年齢も、経歴も異色な学生たちが混じっておりました。それまで（少数例を除いては）一般的には女人禁制だった旧帝国大学もアメリカのお蔭で女子にも開放され、私もオンナの子のハシリで、皆さんに可愛がって頂きました。

学生たちも学問に対して鼻息荒く、自分の実力も顧みず、数学界の最先端の問題に意欲的に立ち向かっていました。２０世紀の初頭にD.Hilbertがパリでの国際会議に、今世紀の課題として提出された２３の未解決問題は、学生たちにさえも憧れの的だったのです。それぞれが自分の興味を持つ分野に分れて２３問題の一つに取組みました。雅宜は第１４問題に、私如きも第5問題のグループの片隅で勉強していました。

学生控室には若い研究者達も、時には中山先生も出入りされました。上下関係のない、学問の前にはみんな平等と、同じ視点に立っての和やかさがありました。

時には碁や将棋に沈思黙考。夜も泊まり込みらしい人達も何人かありました。

昔懐かしい渦巻状のコイルむき出しの電熱器で、よく鯖を焼いていたのが永田。だからみんなにネコと呼ばれていましたが、ネコの本当の意味はネコ舌とか・・・。

そんな雰囲気の中で、彼ははじめ第１４問題は肯定的に解決されると思い込んで打ち込んでいました。何らかの付帯条件があれば、肯定的に解決されることを幾つか証明し、付帯条件をだんだん緩めようと考え続けていました。彼は机に向って「勉強している」という姿勢は見せない人です。人と議論している以外は、何時勉強しているのか全く解らない。遊んだり、歩いたり、眠ったり、他の事をやりながら、頭の中で問題をあたため続け、始終反芻していたようです。

第14問題がかなり煮詰まって、イイ線、行っている頃に秋月康夫先生から京都大学に招かれました。秋月先生を囲んで、井草準一さんを始め、中野茂男さん、中井喜和さん達との交流で視野が開けて行ったようで、永田を含め、スリーＮと呼ばれていました。

昭和20年代の終わり頃に、日光での国際シンポジウムをきっかけにして、外国の研究者達との交流も密になり、Ｚariski先生からHarvard University に招かれて渡米することになりました。当時は殆どの方が単身で行かれたのですが、その頃子供も2人になり私も職を去って貧しい生活でしたので、心配した彼は妻と幼児2人を強引に連れて行く決心をしました。氷川丸でシアトルに着き大陸横断鉄道を利用しての長い旅で漸くボストンへ。最初の頃は生活も苦しく様々のエピソードが思い浮かびます。

アメリカでの研究も踏まえて、第14問題は永田の作った反例により予想は否定的に解決されました。・・・その後の事は今日ご参加下さった皆様の中にも私よりよくご存じの方が沢山いらっしゃることと思います。

一つだけつけ加えたいのは、子供時代の永田は特別成績優秀な子ではなく、ごく普通の子だったようです。有名校に入るための受験勉強など考えた事もなく、将来の夢は小学校の先生になる事でした。（旧制）刈谷中学の先生から、中学校の先生にもなれるように（旧制）八高に進む方がよいとのアドバイスを受けたことから道が開けたのです。ただ日常生活においても、何でも自分で考えたり工夫したりするのが好きでした。父親から受け継いだ性格のようです。小さい頃から数学の問題を解くのは好きだったようで、何時も未解決の問題を幾つか温めて考えるのが好きだったそうです。

70歳台になり、京大退官後に勤めた岡山理科大をも退職した後、キッカケがあって、兵庫県八千代町の教育委員会のご好意で、3年間小学校、中学校の教壇に立つ機会を頂きました。キッズランドの名誉園長もさせて頂きました。今まで見た事もないような穏やかな顔つきで嬉しさ一杯。幼い頃の夢をかなえて、子供たちに接しさせて頂きました。幸せな一生だったと思っております。

皆様本当にありがとうございました。　　　　　妻　永田千種

代数幾何学城崎シンポジウムのご案内

2008 年 8 月 13 日

残暑厳しいおりから、皆様ますますご清栄のこととお喜び申し上げます。さて、下記の要領で代数幾何学城崎シンポジウムを開催いたしますので、ご案内申し上げます。貴方のご予定を同封の回答用紙にご記入の上、9 月 19 日 (金) 必着で、

606-8501 京都市左京区吉田本町
京都大学大学院工学研究科電気工学専攻
前野 俊昭 宛

に御返送下さい。

世話係　小木曽 啓示（慶應大経済）　oguiso@hc.cc.keio.ac.jp
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記

1. 日時：2008 年 10 月 20 日 (月) 夕刻 — 10 月 24 日 (金) 午前
   但し、20 日と 24 日は旅行日です。
2. 場所：城崎大会議館 (〒 669 - 6101 兵庫県豊岡市城崎町湯島 1062)
   電話：0796-32-3888 (つたや晴嵐亭：0796-32-2511)
3. プログラム：

	21 日 (火)	22 日 (水)	23 日 (木)	24 日 (金)
9:30–10:30 (9:00–10:00)	Kontsevich	高木	Lee	Kaledin
10:45–11:45 (10:15–11:15)	寺杣	Dolgachev	Zhang	並河
13:30–14:30 (11:30–12:30)	渡邉	川ノ上	岩成	
14:45–15:45	Hwang	自由討論	高橋	
16:00–17:00	松下	自由討論	戸田	

() 内は 22 日, 24 日の時間です．21 日 17:15 からポスター紹介があります.

Kinosaki Algebraic Geometry Symposium

October 21–24, 2008

Kinosaki Conference Hall

Organizers:
Keiji Oguiso (Keio), oguiso@hc.cc.keio.ac.jp
Toshiaki Maeno (Kyoto), maeno@kuee.kyoto-u.ac.jp
Masahiko Yoshinaga (Kobe), myoshina@math.kobe-u.ac.jp

For lectures: For your lecture, you may use three white boards, OHP projector, projector from your own or our PC. If you have a special request, please let us know.

Program

- Oct. 20

ARRIVAL (No Lecture)

- Oct. 21

9:30–10:30	Maxim Kontsevich (IHES) Generalized Tian-Todorov theorems
10:45–11:45	Tomohide Terasoma (Tokyo) Thomae's formula and binary tree
13:30–14:30	Kenta Watanabe (Osaka) A counterexample to a conjecture of complete fan
14:45–15:45	Jun-Muk Hwang (KIAS) Base manifolds for fibrations of projective irreducible symplectic manifolds
16:00–17:00	Daisuke Matsushita (Hokkaido) On deformation of Lagrangian fibrations
17:15–	Introduction for Poster Session

• <u>Oct. 22</u>

9:00–10:00	Hiromichi Takagi (Tokyo) Scorza quartics of trigonal spin curves and their varieties of power sums
10:15–11:15	Igor Dolgachev (Michigan) Finite subgroups of the plane Cremona group over perfect fields
11:30–12:30	Hiraku Kawanoue (RIMS) Toward resolution of singularities for arbitrary characteristics
Free Afternoon	
18:30–	Reception

• <u>Oct. 23</u>

9:30–10:30	Yongnam Lee (Sogang U.) Construction of surfaces of general type with $p_g = 0$ via Q-Gorenstein smoothings
10:45–11:45	De-Qi Zhang (U. Singapore) Polarized endomorphisms on normal projective varieties
13:30–14:30	Isamu Iwanari (Kyoto) Stable points on stacks
14:45–15:45	Atsushi Takahashi (Osaka) Homological Mirror Symmetry for Singularities
16:00–17:00	Yukinobu Toda (IPMU) Pandharipande-Thomas theory and wall-crossings in derived categories

• <u>Oct. 24</u>

9:00–10:00	Dmitry Kaledin (Stekolov) Cyclotomic complexes and Dieudonne modules
10:15–11:15	Yoshinori Namikawa (Kyoto) Induced nilpotent orbits and birational geometry

Kinosaki Algebraic Geometry Symposium,
October 21–24, Kinosaki
Contributed posters.

Yuhi Sekiya (Graduate School of Mathematics Nagoya University)
Abelian G-Hilbert schemes via Gröbner bases

Shingo TAKI (Graduate School of Mathematics Nagoya University)
Non-symplectic automorphisms of prime order on $K3$ surfaces

Kyouko Kimura (Nagoya University)
On the arithmetical rank of squarefree monomial ideals concerned with the complete bipartite graph $K_{2,n}$

Ma Shouhei (Tokyo)
Twisted Fourier-Mukai number of a K3 surface

Kentaro Nagao (Kyoto, RIMS)
Counting invariants of perverse coherent sheaves and wall-crossings

Hisanori Ohashi, Kyoto University (RIMS)
Enriques surfaces covered by Jacobian Kummer surfaces

Kazuki Utsumi (Hiroshima University)
On the structure of certain K3 surfaces

Kotaro Kawatani (Osaka University)
On finite group actions on an irreducible symplectic 4-fold

Kimiko Yamada (Kyoto University)
Flips and variation of moduli of sheaves

Satoru FUKASAWA (Waseda University/JSPS)
The reflexivity of a Segre product of projective varieties (Joint work with Hajime Kaji)

Kiwamu WATANABE (Waseda University)
Actions of linear algebraic groups of exceptional type on projective varieties

Katsuhisa FURUKAWA (Waseda University)
Rational Curves on Hypersurfaces

Takuzo Okada (RIMS, Kyoto University)
On the birational unboundedness of higher dimensional $\mathbb{Q}$-Fano varieties.

Ryo Ohkawa (Tokyo Institute of Technology)
Moduli of Bridgeland semistable objects on the projective plane

Shinobu FUJII, (Hiroshima University)
Homogeneous isoparametric hypersurfaces with four distinct principal curvatures and moment maps

Hiroko Yanaba (Tokyo Denki University)
On Mixed Plurigenera of Algebraic Plane Curves

Michael Wemyss (Nagoya University, JSPS)
The GL(2) McKay Correspondence

Justin Sawon (Colorado State University)
A classification of Lagrangian fibrations by Jacobians

Hiroyuki Minamoto (Kyoto-U.)
Generalized Koszul duality and its application (joint work with A. Takahashi)

Generalized Tian-Todorov theorems

M.Kontsevich

1 The classical Tian-Todorov theorem

Recall the classical Tian-Todorov theorem (see [4],[5]) about the smoothness of the moduli spaces of Calabi-Yau manifolds:

Theorem 1.1 *If X is a compact Kähler manifold with $c_1(X) = 0 \in \mathrm{Pic}(X)$, then the Kuranishi space of deformations of complex structures on X is smooth of dimension $h^{n-1,1}(X) := \mathrm{rk}\, H^{n-1,1}(X)$ where $n = \dim(X)$. Manifold X with deformed complex structure is again a Kähler manifold with $c_1(X) = 0 \in \mathrm{Pic}(X)$. Similarly, if X is projective and $\omega \in H^2(X, \mathbb{Z})$ is an ample class, then the Kuranishi space of deformations of X which polarization ω is also smooth, of dimension $\mathrm{rk}\, H^{n-1,1}_{\mathrm{prim}}(X)$ of the primitive cohomology. Moreover, any choice of a splitting of the Hodge filtration on $H^n(X)$ (resp. of $H^n_{\mathrm{prim}}(X)$) defines an analytic affine structure on the Kuranishi space.*

The goal of my talk is to explain that there are many generalizations of this theorem. First, I present a sketch of a proof.

2 Smoothness via dg BV algebras

Definition 2.1 *A differential graded Batalin-Vilkovisky algebra A (a dg BV algebra for a short) over $\mathbb{C}$ is a commutative unital super-algebra endowed with two odd operators d, Δ satisfying*

- $d^2 = \Delta^2 = d\Delta + \Delta d = 0$,
- $d(1_A) = \Delta(1_A) = 0$,
- *operator d is a differential operator of order ≤ 1,*

- *operator Δ is a differential operator of order ≤ 2.*

The vector space $\mathfrak{g} := \Pi A$ obtained from A by the changing of parity, carries a natural structure of Lie super-algebra:

$$[a, b] = \Delta(ab) - \Delta(a)b - (-1)^{\deg a} a\Delta(b)\,.$$

Operators d, Δ on $\mathfrak{g}$ are odd derivations with respect to the Lie bracket.

Proposition 2.2 *Let us assume that $H^\bullet(A[[u]], d + u\Delta)$ is a free $\mathbb{C}[[u]]$-module, where u is a formal even variable. Then the formal moduli space associated with dg Lie algebra $(\mathfrak{g}, d)$ is smooth. Any trivialization of $\mathbb{C}[[u]]$-module $H^\bullet(A[[u]], d+u\Delta)$ gives a formal affine structure ("flat coordinates") on the moduli space.*

The proof of the above proposition can be found e.g. in [3], (also see [1] for a slightly weaker result). The Tian-Todorov theorem follows from the Proposition, applied to

$$A_X := \Gamma(X, \Omega^{0,\bullet} \otimes_{\mathcal{O}_X} \wedge^\bullet T_X)$$

which is the algebra of $\bar\partial$-forms on X with values in polyvector fields. The differential d is $\bar\partial$, and the operator Δ is the divergence with respect to the holomorphic volume form on X. The freeness property of the cohomology with respect to the deformed differential follows from the $\partial\bar\partial$-lemma.

3 Generalizations

Instead of an individual Calabi-Yau manifold X we can consider:

1. a pair (X, D) where X is smooth projective variety (typically X is Fano), and $D \subset X$ is a divisor with normal crossing such that $[D] = -c_1(X) \in \mathrm{Pic}(X)$,

2. a pair (X, D) where X is a Calabi-Yau manifold, $c_1(X) = 0 \in \mathrm{Pic}(X)$, and $D \subset X$ is a divisor with normal crossings,

3. a triple $(X, (D_i)_{i\in I}, (a_i)_{i\in I})$ where X is a smooth projective variety, $(D_i)_{i\in I}$ is a finite collection of irreducible divisors whose union is a divisor with normal crossings, and $(a_i)_{i\in I}$ is a collection of rational numbers $0 < a_i < 1 \ \forall i \in I$ such that

$$\sum_{i\in I} a_i[D_i] = -c_1(X) \in \mathrm{Pic}(X) \otimes \mathbb{Q}$$

4. a pair (X, W) where X is a smooth quasi-projective variety with $c_1(X) = 0 \in \mathrm{Pic}(X)$ and $W : X \to \mathbb{A}^1$ is a proper map.

5. "broken Calabi-Yau variety" X, a singular projective scheme which is a reduced divisor with normal crossing in a larger smooth non-proper variety Y with $c_1(Y) = 0$, given by $X = W_Y^{-1}(0)$ where $W_Y : Y \to \mathbb{A}^1$ is a proper map.

All these examples can be merged together, i.e. one can consider broken non-compact X with a proper map to $\mathbb{A}^1$ and a fractional divisor with weights in $[0,1] \cap \mathbb{Q}$ representing $-c_1(X)$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$.

The proof of the classical Tian-Todorov theorem presented in the previous section, extends immediately to all cases. The dg BV algebra in cases 1,2,3 is

$$A_{X,D} := \Gamma(X, \Omega^{0,\bullet} \otimes_{\mathcal{O}_X} \wedge^\bullet T_{X,D})$$

where $T_{X,D}$ is the sheaf of holomorphic vector fields on X preserving D. The differential d is given by $\bar{\partial}$, and operator Δ is the divergence with respect to a (multi-valued) holomorphic volume form on $X \setminus D$. The contraction of these polyvector fields with the volume form gives the $\bar{\partial}$-resolution of the sheaf of holomorphic forms on X which either have poles of first order on D (case 1), vanish on D (case 2), or take values in a local system with finite monodromy (case 3). The freeness property of cohomology follows from the theory of mixed Hodge structures.

The mirror symmetry for Calabi-Yau manifolds generalizes to some of our examples. The case 1 with smooth D is dual to the case 4, e.g. $X = \mathbb{CP}^n$ with a smooth anticanonical hypersurface $D \subset X$ of degree $n+1$, is mirror dual to $(X^\vee, W^\vee)$ where $X^\vee$ is a partial compactification of $\mathbb{G}_m^n$ endowed with a function

$$W(x_1, \dots, x_n) = x_1 + \dots + x_n + \frac{1}{x_1 \dots x_n}.$$

Similarly, the case 2 with smooth D is dual to the case 5, e.g. the pair (X, D) where X is an elliptic curve and $D \subset X$ is a collection of k points, it mirror dual to a singular elliptic curve $X^\vee$ with double points, which is a wheel of k copies of $\mathbb{CP}^1$. One of the corollaries of the mirror symmetry is that the mapping class group of the open surface $X - D$ acts by automorphisms of $D^b(\mathrm{Coh}(X^\vee))$ (modulo powers of the shift functor).

I do not know what are mirror partners for cases 1 and 2 with a non-smooth divisor D, and also for the case 3.

4 Non-compact Calabi-Yau manifolds

Let X be a smooth projective manifold with a section of its anti-canonical bundle which vanish with multiplicities strictly > 1 at a divisor $D \subset X$ with normal crossings. On the complement $X \setminus D$ we have a non-vanishing holomorphic volume element Ω. We can define a dg BV algebra associated with X and Ω to be a subalgebra of $A_{X,D}$ consisting of such elements for which the contraction with Ω produces a form with logarithmic poles at D. Hence we obtain again certain smooth moduli spaces. Here is one important class of examples: let $f = f(x, y)$ be polynomial defining a smooth curve in $\mathbb{C}^2$. We associate with it a non-compact 3-dimensional Calabi-Yau manifold $Y \subset \mathbb{C}^4$ given by the equation

$$uv = f(x, y)\,.$$

One can show that Y can be represented as a complement $X \setminus D$ of the type described above. Hence we obtain a smooth moduli space. E.g. for the case of hyperelliptic curve $f(x, y) = y^2 + a_0 + a_1 x + \cdots + a_{2g} x^{2g} + x^{2g+2}$ the universal family is obtained by variations of coefficients $a_0, \ldots, a_g$. The flat coordinates on the moduli space are associated with an appropriate splitting of the Hodge filtration, and are exactly those which appear in the matrix models, see e.g. [2].

5 Speculations about Calabi-Yau motives

The construction presented above gives many examples of variations of (mixed) Hodge structures of Calabi-Yau type over smooth bases. This leads to the following question, which I formulate for simplicity only in the pure case.

Question 5.1 *Let H be an absolutely indecomposable pure Hogde structure of weight w of algebro-geometric origin with coefficients in a number field (i.e. H is a direct summand of the cohomology space of some smooth projective variety), and such that there exists $k \in \mathbb{Z}$ such that H is of Calabi-Yau type, i.e.*

$$\operatorname{rk} H^{k,w-k} = 1, \;\; H^{k',w-k'} = 0 \quad \forall k' > k\ .$$

Does there exists a smooth universal family of variations of H of an algebro-geometric origin, of dimension equal to $\operatorname{rk} H^{k-1,w-k+1}$?

There are many examples supporting this, e.g. one can take H to be the primitive part of the middle cohomology of hypersurface $X \subset \mathbb{CP}^{N-1}$ of degree $d|N$. The proof of the generalized Tian-Todorov theorems does not apply in this case, but still the dimension of the moduli space and of the corresponding Hodge component match. It would be wonderful if the answer to the question is positive. It means that we have nice smooth moduli stacks of Calabi-Yau motives (generalizations of Shimura varieties). With any pure Hodge structure H one can associate another Hodge structure of Calabi-Yau type (maybe decomposable), by taking the exterior power $\wedge^m H$ where $m \in \mathbb{Z}_+$ is the dimension of a term $F^l H$ of the Hodge filtration of H, i.e. $m = \mathrm{rk} \oplus_{k' \geq l} H^{k', w-k'}$. In the case $H = H^1(C)$ where C is a smooth projective curve of genus g, the absolutely indecomposable summand H' of $\wedge^g H$ containing the one-dimensional component $\wedge^g H^{1,0}$, is a Hodge structure of Calabi-Yau type varying over an appropriate Shimura variety. One can check that the dimension of this variety always coincides with the corresponding Hodge number of H'.

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THOMAE'S FORMULA FOR TRIPLE COVERING AND BINARY GRAPH (松本圭司氏との共同研究)

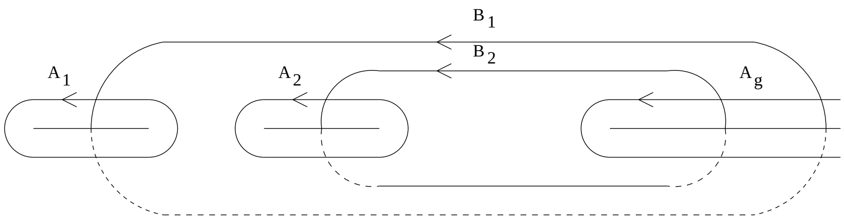
寺杣 友秀

1. Introduction-超楕円曲線の Thomae の公式

g を 2 以上の自然数とし $\lambda_1, \ldots, \lambda_{2g+1}$ を $\lambda_1 < \cdots < \lambda_{2g+1}$ なる実数とする。C を

$$C : y^2 = \prod_{i=1}^{2g+1}(x - \lambda_i)$$

で定まる超楕円曲線とし、C の上の homology の symplectic base $A_1, \ldots, A_g, B_1, \ldots, B_g$ を下の図のように定める。



また $\omega_i = \dfrac{x^{i-1}dx}{y}$ とおくと $\omega_1, \ldots, \omega_g$ は正則微分形式の基底となる。さらにこれらの基底に関する周期行列 P_A, P_B を

$$P_A = \left(\int_{A_i} \omega_j\right)_{i,j=1,\ldots,g}, \quad P_B = \left(\int_{B_i} \omega_j\right)_{i,j=1,\ldots,g} \tag{1.1}$$

として定め、正規化された周期行列 τ を

$$\tau = P_A \cdot P_B^{-1} \tag{1.2}$$

と定める。このとき τ は $\tau = {}^t\tau, Im(\tau) > 0$ を満たし、g 次の Siegel 上半空間の点を定める。$\Lambda = (\Lambda_1, \ldots, \Lambda_{2g}) = (\Lambda', \Lambda'') \in \mathbf{Q}^g \oplus \mathbf{Q}^g$ に対して characteristic が Λ の Theta constant$\vartheta(\tau, \Lambda)$ を

$$\vartheta(\tau, \Lambda) = \sum_{n \in \mathbf{Z}^g} \exp(2\pi\sqrt{-1}(\frac{1}{2}(n + \Lambda')\tau^t(n + \Lambda') + (n + \Lambda')^t\Lambda''))$$

と定める。$2\Lambda \in \mathbf{Z}^{2g}$ であれば $\vartheta^4(\tau, \Lambda)$ は Λ の $\mathbf{Z}^{2g}$ による剰余類のみで定まる。$2\Lambda \in \mathbf{Z}^{2g}$ のとき $4\Lambda' \cdot {}^t\Lambda''$ の偶奇によりそれぞれ Λ を even theta characgeristic, odd theta charcteristc という。odd theta characteristic Λ に対する theta constant は消えている。even theta characteristic のうちで消えてい

ないものの値を与えるのがThomaeの公式である。半整数の theta characteristic に対して $\{1,\ldots,2g+1\}$ の部分集合集合 S が [M],p.106 によるやり方でただ一通りに定まる。$U=\{1,3,5,\cdots,2g+1\}$ とおき、$\{1,2,\cdots,2g+1\}$ の部分集合 P,Q に対しての対称差 $P\circ Q$ を $(P-Q)\cup(Q-P)$ によって定義する。

定理 1.1 (Thomae の公式 [T],[F],[M])**.** $\#S\circ U=g+1$ とする。以上の記号のもとで次の式が成立する。

$$\vartheta(\tau,\Lambda)^4=\frac{\pm 1}{(2\pi)^g}\det(P_B)^2\prod_{p,q\in S\circ U,p<q}(\lambda_p-\lambda_q)\cdot\prod_{p,q\notin S\circ U,p<q}(\lambda_p-\lambda_q)$$

2. Bershadski-Radul-Nakayashiki による triple cyclic covering に対する Thomae の公式

Bershadski-Radul-Nakayashiki により $\mathbf{P}^1$ の特別なタイプの d-重巡回被覆における Thomae の公式の類似物が与えられた。ここでは $d=3$ のときに限ってそれを紹介しよう。

$\lambda_1,\ldots,\lambda_{3n}$ を $\mathbf{C}$ の異なる点として $\mathbf{P}^1$ の 3 次の巡回被覆 C を

$$C:y^2=\prod_{i=1}^{3n}(x-\lambda_i)$$

によって定める。このようなタイプの 3 次被覆を branching index $(1/3^{3n})=(1/3,\ldots,1/3)$ の被覆という。このとき C の種数は $g=3n-2$ となる。C の正則微分形式の基底として次の微分形式 $\omega_1,\cdots,\omega_{3n-2}$ がとれる。

$$\omega_i=\begin{cases}\dfrac{x^{i-1}dx}{y} & \text{for } i=1,\ldots,n-1\\ \dfrac{x^{i-n}dx}{y^2} & \text{for } i=n,\ldots,3n-2.\end{cases}\tag{2.1}$$

C には $\mu_3=\{\omega\in\mathbf{C}\mid\omega^3=1\}$ が $x\mapsto x,y\mapsto\omega y$ によって作用するので $H_1(C,\mathbf{Q}/\mathbf{Z})$ にも μ_3 が作用する。$\omega=\dfrac{-1+\sqrt{-3}}{2}$ として定まる自己同型を ρ と書き、$H_1(C,\mathbf{Q}/\mathbf{Z})$ の $(1-\rho)$-torsion point を $H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ と書く。$H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ 構造は次のように記述される。$\mathbf{P}^1$ の起点 b をとりそれを C に持ち上げたものを $\tilde{b}$ と書く。b を起点として λ_i のまわりを反時計廻りに小さくまわってもとに戻る道を γ_i とする。ただし $\gamma_1,\cdots,\gamma_{3n}$ を合成したものは可縮であるようにとっておく。γ_i を $\tilde{b}$ を起点として C に持ち上げたものを $\widetilde{\gamma_i}$ とかく。このとき $\frac{1}{3}(1-\rho)(\widetilde{\gamma_i}-\widetilde{\gamma}_{3n})$ は $H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ の元を与える。$e_i(i=1,\ldots,3n)$ を基底とする $\mathbf{F}_3$ 上のベクトル空間 $\oplus_{i=1}^{3n}\mathbf{F}_3e_i$ の部分空間

$$Z=\{k_ie_i\mid\sum_i k_i\equiv 0(\bmod 3)\}$$

の元 e_i-e_{3n} に対して $\frac{1}{3}(1-\rho)(\widetilde{\gamma_i}-\widetilde{\gamma}_{3n})$ を対応させることにより $Z\to H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ なる写像が定まる。この対応により次の同型写像を得る。

$$H(\mathbf{F}_3\xrightarrow{\alpha}\oplus_{i=1}^{3n}\mathbf{F}_3e_i\xrightarrow{\beta}\mathbf{F}_3)\simeq H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$$

ここで α は $1 \mapsto \sum_{i=1}^{3n} e_i$ で与えられる写像、β は $\sum_i k_i e_i \mapsto \sum_i k_i$ で与えられる写像で H は homology を表す。$\widetilde{\Lambda} = \sum_i k_i e_i$ を Z の元とするとき $\widetilde{\Lambda}_j \subset \{1, \dots, 3n\}$ $(j = 0, 1, 2)$ を

$$\widetilde{\Lambda}_j = \{i \in \{1, \dots, 3n\} \mid k_i \equiv j \pmod 3\}$$

と定める。

次に $\Lambda \in H_1(C, \mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ に対応する差積 $\Delta(\Lambda)$ を次のように定義する。まず Λ の Z へのもちあげ $\widetilde{\Lambda}$ をとってきて $\widetilde{\Lambda}_j$ $(j = 0, 1, 2)$ を上のように定義する。そして $(\widetilde{\Lambda}_i, \widetilde{\Lambda}_j)$ を

$$(\widetilde{\Lambda}_i, \widetilde{\Lambda}_j) = \begin{cases} \prod_{p \in \widetilde{\Lambda}_i, q \in \widetilde{\Lambda}_j} (\lambda_p - \lambda_q) & \text{if } (i \neq j) \\ \prod_{p, q \in \widetilde{\Lambda}_i, p < q} (\lambda_p - \lambda_q) & \text{if } (i = j) \end{cases}$$

と定義し、

$$\Delta(\Lambda) = (\widetilde{\Lambda}_0, \widetilde{\Lambda}_1)(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2)(\widetilde{\Lambda}_2, \widetilde{\Lambda}_0)(\widetilde{\Lambda}_0, \widetilde{\Lambda}_0)^3(\widetilde{\Lambda}_1, \widetilde{\Lambda}_1)^3(\widetilde{\Lambda}_2, \widetilde{\Lambda}_2)^3$$

と定めると、これは持ち上げ $\widetilde{\Lambda}$ の取り方によらない事がわかる。

最後に C の symplectic base $A_1, \cdots, A_g, B_1, \dots, B_g$ を固定して theta constant との関係をのべることにしよう。

$$\mathbf{Z}^{2g} \xrightarrow{\simeq} H_1(C, \mathbf{Z}) : (a_i, b_i)_i \mapsto \sum_{i=1}^{g} (a_i A_i + b_i B_i)$$

なる同型が定まるので、この同型により

$$H_1(C, \mathbf{Q}/\mathbf{Z})_{(1-\rho)} \subset (\frac{1}{3}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$$

なる単射ができる。このときリーマン定数 ρ が $(\frac{1}{2}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$ の元として定まる。上の symplectic base $A_1, \dots, A_g, B_1, \dots, B_g$ と正則微分形式 (2.1) を使って式 (1.1) により P_A, P_B を定め、これらを用いて正規化された周期行列 τ を式 (1.2) により定める。

上の記号の準備のもとで Bershadski-Radul-Nakayashiki による Thomae の定理の類似は次のように述べられる。

定理 2.1 (Bershadski-Radul-Nakayashiki)**.** 上記の記号のもとで $\#\Lambda_0 = \#\Lambda_1 = \#\Lambda_2 (= n)$ とする。このとき

(1) $\Lambda + \rho \in (\frac{1}{6}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$ の $\mathbf{Q}^{2g}$ への持ち上げを $\tilde{l}$ とすると $\vartheta(\tau, \tilde{l})^6$ はその持ち上げ方によらない。この $\vartheta(\tau, \tilde{l})^6$ を $\vartheta(\tau, \Lambda + \rho)^6$ と書く。

(2)
$$\vartheta(\tau, \Lambda + \rho)^6 = C_\Lambda \det(P_B)^3 \cdot \Delta(\Lambda)$$
が成り立つ。ここで C_Λ は曲線 C によらない絶対定数である。

この報告では次の問題を考える。

Problem 2.2. (1) 上の定理に現れる絶対定数はいくつになるか？

(2) *branching index* が $(1/3^{3n})$ でない一般の場合ではどうなっているか？

3. $\mathbf{P}^1$ の巡回 3 重被覆と BINARY TREE

この章以降は一般の branching index の場合を考える。$\lambda_1, \ldots, \lambda_n$ を相異なる複素数とする。さらに $a_1, \ldots, a_n \in \{1, 2\}$ で $\sum_{i=1}^n a_i \equiv 0 \pmod 3$ を満たすものとする。C を

$$y^3 = \prod_{i=1}^n (x - \lambda_i)^{a_i} \tag{3.1}$$

によって定義された曲線とする。この曲線の λ_i における branching index を $a_i/3$ と定義する。このタイプの曲線を branch index が $(a_1/3, \ldots, a_n/3)$ の曲線という。

前章と同様にして $H_1(C, \mathbf{Z})$ に μ_3 の作用が自然に定まるので、その生成元の作用 ρ も同様に定まる。$H_1(C, \mathbf{Q}/\mathbf{Z})$ の $(1-\rho)$-torsion element の集合をやはり同様に $H_1(C, \mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ と書く。まずこの群を分岐点の集合を用いて記述する。$\mathbf{F}_3$ 線形写像 α, β を

$$\alpha : \mathbf{F}_3 \to \oplus_{i=1}^n \mathbf{F}_3 e_i : 1 \mapsto \sum_{i=1}^n e_i$$

$$\beta : \oplus_{i=1}^n \mathbf{F}_3 e_i \to \mathbf{F}_3 : \sum_{i=1}^n k_i e_i \mapsto \sum_{i=1}^n a_i e_i$$

によって定めると列

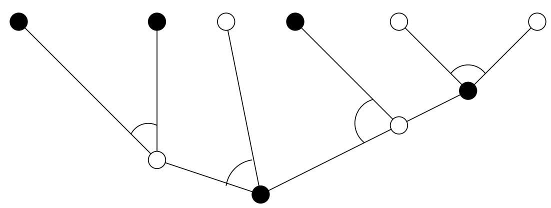
$$\mathbf{F}_3 \xrightarrow{\alpha} \oplus_{i=1}^n \mathbf{F}_3 e_i \xrightarrow{\beta} \mathbf{F}_3$$

は complex となるが、この complex を用いて

$$H(\mathbf{F}_3 \xrightarrow{\alpha} \oplus_{i=1}^n \mathbf{F}_3 e_i \xrightarrow{\beta} \mathbf{F}_3) \simeq H_1(C, \mathbf{Q}/\mathbf{Z})_{(1-\rho)} \tag{3.2}$$

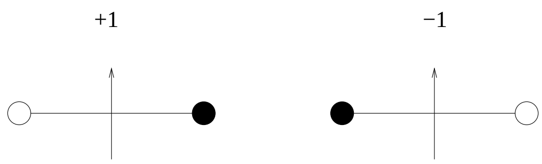
なる同型を得る。

次に binary tree と C の homology の symplectic base について述べよう。ここで tree は planar tree つまり各頂点には巡回順序が定まっている tree であって、trivalent、すなわち各内頂点 (inner vertex) からは丁度 3 本の edge が出ていているものを言う。さらに binary というのは各頂点に 2 つの色 (ここでは白と黒) がついていて edge で結ばれている頂点には異なる色がつけられているものとする。また binary tree の marking とは各内頂点から出ている 3 本の edge のうちの 2 つの edge が指定されているものとする。この指定は図形的には孤を結ぶことによってあらわす。下は maked binary tree の例である。

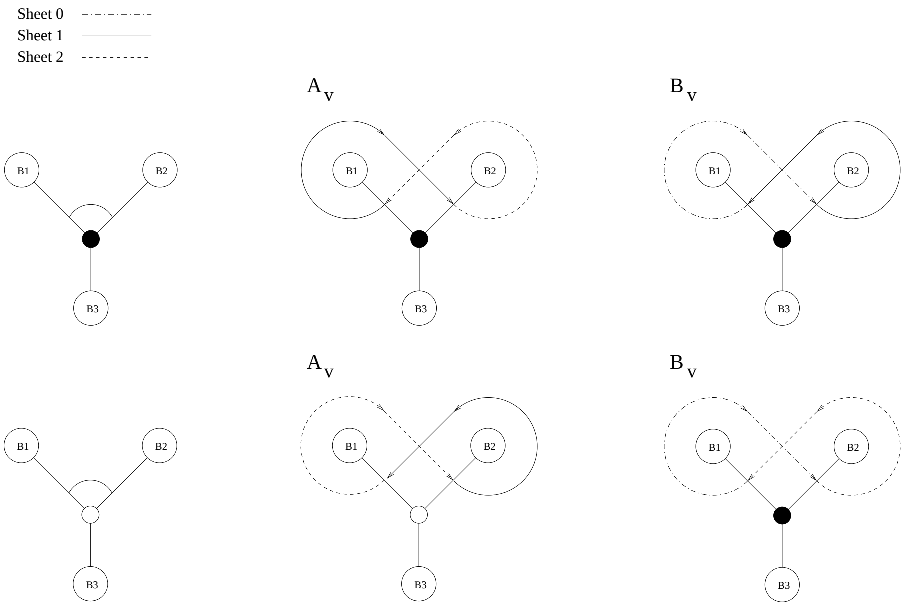


marked binary tree を端頂点 (outer vertex) が丁度 $\lambda_1, \ldots, \lambda_n$ になるように $\mathbf{C}$ の上に書く。ただし λ_i の branching index が 1/3 の時は白い頂点、2/3 の時は黒い頂点が書かれているようになっているものとする。$\mathbf{P}^1$ のコピー 3 枚を (コ

ピーは $\mathbf{Z}/3$ によって番号付けされているものとする) このグラフに沿って切れ目をいれて、貼り付ける。このときの貼り合わせのルールは黒い頂点を左に見ながら edge を横切る時にはコピーの番号を一つ増やし、右に見ながら横切る時にはひとつ減らすように別のシートに移ってゆくというものである。(下図) このようにして貼り付けたものが C を与えている。



上の表示を用いて C の symplectic base を構成しよう。まず、各 inner vertex v に対して、そこでの marking を用いて C の topological cycle A_v, B_v を定義する。vertex v の色に応じて定義は異なるが、その定義を下に図示する。ここで一点鎖線、実線、点線はそれぞれ 0 番目、1 番目、2 番目のシートの上の道を表す。



このとき $\{A_v, B_v\}_v$ は symplectic base となり、v が白い inner vertex であれば $\rho^2(A_v) = B_v$、黒ならば $\rho(A_v) = B_v$ となる。inner vertex の数が C の genus $g(C)$ となり、下の同型が得られる。

$$\iota : \mathbf{Z}^g \oplus \mathbf{Z}^g \to H_1(C, \mathbf{Z}) : (p_v, q_v)_{v:\text{inner vertex}} \mapsto \sum_v (p_v A_v + q_v B_v) \tag{3.3}$$

また Riemann constant は $\rho = \dfrac{1}{2}\sum_v (A_v + B_v)$ で与えられる。A_v, B_v からなる symplectic base は Riemann constant ρ は marked binary tree の取り方に depend する。以下 marked binary tree は fix して考える。

4. 微分形式と周期行列

C の微分形式について述べる。λ_i における branching index を $a_i/3$ とし、$b_i = 3 - a_i$ とおく。方程式 (3.1) における y を y_1 とおいて、$y_2 = \dfrac{1}{y}\prod_i (x - \lambda_i)$ とおく。すなわち

$$y_2^3 = \prod_{i=1}^{n} (x - \lambda_i)^{b_i}$$

となる。いま $d_1 = \sum_i (a_i/3) - 1, d_2 = \sum_i (b_i/3) - 1$ とおくと、$d_1 + d_2$ 個の微分形式 η_j を

$$\begin{aligned} \eta_j &= \frac{x^{i-1}dx}{y_1} \quad \text{for } j = 1, \ldots, d_1 \\ \eta_{j+d_1} &= \frac{x^{i-1}dx}{y_2} \quad \text{for } j = 1, \ldots, d_2. \end{aligned}$$

と定めると、これらは C の正則微分形式の基底となることがわかる。したがって $H^0(C, \Omega_C^1)$ の次元は $d_1 + d_2$ で $\dim(H^0(C, \Omega_C^1)(\chi) = d_2, \dim(H^0(C, \Omega_C^1)(\overline{\chi}) = d_1$ となる。ここで χ は μ_3 の自然な指標、$H^0(C, \Omega_C^1)(\chi)$ は指標 χ で作用する $H^0(C, \Omega_C^1)$ の部分空間等である。d_1, d_2 は白あるいは黒の inner vertex の数にもなっている。前章で定義した symplectic base および微分形式 η_i $(i = 1, \ldots, g = d_1 + d_2)$ を用いて周期行列 P_A, P_B、および正規化された周期行列 τ を式 (1.1),(1.2) によって定義する。

5. 3次巡回被覆に対する Thomae の公式

それでは Thomae の公式を述べることにしよう。同型 (3.2), (3.3) より得られる次の写像を考える。

$$\text{(5.1)} \qquad \mathrm{Ker}(\beta) \to H_1(C, \mathbf{Q}/\mathbf{Z})_{(1-\rho)} \xrightarrow{\subseteq} H_1(C, \frac{1}{3}\mathbf{Z}/\mathbf{Z})_{(1-\rho)} \overset{\iota^{-1}}{\simeq} (\frac{1}{3}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$$

$\widetilde{\Lambda} = \sum_i k_i e_i$ を $\mathrm{Ker}(\beta)$ の元とする。$\widetilde{\Lambda}^{\pm}$ を

$$\widetilde{\Lambda}^{\pm} = \{i \in \{1, \ldots, n\} \mid a_i = \pm 1 \pmod 3\}$$

と定義し、さらに $\widetilde{\Lambda}_j^{\pm}$ を

$$\widetilde{\Lambda}_j^{\pm} = \{i \in \widetilde{\Lambda}^{\pm} \mid k_i = j \pmod 3\}$$

と定義する。S_1, S_2 を $\widetilde{\Lambda}_j^{\pm}$ のどれかとして (S_1, S_2) を

$$(S_1, S_2) = \begin{cases} \prod_{p \in S_1, q \in S_2} (\lambda_p - \lambda_q) & \text{if } S_1 \neq S_2 \\ \prod_{p, q \in S_1, p < q} (\lambda_p - \lambda_q) & \text{if } S_1 = S_2 \end{cases}$$

と定義する。このとき $\widetilde{\Lambda}\in \mathrm{Ker}(\beta)$ に対して

$$\Delta(\widetilde{\Lambda})=\prod_i(\tilde{\Lambda}_i^+,\tilde{\Lambda}_i^+)^3(\tilde{\Lambda}_i^-,\tilde{\Lambda}_i^-)^3\prod_{i\neq j}(\tilde{\Lambda}_i^+,\tilde{\Lambda}_j^-)^2$$
$$\prod_{i<j}(\tilde{\Lambda}_i^+,\tilde{\Lambda}_j^+)(\tilde{\Lambda}_i^-,\tilde{\Lambda}_j^-)$$

とおくと $\Delta(\widetilde{\Lambda})$ は $H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ における $\widetilde{\Lambda}$ の類 Λ のみによることがわかる。これを $\Delta(\Lambda)$ と書くことにする。

$\mathrm{Ker}(\beta)$ の元 $\widetilde{\Lambda}$ が条件

$$\#\widetilde{\Lambda}_0^+-\#\widetilde{\Lambda}_0^-=\#\widetilde{\Lambda}_1^+-\#\widetilde{\Lambda}_1^-=\#\widetilde{\Lambda}_2^+-\#\widetilde{\Lambda}_2^- \tag{5.2}$$

を満たしているか否かは、その $H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ における類のみにより決定される。$H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ の元 Λ の持ち上げ $\widetilde{\Lambda}$ がこの条件を満たしているとき Λ は equidistributed であるという。

以上の準備のもとで、次の定理が成立する。

定理 5.1 (Matsumoto-T.)**.**

Λ を equidistributed な $H_1(C,\mathbf{Q}/\mathbf{Z})_{(1-\rho)}$ の元とする。

(1) Λ を写像 (5.1) によって $(\frac{1}{3}\mathbf{Z})^{2g}/\mathbf{Z}^{2g}$ の元とみる。$\hat{\Lambda}$ を Λ の $(\frac{1}{3}\mathbf{Z})^{2g}$ への持ち上げとする。このとき $\vartheta(\tau,\hat{\Lambda}+\rho)^6$ はその持ち上げの仕方によらない。ここで ρ はリーマン定数である。これを単に $\vartheta(\Lambda+\rho)$ と書く。

(2) (Thomae の公式の類似)

$$\vartheta(\tau,\Lambda+\rho)^6=\pm C_\Lambda\cdot\det(P_B)^3\cdot\Delta(\Lambda)$$

が成立する。ここで C_Λ は Λ のみによる absolute constant で

$$C_\Lambda^6=[(2\pi)^3 3^{3/4}\exp(\frac{11}{12}\pi\sqrt{-1})]^{-6g}.$$

となる。

6. 証明のアウトライン

証明は Bershadski-Radul-Nakayashiki の場合の 3 重被覆の曲線の退化を考え一般の branching index の場合の定理を absolute constant の部分を除いて証明する。Absolute constant を決定する部分についても 3 重被覆の退化を考え楕円曲線の場合に帰着させる。楕円曲線の場合は 1 の 3 乗根における theta constant とガンマ関数の比の計算に帰着する。これは Chowla-Selberg の公式として知られている。

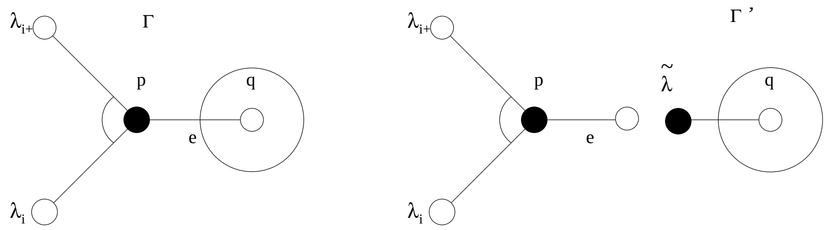
ここでは証明の中心となる曲線の退化の様子についてのべることにしよう。t を 0 に近い複素数として分岐点 $\lambda_1,\dots,\lambda_n$ およびそこに outer vertex をもつ marked binary tree を変形することを考える。ここで簡単のため $\lambda_1,\cdots,\lambda_n$ は、この順番で tree の planar graph より導かれる outer vertex の cyclic order の順番になっていて λ_i と λ_{i+1} は inner vertex p に edge で結ばれているとする。$\mathbf{C}$ の点 $\tilde{\lambda}$ を一つ選んで、$k=i,i+1$ として分岐点 $\lambda_k(t)$ を

$$\lambda_i(t)=\tilde{\lambda}+t(\lambda_k-\tilde{\lambda})$$

と変形する。このとき分岐点の変形 $\lambda_1, \cdots, \lambda_i(t), \lambda_{i+1}(t), \cdots, \lambda_n$ に従ってその3次被覆 C も変形する。この変形は原点にも安定曲線に完備化され、特異ファイバーは二つの連結成分をもつ。連結成分の一つは上の座標でただ $t=0$ の極限をとったもので、$\mathbf{P}^1$ 内の $\lambda_1, \cdots, \lambda_{i-1}\tilde{\lambda}, \lambda_{i+2}, \cdots, \lambda_n$ で分岐した3次巡回被覆でこれを C_2 とおく。もう一つの成分は $\xi = (x-\tilde{\lambda})/t$ なる $\mathbf{P}^1$ の座標に変換することにより見やすくなる。この座標変換で分岐点 $\lambda_k(t)$ は $\lambda_k'(t)$ に変換される。ここで

$$\lambda_k'(t) = \begin{cases} (\lambda_k - \tilde{\lambda})/t & \text{if } k \neq i, i+1 \\ \lambda_i - \tilde{\lambda} & \text{if } k = i, i+1 \end{cases}$$

である。従って、特異ファイバーのもう一つの成分は $\xi = \lambda_i - \tilde{\lambda}, \lambda_{i+1} - \tilde{\lambda}, \infty$ で分岐する3次巡回被覆 C_1 となる。特異ファイバーはこの二つの成分を C_1(これは楕円曲線となる) の ∞ 上にある分岐点と C_2 の $\tilde{\lambda}$ 上にある分岐点でくっつけたものとなる。このとき曲線 C 上にある marked binary tree も下の図のように分裂する。



従ってこの分解に従って symplectic base もそれぞれの成分上の symplectic base に分解する。

微分形式の方も極限を考えると、差積 $\Delta(\Lambda)$ とキャンセルして有限の値となる。たとえば λ_i, λ_{i+1} が白の outer vertex であるとして、上の図の状況のように innter vertex p につながっていたとすると、B_p 上の $\dfrac{dx}{y_2}$ の積分は

$$\int_{B_p} \frac{dx}{y_2(t)} = \omega^i(1-\omega) \int_{\lambda_i(t)}^{\lambda_{i+1}(t)} \frac{dx}{(x-\lambda_i(t))^{2/3}(x-\lambda_{i+1}(t))^{2/3}\prod_{k\neq i,i+1}(x-\lambda_k)^{b_k/3}}$$

と計算されるが、さらに、上の座標 ξ で変換することにより

$$\begin{aligned} \lim_{t\to 0} t^{1/3} \int_{B_p} \frac{dx}{y_2(t)} =& \frac{\omega^i(1-\omega)}{\prod_{k\neq i,i+1}(\tilde{\lambda}-\lambda_k)^{b_k/3}} \int_{\lambda_i}^{\lambda_{i+1}} \frac{d\xi}{(\xi-\lambda_i)^{2/3}(\xi-\lambda_{i+1})^{2/3}} \\ =& \frac{-\omega^{i'}(1-\omega)}{(\lambda_i-\lambda_{i+1})^{1/3}\prod_{k\neq i,i+1}(\tilde{\lambda}-\lambda_k)^{b_k/3}} B(\frac{1}{3}, \frac{1}{3}). \end{aligned}$$

なる極限をえる。P_B のそれぞれの成分に対して上と同様の計算を行うと $\det(P_B)$ は適当な t のべきで割ると有限の値に近づく。差積のほうからも t のべきがでてキャンセルする。

正規化された周期は Siegel 上半空間の中の $(g-1)\times(g-1)$ 行列と 1×1 行列の直和行列に収束するので theta constant はある有限の値に収束する。このことから C に関する Thomae の公式から C_2 に関する Thomae の公式を (absolute constant の部分を除いて) 得る。さらに C と C_2 の Thomae の公式に関する absolute constant 同士の関係式が得られるので、absolute constant が決定は marked binary tree が inner vertex が 1 個のものに完全に分解されてしまったとき、すなわち $\dfrac{-1+\sqrt{-3}}{2}$ を虚数乗法にもつ楕円曲線の時に帰着される。この時は、この公式は Chowla-Selberg の公式に帰着される。

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A counterexample to a conjecture of complete fan

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Abstract

If a Griffiths domain D is a symmetric Hermitian domain, the toroidal compactification of the quotient space $\Gamma\backslash D$, associated to a projective fan and a discrete subgroup Γ of $\mathrm{Aut}(D)$, was constructed by Mumford et al. Kazuya Kato and Sampei Usui studied extensions of $\Gamma\backslash D$ for a Griffiths domain D in general, and introduced a notion of "complete fan" as a generalization of a notion of projective fan. The existence of complete fans is expected. In this paper, we give an example of D which has no complete fan.

1 Introduction

Let D be a Griffiths domain, let Γ be a "neat" discrete subgroup of $\mathrm{Aut}(D)$, and let Σ be a fan consisting of rational nilpotent cones in $\mathrm{Lie}(\mathrm{Aut}(D))$ which is "strongly compatible" with Γ. Kazuya Kato and Sampei Usui [KU] introduced the notion of "polarized logarithmic Hodge structure" and enlarged the space $\Gamma\backslash D$ to the space $\Gamma\backslash D_\Sigma$ by adding the classes modulo Γ of nilpotent orbits in the directions of cones contained in Σ as the boundary points. They proved that the space $\Gamma\backslash D_\Sigma$ is the fine moduli space of polarized logarithmic Hodge structures of type $\Phi := (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_\mathbb{Z}, \langle\ ,\ \rangle, \Gamma, \Sigma)$ ([KU] 4.2.1, Theorem B), and that $\Gamma\backslash D_\Sigma$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" caused by "Griffiths transversality" condition at the boundary ([KU] 4.1.1, Theorem A).

In the classical situation, that is, D is a symmetric Hermitian domain, the toroidal projective compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ was constructed with a sufficiently big fan Σ, called a projective fan, by A. Ash, D. Mumford, M. Rapoport and Y. S. Tai [AMRT].

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For general D, Kato and Usui introduced in [KU] a “complete fan” as a generalization of a projective fan, and they gave a conjecture of the existence of such fans ([KU] 12.6.3). As an example, they gave a concrete description of the space $\Gamma\backslash D_\Sigma$ for Hodge type $h^{2,0} = h^{0,2} = 2,\ h^{1,1} = 1$ and for $\Sigma = \Xi$; i.e., the fan consisting of all rational nilpotent cones whose rank are less than or equal to one in Lie(Aut(D)) in [KU] 12.2.2. In this case, the fan $\Sigma = \Xi$ is complete.

In the present work, we started to generalize the description of the above example, but in fact we encounter a counterexample to the conjecture of existence of complete fans. We show that D with $h^{2,0} = h^{1,1} = h^{0,2} = 2$ has no complete fans.

After the present work, a modified version of the conjecture about complete fan is added at the end of 12.7 in [KU].

We fix a 4-tuple $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_\mathbb{Z}, \langle\ ,\ \rangle)$, consisting of an integer w, Hodge number $(h^{p,q})_{p,q\in\mathbb{Z}}$, a free $\mathbb{Z}$-module $H_\mathbb{Z}$ of rank $\Sigma_{p,q}h^{p,q}$, and a non-degenerate bilinear form $\langle\ ,\ \rangle$ on $H_\mathbb{Q} := \mathbb{Q} \otimes_\mathbb{Z} H_\mathbb{Z}$ which is symmetric if w is even and skew-symmetric if w is odd. Then, let D be a classifying space of polarized Hodge structure of type Φ_0 (This is also called Griffiths domain), and let $\check{D}$ be a compactdual of D.

Let

$$G_\mathbb{Z} := \mathrm{Aut}(H_\mathbb{Z}, \langle\ ,\ \rangle),$$

and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let

$$\begin{aligned} &H_R := R \otimes_\mathbb{Z} H_\mathbb{Z},\ G_R := \mathrm{Aut}(H_R, \langle\ ,\ \rangle),\\ &\mathfrak{g}_R := \mathrm{Lie}(G_R)\\ &\quad = \{N \in \mathrm{End}_R(H_R) \mid \langle Nx, y\rangle + \langle x, Ny\rangle = 0 \ for\ all\ x, y \in H_R\}. \end{aligned}$$

2 Nilpotent orbit

In this section, we recall the definition of nilpotent orbits after [KU].

We fix $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_\mathbb{Z}, \langle\ ,\ \rangle)$ as above.

Definition 2.1 ([KU] 0.4.2, 1.3.1) *A subset σ of $\mathfrak{g}_\mathbb{R}$ is said to be a* nilpotent cone*, if the following conditions are satisfied.*

(1) $\sigma = \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ *for some* $n \geq 1$ *and for some* $N_1, \ldots, N_n \in \sigma$.

(2) *Any element of* σ *is nilpotent as an endomorphism of* $H_\mathbb{R}$.

(3) $[N, N'] = 0$ *for any* $N,\ N' \in \sigma$ *as endomorphisms of* $H_\mathbb{R}$,
where $[N, N'] := NN' - N'N$.

We recall some notion about nilpotent cones in [KU] 0.4.3, 1.3.2.

A nilpotent cone is said *rational*, if we can take $N_1, \dots, N_n \in \mathfrak{g}_{\mathbb{Q}}$ in Definition 2.1 (1).

For a nilpotent cone σ, a *face* of σ is a non-empty subset τ of σ which satisfies the following two conditions.

(1) If $x, y \in \tau$ and $a \in \mathbb{R}_{\geq 0}$, then $x + y,\ ax \in \tau$.
(2) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

Definition 2.2 ([KU] 0.4.4, 1.3.3) *A* fan in $\mathfrak{g}_{\mathbb{Q}}$ *is a non-empty set* Σ *of rational nilpotent cones in* $\mathfrak{g}_{\mathbb{R}}$ *satisfying the following three conditions:*

(1) *If* $\sigma \in \Sigma$, *any face of* σ *belongs to* Σ.
(2) *If* $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ *is a face of* σ *and of* σ'.
(3) *Any* $\sigma \in \Sigma$ *is sharp. That is,* $\sigma \cap (-\sigma) = \{0\}$.

Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$. For $R = \mathbb{R}, \mathbb{C}$, we denote by σ_R the R-linear span of $\sigma \subset \mathfrak{g}_{\mathbb{R}}$.

Definition 2.3 ([KU] 0.4.7, 1.3.7) *Let* $\sigma = \Sigma_{1 \leq j \leq r}(\mathbb{R}_{\geq 0})N_j$ *be a rational nilpotent cone. A subset* Z *of* $\check{D}$ *is said to be a* σ-nilpotent orbit *if there is* $F \in \check{D}$ *which satisfies* $Z = \exp(\sigma_{\mathbb{C}})F$ *and satisfies the following two conditions.*

(1) $N_j F^p \subset F^{p-1}$ $(1 \leq j \leq r, p \in \mathbb{Z})$.
(2) $\exp(\sum_{1 \leq j \leq r} z_j N_j)F \in D$ *if* $z_j \in \mathbb{C}$ *and* $\mathrm{Im}(z_j) \gg 0$.

The conditions (1) and (2) are called *Griffiths transversality* and *positivity*, respectively.

We say that the pair (σ, F), consisting of a rational nilpotent cone $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ and of $F \in \check{D}$, generates a nilpotent orbit if $Z = \exp(\sigma_{\mathbb{C}})F$ is a σ-nilpotent orbit.

Example 2.1 Let $w = 2$, $h^{2,0} = h^{1,1} = h^{0,2} = 2$, $h^{p,q} = 0$ otherwise, and $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with a basis $(e_j)_{1 \leq j \leq 6}$. Let $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the $\mathbb{Q}$-bilinear form defined by

$$(\langle e_i, e_j \rangle)_{1 \leq i,j \leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $H'_{\mathbb{Q}} := \bigoplus_{1 \leq j \leq 4} \mathbb{Q}e_j$. For $a \in H'_{\mathbb{Q}}$, let $N_a : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ be the nilpotent endomorphism given by

$$N_a(b) = -\langle a, b\rangle e_5 \;\; (b \in H'_{\mathbb{Q}}), \quad N_a(e_5) = 0, \quad N_a(e_6) = a.$$

Note that, for all $a, a' \in H_{\mathbb{Q}}$, $N_a, N_{a'} \in \mathfrak{g}_{\mathbb{Q}}$ and $[N_a, N_{a'}] = 0$. Let $F \in \check{D}$ be given by $F^2 = \mathbb{C}(ie_1 + e_2) \oplus \mathbb{C}e_6$, and $F^1 = (F^2)^{\perp}$. Let $\sigma = \mathbb{R}_{\geq 0}(-N_{e_3}) + \mathbb{R}_{\geq 0}N_{e_4}$. Then, (σ, F) generates a nilpotent orbit.

Definition 2.4 ([KU] 0.4.8, 1.3.8) *Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$. As a set, we define D_{Σ} by*

$$D_{\Sigma} := \{(\sigma, Z) \mid \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Note that we have the inclusion map

$$D \hookrightarrow D_{\Sigma}, \;\; F \mapsto (\{0\}, \{F\}).$$

Definition 2.5 ([KU] 0.4.10, 1.3.10) *Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$ and let Γ be a subgroup of $G_{\mathbb{Z}}$.*

(i) *We say Γ is* compatible with Σ *if the following condition* (1) *is satisfied.*

(1) *If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$. Here, $\mathrm{Ad}(\gamma)(\sigma) = \gamma\sigma\gamma^{-1}$. Note that, if Γ is compatible with Σ, Γ acts on D_{Σ} by*

$$\gamma : (\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)(\sigma), \gamma Z) \; (\gamma \in \Gamma).$$

(ii) *We say Γ is* strongly compatible with Σ *if it is compatible with Σ and the following condition* (2) *is also satisfied. For $\sigma \in \Sigma$, define*

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma).$$

(2) *The cone σ is generated by $\log \Gamma(\sigma)$, that is, any element of σ can be written as a sum of $c \log(\gamma)$ $(c \in \mathbb{R}_{\geq 0}, \; \gamma \in \Gamma(\sigma))$.*

Assume that Γ is "neat" and strongly compatible with Σ. $\Gamma\backslash D_{\Sigma}$ is a "logarithmic manifold" which is nearly a complex analytic manifold but has "slits" (see [KU]).

3 Complete fan

In this section, we recall the definition of a space D_{val} and the definition of a complete fan after [KU].

Definition 3.1 ([KU] Definition 5.3.1) *We define*

$$\mathcal{V} := \left\{ (A, V) \;\middle|\; \begin{array}{l} A \text{ is a } \mathbb{Q}\text{-linear subspace of } \mathfrak{g}_{\mathbb{Q}} \text{ consisting of} \\ \text{mutually commutative nilpotent elements,} \\ V \text{ is a valuative submonoid of } A^* := \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q}) \\ \text{with } V \cap (-V) = \{0\} \end{array} \right\}.$$

Here a submonoid V of A^* is said to be a valuative *submonoid, if* $V \cup (-V) = A^*$.

For $(A, V) \in \mathcal{V}$, let $\mathcal{F}(A, V)$ be the set of all rational nilpotent cones $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ satisfying the following (1) and (2).

(1) $\sigma_{\mathbb{R}} = A_{\mathbb{R}}$.
(2) Let $(\sigma \cap A)^{\vee} := \{h \in A^* \mid h(\sigma \cap A) \subset \mathbb{Q}_{\geq 0}\}$. Then $(\sigma \cap A)^{\vee} \subset V$.

Definition 3.2 ([KU] Definition 5.3.3) (i) *We define*

$$\check{D}_{\mathrm{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V) \in \mathcal{V}, \\ Z \textit{ is an } \exp(A_{\mathbb{C}})\textit{-orbit in } \check{D} \end{array} \right\}.$$

(ii) *We define*

$$D_{\mathrm{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V, Z) \in \check{D}_{\mathrm{val}}, \\ \textit{there exists } \sigma \in \mathcal{F}(A, V) \textit{ such that} \\ Z \textit{ is a } \sigma\textit{-nilpotent orbit} \end{array} \right\}.$$

Definition 3.3 *Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$. For $(A, V) \in \mathcal{V}$, we define*

$$X_{A,V,\Sigma} := \{\sigma \in \Sigma \mid \sigma \cap A_{\mathbb{R}} \in \mathcal{F}(A, V)\}.$$

It is known that, if $X_{A,V,\Sigma}$ is not empty, then there exists the smallest element σ_0 of $X_{A,V,\Sigma}$ ([KU] Lemma 5.3.4).

Definition 3.4 ([KU] Definition 5.3.5) *For a fan Σ in $\mathfrak{g}_{\mathbb{Q}}$, we define*

$$D_{\Sigma,\mathrm{val}} := \left\{ (A, V, Z) \;\middle|\; \begin{array}{l} (A, V, Z) \in \check{D}_{\mathrm{val}},\ X_{A,V,\Sigma} \textit{ is not empty,} \\ \exp(\sigma_{0,\mathbb{C}})Z \textit{ is a } \sigma_0\textit{-nilpotent orbit} \end{array} \right\}.$$

Here σ_0 is just as above.

Definition 3.5 ([KU] Definition 12.6.1) *A fan Σ in $\mathfrak{g}_{\mathbb{Q}}$ is* complete, *if* $D_{\mathrm{val}} = D_{\Sigma,\mathrm{val}}$.

In the case where a Griffiths domain D is a symmetric Hermitian domain, a fan Σ, used in the construction of the toroidal projective compactification $G_{\mathbb{Z}} \backslash D_{\Sigma}$ in [AMRT], is complete ([KU] 12.6.4). For general D, the existence of complete fans which are strongly compatible with $G_{\mathbb{Z}}$ was expected in [KU] conjecture 12.6.3. In the next section, we give a counterexample to that conjecture.

4 Counterexample (main result)

In this section, we state our main result. Let $w = 2$, and let $h^{p,q} = 2$ $(p+q = 2,\ p, q \geq 0)$, and $h^{p,q} = 0$ otherwise. We consider about the existence of the complete fans in this case. Let $(e_j)_{1\leq j\leq 6}$ be a free basis of $H_{\mathbb{Z}}$ and $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$ be the bilinear form on $H_{\mathbb{Q}}$ given by

$$(\langle e_i, e_j\rangle)_{1\leq i,j\leq 6} = \begin{pmatrix} -1_2 & O & O \\ O & E & O \\ O & O & E \end{pmatrix}, \text{ where } 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 4.1 *In this case, there is no complete fan.*

For the proof of Theorem 4.1, we first show that the rank of any rational nilpotent cone, which appears in a nilpotent orbit, is less than or equal to two. Next, assuming the existence of a complete fan Σ on D, we derive a contradiction: Σ has two different cones of rank two which have a common point as in each of their interiors.

5 Modified version of complete fan

In this section, we introduce the definition of modified version of complete fan. Recently, the definition of complete fan was modified by Chikara Nakayama as follows.

Definition 5.1 (Chikara Nakayama) *Let N be a set of all rational nilpotent cones which appear in a nilpotent orbit. Then, a fan Σ in $\mathfrak{g}_{\mathbb{Q}}$ is said to be* complete *if it satisfies following condition.*

$$\bigcup_{\sigma\in\Sigma} \sigma = \bigcup_{\sigma\in N} \sigma$$

By this definition, the conjecture of the existence of complete fan was modified as follows.

Conjecture 5.1 *There exists a fan in $\mathfrak{g}_{\mathbb{Q}}$ which satisfies the condition in* Definition 5.1*, and is strongly compatible with $G_{\mathbb{Z}}$.*

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Base manifolds for fibrations of projective irreducible symplectic manifolds

Jun-Muk Hwang

A connected complex manifold M of dimension $2n$ equipped with a holomorphic symplectic from $\omega \in H^0(M, \Omega^2_M)$ is called a *holomorphic symplectic manifold*. A subvariety V of M is said to be *Lagrangian* if V has dimension n and the restriction of ω on the smooth part of V is identically zero. A simply connected projective algebraic manifold M is called a *projective irreducible symplectic manifold* if M has a symplectic form ω such that $H^0(M, \Omega^2_M) = \mathbf{C}\omega$. It is remarkable that fibrations of projective irreducible symplectic manifolds are of very special form, as described in the following theorem due to D. Matsushita.

Theorem 1 *Let M be a projective irreducible symplectic manifold of dimension $2n$. For a projective manifold X and a surjective holomorphic map $f : M \to X$ with connected fibers of positive dimension, the following holds.*

(1) X is a Fano manifold of dimension n with Picard number 1.

(2) A general fiber of f is biholomorphic to an abelian variety.

(3) The underlying subvariety of every fiber of f is Lagrangian.

(4) All even Betti numbers of X are equal to 1 and all odd Betti numbers of X are equal to 0.

(1), (2) and (3) in Theorem 1 were proved in [Ma1] and [Ma2]. These results led to the question whether the base manifold X is the complex projective space (cf. [Hu, 21.4]). The result of [Ma3] verifies Theorem 1 (4), i.e., that the Betti numbers of X are indeed equal to those of $\mathbf{P}_n$.

Our goal is to give an affirmative answer to the question as follows.

Theorem 2 *In the setting of Theorem 1, X is biholomorphic to $\mathbf{P}_n$.*

There are two geometric ingredients in the proof of Theorem 2: the theory of varieties of minimal rational tangents and the theory of Lagrangian fibrations. On the one hand, the theory of varieties of minimal rational tangents describes a certain geometric structure arising from minimal rational curves at general points of a Fano manifold X with $b_2(X) = 1$ (cf. [HwMo1], [HwMo2]). This geometric structure has differential geometric properties reflecting special features of the deformation theory of minimal rational curves. On the other hand, the theory of Lagrangian fibrations, or equivalently, the theory of completely integrable Hamiltonian systems, provides an affine structure at general points of the base manifold X via the classical action variables (cf. [GuSt, Section 44]). Our strategy to prove Theorem 2 is to exploit the interplay of these two geometric structures on the base manifold X. Under the assumption that X is different from $\mathbf{P}_n$, the condition $b_2(X) = 1$ forces the geometric structure arising from the variety of minimal rational tangents to be 'non-flat', while the affine structure arising from the action variables is naturally 'flat'. These two structures interact via the monodromy of the Lagrangian fibration, leading to a contradiction. To be precise, two separate arguments are needed depending on whether the dimension p of the variety of minimal rational tangents is positive or zero. The easier case of $p > 0$ is handled by a topological argument using $b_4(X) = 1$, using the result of [Hw]. The more difficult case of $p = 0$ needs a deeper argument, depending on the local differential geometry of the variety of minimal rational tangents.

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ON DEFORMATIONS OF LAGRANGIAN FIBRATIONS

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ABSTRACT. Let X be an irreducible symplectic manifold and $\mathrm{Def}(X)$ the Kuranishi family. Assume that X admits a Lagrangian fibration. We prove that there exists a smooth hypersurface H of $\mathrm{Def}(X)$ such that the restriction family $\mathscr{X} \times_{\mathrm{Def}(X)} H$ admits a family of Lagrangian fibrations over H.

1. INTRODUCTION

A compact Kähler manifold X is said to be *symplectic* if X carries a holomorphic symplectic form. Moreover X is said to be *irreducible symplectic* if X satisfies the following two properties:

(1) $\dim H^0(X.\Omega_X^2) = 1$ and;
(2) $\pi_1(X) = \{1\}$.

A surjective morphism between Kähler spaces is said to be *fibration* if it is surjective and has only connected fibres. A fibration from a symplectic manifold is said to be *Lagrangian* if a general fibre is a Lagrangian submanifold. The plainest example of an irreducible symplectic is a $K3$ surface. An elliptic fibration from a $K3$ surface gives an example of a Lagrangian fibration. It is expected that a $K3$ surface and an irreducible symplectic manifold share many geometric properties. Let S be a $K3$ surface and $g : S \to \mathbb{P}^1$ an elliptic fibration. Kodaira proves that there exists a smooth hypersurface H_S in the Kuranishi space $\mathrm{Def}(S)$ of S which has the following three properties:

(1) The hypersurface H_S passes the reference point.
(2) For the Kuranishi family $\mathscr{S}$ of S, the base change $\mathscr{S} \times_{\mathrm{Def}(S)} H_S$ admits a surjective morphsim over $\mathbb{P}^1_{H_S}$. Moreover they satisfy the following diagram:

$$\begin{tikzcd} \mathscr{S} \times_{\mathrm{Def}(S)} H_S \arrow[r] \arrow[d] & \mathbb{P}^1_{H_S} \arrow[dl] \\ H_S. & \end{tikzcd}$$

(3) The original fibration g coincides with the restriction of the above diagram over the reference point. The restriction of the diagram over a every point of H_S gives an elliptic fibration.

The following is the main theorem, which induces a higher dimensional analog of the above statement.

THEOREM **1.1.** *Let X be an irreducible holomorphic symplectic manifold and $\mathscr{X} \to \mathrm{Def}(X)$ the Kuranishi family of X. Assume that X admits a Lagrangian fibration $f : X \to B$ over a projective variety B. Let L be a line bundle which is a pull back of an ample line bundle on B. Then we have a smooth hypersurface H of $\mathrm{Def}(X)$ and a line bundle $\mathscr{L}$ on $\mathscr{X} \times_{\mathrm{Def}(X)} H$ which satisfies the following two properties:*

(1) *The hypersurface H passes the reference point.*
(2) *The restriction of $\mathscr{L}$ to X is isomorphic to L.*
(3) *For the projection $\pi : \mathscr{X} \times_{\mathrm{Def}(X)} H \to H$, $R^i\pi_*\mathscr{L}$ is locally free for every i.*

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COROLLARY **1.2.** *Let $f : X \to B$ be as in Theorem 1.1. We also let L be a pull back of a very ample line buncle of S. The symbols π, $\mathscr{X}$, H and $\mathscr{L}$ denote same objects as in Theorem 1.1. Then there exists a morphism $f_H : \mathscr{X} \times_{\mathrm{Def}(X)} H \to \mathbb{P}(\pi_* \mathscr{L})$. Together with π, they form the following diagram:*

$$\begin{tikzcd} \mathscr{X} \times_{\mathrm{Def}(X)} H \arrow[r, "f_H"] \arrow[d, "\pi"'] & \mathbb{P}(\pi_* \mathscr{L}) \arrow[dl] \\ H, & \end{tikzcd}$$

The orginal fibration f coincides with the restriction of the above diagram over the reference point. The restriction of the diagram over a every point of H gives a Lagrangian fibration.

REMARK **1.3.** *If X be an irreducible symplectic manifold. Assume that X admits a surjective morphism $f : X \to S$ such that f has connected fibres and $0 < \dim S < \dim X$. If X and S are projective or X and S are smooth and Kähler then f is Lagrangian over a projective base S by* [8], [9] *and* [5, Proposition 24.8].

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2. PROOF OF THEOREM

PROPOSITION **2.1.** *Let $f : X \to B$ and L be as in Theorem 1.1. We denote by A a general fiber of f. Then there exists a smooth hypersurface H of $\mathrm{Def}(X)$ such that*

(1) *The base change $\mathscr{X} \times_{\mathrm{Def}(X)} H$ carries the line bundle $\mathscr{L}$ on $\mathscr{X} \times_{\mathrm{Def}(X)} H$ whose restiction to the fibre over the reference point is isomorphic to L.*
(2) *The relative Douady space $D(\mathscr{X}/\mathrm{Def}(X))$ of the morphism $\mathscr{X} \to \mathrm{Def}(X)$ is smooth at A.*
(3) *Let $D(\mathscr{X}/\mathrm{Def}(X))_A$ be the irreducible component of $D(\mathscr{X}/\mathrm{Def}(X))$ which contains A. The image of the induced morphism $D(\mathscr{X}/\mathrm{Def}(X))_A \to \mathrm{Def}(X)$ coincides with H.*

Proof of Proposition 2.1. (1) By [6, (1.14)], there exists a universal deformation $(\mathscr{X}, \mathscr{L})$ of the pair (X, L). The parameter space of the universal family forms a smooth hypersurface H of $\mathrm{Def}(X)$. The hypersurface H and the line bundle $\mathscr{L}$ satisfy the assertion (1) of Proposition 2.1.
(2) Let $D(\mathscr{X}/\mathrm{Def}(X))$ be the relative Douady space of the morphism $\mathscr{X} \to \mathrm{Def}(X)$. Since A is smooth and Lagrangian, $D(\mathscr{X}/\mathrm{Def}(X))$ is smooth at A by [11, Theorem 0.1].
(3) We need the following Lemma.

LEMMA **2.2.** *Let X, L and A be as in Propositon 2.1. For an element z of $H^1(X, \Omega^1_X)$, the restriction $z|_A = 0$ in $H^1(A, \Omega^1_A)$ if $q_X(z, L) = 0$, where q_X is the Beauville-Bogomolov-Fujiki form on X.*

Proof. Let σ be a Kähler class of X. It is enough to prove that

$$z\sigma^{n-1}L^n = z^2\sigma^{n-2}L^n = 0,$$

where $2n = \dim X$. By [3, Theorem 4.7], we have the following equation;

$$c_X q_X(z + s\sigma + tL)^n = (z + s\sigma + tL)^{2n}, \tag{1}$$

where s and t are indeterminacy and c_X is a constant only depending on X. By the assumption,

$$c_X q_X(z + s\sigma + tL)^n = c_X(q_X(z) + s^2 q_X(\sigma) + 2s q_X(z, \sigma) + 2st q_X(\sigma, L))^n.$$

If we compare the $s^{n-1}t^n$ and $s^{n-2}t^n$ terms of the both hand sides of the above equation (1), we obtain the assertions. $\square$

We go back to the proof of the assertion (3) of Proposition 2.1. Let $j : H^2(X,\mathbb{C}) \to H^2(A,\mathbb{C})$ be the natural induced morphism by the inclusion $A \to X$. We denote by L_X the intersection of $H^2(X,\mathbb{Q})$ and the orthogonal space of $\mathrm{Ker}(j)$ with respect to the Beauville-Bogomolov-Fujiki form. Since A is Lagrangian, the image of the natural projection $D(\mathscr{X}/\mathrm{Def}(X))_A \to \mathrm{Def}(X)$ is a smooth proper subanalytic space H_A of $\mathrm{Def}(X)$ by [11, 0.1 Theorem]. Moreover, the family over H_A perserves the subspace L_X of $\mathrm{NS}(X)\otimes\mathbb{Q}$ by [11, 0.2 Corollary]. The tangent space of H_A is $\mathrm{Ker}(j) = \mathrm{Ker}\{H^1(X,\Omega^1_X) \to H^1(A,\Omega^1_A)\}$ by [11, 0.1 Theorem]. Let $L^{\perp}$ be the orthogonal space of L in $H^1(X,\Omega^1_X)$ with the Beauville-Bogomolov-Fujiki form. We note that $L^{\perp}$ is tha tangent space of H at the reference point. By Lemma 2.2, $L^{\perp}$ is contained in $\mathrm{Ker}(j)$. This implies that $\mathrm{Ker}(j) = L^{\perp}$. Moreover L_X is spaned by L, because the Beauville-Bogomolov-Fujiki form is nondegenerate. Since H is the universal family of the pair (X,L), we obtain that $H_A \subset H$. Comparing the dimension of the tangent spaces, we have $H_A = H$. $\square$

PROPOSITION **2.3.** *Let $\mathscr{X}$, $\mathrm{Def}(X)$, $\mathscr{L}$ and H be as in Proposition 2.1. We also let Δ be a unit disk in H which has the following two properties:*

(1) *Δ passes the reference point of* $\mathrm{Def}(X)$.
(2) *For a very general point t of Δ, the Picard number of the fibre $\mathscr{X}_t$ of π over t is one.*

The symbols $\mathscr{X}_\Delta$, π_Δ and $\mathscr{L}_\Delta$ denote the base change $\mathscr{X}\times_H \Delta$, the induced morphism $\mathscr{X}_\Delta \to \Delta$ and the restriction $\mathscr{L}$ to $\mathscr{X}_\Delta$, respectively. Then

$$R^i(\pi_\Delta)_*\mathscr{L}_\Delta,$$

are locally free for all i at the reference point.

Proof. For a point u of Δ, $\mathscr{X}_u$ and $\mathscr{L}_u$ denote the fibre of π_Δ over u and the restriction of $\mathscr{L}_\Delta$ to $\mathscr{X}_u$, respectively. We consider whether $\mathscr{L}_\Delta$ has the following property:

$$\text{For every } u \neq o,\ \mathscr{L}_u \text{ is semi-ample} \tag{2}$$

If $\mathscr{L}_u$ has the above property, the assertion of Proposition 2.3 follows from [10, Corollary 3.14]. To prove it, we need the following two lemmata.

LEMMA **2.4.** *For a very general point u of Δ, $\mathscr{L}_u$ is semi-ample.*

Proof. We start with proving the following claim.

CLAIM **2.5.** *There exists a dominant meromorphic map $\Phi : \mathscr{X}_u \dashrightarrow B_u$ such that a general fibre of Φ is compact, B_u is a Kähler manifold and* $\dim B_u > 0$.

Proof. We use the notation as in Proposition 2.1. By Propositon 2.1 (2), there exists a smooth open neighborhood V of A in $D(\mathscr{X}/\mathrm{Def}(X))$. Let $D(\mathscr{X}_u)$ be the irreducible component of $D(\mathscr{X}/\mathrm{Def}(X))\times_H \{u\}$ which intersects V. We note that $D(\mathscr{X}_u)$ is an irreucible component of the Douady space of $\mathscr{X}_u$. We take a resolution $D(\mathscr{X}_u)^{\sim} \to D(\mathscr{X}_u)$ and denote by $U(\mathscr{X}_u)^{-}$ the normalization of $U(\mathscr{X}_u)\times_{D(\mathscr{X}_u)} D(\mathscr{X}_u)^{\sim}$, where $U(\mathscr{X}_u)$ is the universal family over $D(\mathscr{X}_u)$. We also denote by by p and q the natural projections $U(\mathscr{X}_u)^{\sim} \to \mathscr{X}_u$ and $U(\mathscr{X}_u)^{\sim} \to D(\mathscr{X}_u)^{\sim}$. The relations of these objects are summerized in the following diagram:

$$\begin{tikzcd}
\mathscr{X}_u & U(\mathscr{X}_u) \arrow[l] \arrow[d] & U(\mathscr{X}_u)^{-} \arrow[l] \arrow[ll, bend right, "p"'] \arrow[d, "q"] \\
 & D(\mathscr{X}_u) & D(\mathscr{X}_u)^{\sim} \arrow[l]
\end{tikzcd}$$

Let a be a point of $\mathscr{X}_u$. We define the subvarieties $G_i(a)$ of $\mathscr{X}_u$ by

$$\begin{aligned} G_0 &:= a \\ G_{i+1} &:= p(q^{-1}(q(p^{-1}(G_i(a))))) \end{aligned}$$

We also define

$$G_\infty(a) := \bigcup_{i=0}^{\infty} G_i(a).$$

Let $B(\mathscr{X}_u)$ be the Barlet space of $\mathscr{X}_u$. By [1, Théorème A.3], $G_\infty(a)$ is compact for a general point a of $\mathscr{X}_u$ and there exists a meromorphic map $\Phi : \mathscr{X}_u \dashrightarrow B(\mathscr{X}_u)$ whose general fibre is $G_\infty(a)$. By [3, (5.2) Theorem], $B(\mathscr{X}_u)$ is of class $\mathscr{C}$. Hence there exists an embedded resolution $B(\mathscr{X}_u)^\sim \to B(\mathscr{X}_u)$ of the image of Φ whose proper transformation is smooth and Kähler. We denote by B_u the proper transformation. The composition map

$$\mathscr{X}_u \dashrightarrow \mathrm{Im}(\Phi) \dashrightarrow B_u$$

gives the desired meromorphic map if the image of Φ is not a point. Hence we show that Φ is not a trivial. Let a be a general point of $\mathscr{X}_u$. Then $G_1(a)$ is a complex torus. Moreover $G_1(a)$ is a Lagrangian submanifold of $\mathscr{X}_u$. Thus $D(\mathscr{X}_u)$ is smooth at $G_1(a)$ and its dimension is half of those of $\mathscr{X}_u$. The normal bundle of $G_1(a)$ is the direct sum of the trivial bundles. Therefore p is locally isomorphic in a neighborhood of $p^{-1}(G_1(a))$ and p is generically finite. If p is bimeromorphic, then $G_\infty(a) = G_1(a)$ and we are done. If p is not bimeromorphic, we consider the branch locus of the Stein factorization of p. Since $\mathscr{X}_u$ is smooth, the branch locus defines an effective divisor E of $\mathscr{X}_u$. We will prove that $G_\infty(a) \cap E = \emptyset$ if a is general. Since the Picard number of $\mathscr{X}_u$ is one, $\mathscr{L}_u$ and E should be numerically propotional. The pull back $p^*\mathscr{L}_u$ is numerically trivial on fibres of q. Hence the restiction of $\mathscr{L}_u$ to $G_1(a)$ is a numerically trivial bundle if a is general. Therefore $q(p^{-1}(E)) \neq D(\mathscr{X}_u)^\sim$. This implies that $E \cap G_1(a) = \emptyset$ for a general point a of $\mathscr{X}_u$. Since E is effective, E is nef. By [10, Lemma 2.15], there exists an effective $\mathbb{Q}$-divisor E' on $D(\mathscr{X}_u)^\sim$ such that

$$p^*E = q^*E'.$$

Hence $G_\infty(a) \cap E = \emptyset$ if $G_1(a) \cap E = \emptyset$. □

We go back to the proof of Lemma. By blowing ups and flattening, we have the following diagram:

$$\begin{tikzcd} \mathscr{X}_u \arrow[d, dotted] & \mathscr{Y}_u \arrow[l] \arrow[d] & \mathscr{Z}_u \arrow[l] \arrow[d] & \mathscr{W}_u \arrow[l] \arrow[d, "r"] \arrow[lll, bend right, "\nu"'] \\ B_u \arrow[r, equal] & B_u & B_u^\sim \arrow[l] \arrow[r, equal] & B_u^\sim, \end{tikzcd}$$

where

(1) $\mathscr{Y}_u \to \mathscr{X}_u$ is a resolution of indeteminancy of Φ.
(2) $\mathscr{Z} \to \mathscr{Y}_u$ and $B_u^\sim \to B_u$ are bimeromorphic.
(3) $B_u^\sim$ is smooth and Kähler.
(4) $\mathscr{Z}_u \to B_u^\sim$ is flat.
(5) $\mathscr{W}_u \to \mathscr{Z}_u$ is the normalization.

We denote by ν and r the induced morphisms $\mathscr{W}_u \to \mathscr{X}_u$ and $\mathscr{W}_u \to B_u^\sim$, respectively. The proof consists of three steps.

Step 1. We prove that $B_u^\sim$ is projective. Since $B_u^\sim$ is Kähler, it is enough to prove that $\dim H^0(B_u^\sim, \Omega^2) = 0$. We derive a contradiction assuming that $\dim H^0(B_u^\sim, \Omega^2) > 0$. Under this assumption, there exists a holomorphic 2-form ω on $B_u^\sim$. The pull back $r^*\omega$ defines a degenerate holomorphic 2-form on

$\mathscr{W}_u$. On the other hand, $H^0(\mathscr{W}_u,\Omega^2) \cong H^0(\mathscr{X}_u,\Omega^2)$ because ν is birational and $\mathscr{X}_u$ is smooth. Hence $\dim H^0(\mathscr{W}_u,\Omega^2) = 1$ and it should be generated by a generically nondegenerate holomorphic 2-form. That is a contradiction.

Step 2. We prove that $\mathscr{L}_u$ is nef. This is [2, 3.4 Theorem]. For the convinience of readers, we copy their arguments. By [7, Proposition 3.2] it is enoght to prove that $\mathscr{L}_u.C \geq 0$ for every effective curve of $\mathscr{X}_u$. Since the Beauville-Bogomolov-Fujiki form $q_{\mathscr{X}_u}$ is non-degenerate and defined over $H^2(\mathscr{X}_u,\mathbb{Q})$, there exists an isomorphic

$$\iota : H^{1,1}(\mathscr{X}_u,\mathbb{C})_{\mathbb{R}} \to H^{2n-1,2n-1}(\mathscr{X}_u,\mathbb{C})_{\mathbb{R}}$$

such that

$$q_{\mathscr{X}_u}(\mathscr{L}_u,\iota^{-1}([C])) = \mathscr{L}_u.C.$$

If $q(\mathscr{L}_u,z) \neq 0$ for an element z of $H^{1,1}(\mathscr{X}_u,\mathbb{Q})$, then there exists a rational number d such that $q(\mathscr{L}_u + dz) > 0$. By [6, Corollary 3.9], this implies that $\mathscr{X}_u$ is projective. That is a contradiction. Thus $\mathscr{L}_u.C = 0$ for every curve C.

Step 3. Let M be a very ample divisor on $B^{\sim}_u$. We prove that there exists a rational number c such that

$$\mathscr{L}_u \sim_{\mathbb{Q}} c\nu_* r^* M.$$

It is enough to prove that

$$q_{\mathscr{X}_u}(\nu_* r^* M) = q_{\mathscr{X}_u}(\mathscr{L}_u) = q_{\mathscr{X}_u}(\nu_* r^* M,\mathscr{L}_u) = 0.$$

Since $\mathscr{X}_u$ is non projective, $q_{\mathscr{X}_u}(\nu_* r^* M) \leq 0$ and $q_{\mathscr{X}_u}(\mathscr{L}_u) \leq 0$ by [6, Corollary 3.8]. On the other hand, $q_{\mathscr{X}_u}(\mathscr{L}_u) \geq 0$ because $\mathscr{L}_u$ is nef. The linear system $|r^*M|$ contains members M_1 and M_2 such that $M_1 \cap M_2$ has a codimension two. By the definition

$$q_{\mathscr{X}_u}(\nu_* r^* M) = \int (\nu_* r^* M)^2 \sigma^{n-1}\bar{\sigma}^{n-1},$$

where σ is a symplectic form on $\mathscr{X}_u$. Thus $q_{\mathscr{X}_u}(\nu_* r^* M) \geq 0$. Therefore $q_{\mathscr{X}_u}(\nu_* r^* M) = q(\mathscr{L}_u) = 0$. Since $\nu_* r^* M$ is effective and $\mathscr{L}_u$ is nef, $q_{\mathscr{X}_u}(\nu_* r^* M,\mathscr{L}_u) \geq 0$. Again by [6, Corollary 3.8], $q_{\mathscr{X}_u}(\nu_* r^* M + \mathscr{L}_u) \leq 0$. Thus $q_{\mathscr{X}_u}(\nu_* r^* M,\mathscr{L}_u) = 0$ and we are done.

Step 4. We prove that $\mathscr{L}_u$ is semi-ample. By [10, Remark 2.11.1] and [10, Theorem 5.5], it is enough to prove that there exists a nef and big divisor M' on $B^{\sim}_u$ such that

$$\nu^* \mathscr{L}_u \sim_{\mathbb{Q}} r^* M'$$

By Step 2 and Step 3, $\nu^* \mathscr{L}_u \sim_{\mathbb{Q}} r^* M + \sum e_i E_i$ where E_i are ν-exceptional divisors and e_i are positive rational numbers. By Step 2, $\sum e_i E_i$ is nef for every irreducible component of every fibre of r. By [10, Lemma 2.15], there exists a $\mathbb{Q}$-effective divisor M_0 such that

$$\sum e_i E_i = r^* M_0.$$

If we put $M' := M + M_0$, we are done.

□

LEMMA **2.6.** *If $\mathscr{L}_u$ is semi-ample for very general point u of Δ, then $\mathscr{L}_u$ is semi-ample for every $u \neq o$.*

Proof. Let $\Delta(k)$ be an open set of Δ which has the following two properties:

(1) $\pi_* \mathscr{L}_{\Delta}^{\otimes k}$ is locally free.
(2) $\mathscr{L}_{\Delta}^{\otimes k} \otimes k(u) \cong H^0(\mathscr{X}_u, \mathscr{L}_{\Delta}|_{\mathscr{X}_u})$.

We also define

$$\Delta^\circ := \{u \in \Delta; \rho(\mathscr{X}_u) = 1.\},$$

where $\mathscr{X}_u$ is the fibre over u and $\rho(\mathscr{X}_u)$ stands for the Picard number of $\mathscr{X}_u$. We fix a compact set K of Δ which contains the reference point. Then $K \setminus (K \cap \Delta(k))$ consists of finite points. Thus

$$\bigcup_{k=1}^{\infty} (K \setminus K \cap \Delta(k))$$

consists of countable infinite points. Hence

$$\left(\bigcap_{k=1}^{\infty} \Delta(k) \right) \cap K \cap \Delta^\circ \neq \emptyset.$$

Thus there exists a point t_0 of Δ and an integer k such that

$$\pi_\Delta^*(\pi_\Delta)_*\mathscr{L}_\Delta \to \mathscr{L}_\Delta$$

is surjective on $\mathscr{X}_{t_0}$. This implies that the support Z of the cokernel sheaf of $\pi_\Delta^*(\pi_\Delta)_*\mathscr{L}_\Delta \to \mathscr{L}_\Delta$ is a proper closed subset of $\mathscr{X}_\Delta$. Hence $\mathscr{L}_u$ is semi-ample if $u \in \Delta \setminus \pi(Z)$. □

We complete the proof of Proposition 2.3. □

Proof of Theorem 1.1. By Proposition 2.1, there exists a smooth hypersurface H of $\mathrm{Def}(X)$ and a line bundle $\mathscr{L}$ which have the properties of (1) and (2) of Theorem 1.1. Assume that $R^i\pi_*\mathscr{L}$ is not locally free. We define the function $\varphi(t)$ as

$$\varphi(t) := \dim H^i(\mathscr{X}_t, \mathscr{L}_t)$$

where $\mathscr{X}_t$ is the fibre of π over t and $\mathscr{L}_t$ is the restriction of $\mathscr{L}$ to $\mathscr{X}_t$. Then

$$\varphi(o) > \varphi(t),$$

where o is the reference point and t is a general point of H. The Picard number of a fibre $\mathscr{X}_t$ over a very general point of H is one. Hence there exists a unit disk Δ such that $o \in \Delta$ and the Picard number of a very general fibre of the induced morphsim $\mathscr{X} \times_H \Delta \to \Delta$ is one. By Proposition 2.3, $R^i(\pi_\Delta)_*\mathscr{L}_\Delta$ is locally free for every i. This implies that $\varphi(o) = \varphi(t)$. That is a contradiction. □

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SPIN CURVES AND SCORZA QUARTICS

HIROMICHI TAKAGI

Contents

This is the joint work with Francesco Zucconi. We give some applications of 3-fold birational geometry to the study of even spin curves. Much explanation is taken from the Dolgachev-Kanev's paper [DK93] and our preprints [TZ08a] and [TZ08b].

1. Even spin curves

Let $\mathcal{H}$ be a smooth projective curve of genus g and θ a theta characteristic on $\mathcal{H}$, namely, $2\theta \sim K_{\mathcal{H}}$. A couple $(\mathcal{H}, \theta)$ is called a *spin curve* and *even* if so is $h^0(\mathcal{H}, \theta)$. Let $\mathcal{S}_g^+$ be the moduli space of even spin curves of genus g. It is known that $h^0(\mathcal{H}, \theta) = 0$ (called *ineffective theta characteristics*) for a general pair $(\mathcal{H}, \theta) \in \mathcal{S}_g^+$.

2. Scorza correspondence

The basic of our study is the following correspondence originally studied by G. Scorza.

Definition 2.0.1. Given an ineffective θ, $h^0(\mathcal{H}, \theta + x) = 1$ for every $x \in \mathcal{H}$ by the Riemann-Roch theorem, hence θ gives the correspondence $I_\theta \subset \mathcal{H} \times \mathcal{H}$ such that $(x, y) \in I_\theta$ if and only if y is in the support of the unique member of $|\theta + x|$. This is called *the Scorza correspondence.*

We denote by $I_\theta(x)$ the fiber of $I_\theta \to \mathcal{H}$ over x. In other words, $I_\theta(x)$ is the unique member of $|\theta + x|$.

We can easily verify the following properties of I_θ by the Riemann-Roch theorem, etc:

(a) $\theta = I_\theta(x) - x$ is (of course) independent of x,
(b) $h^0(\mathcal{H}, \theta + x) = 1$ for any $x \in \mathcal{H}$,
(c) I_θ is disjoint from the diagonal,
(d) I_θ is symmetric, and
(e) I_θ is a (g, g)-correspondence.

By [DK93, Lemma 7.2.1], conversely, for any reduced correspondence I' satisfying the above conditions, there exists a unique ineffective theta characteristic such that $I' = I_\theta$.

Here we mention two known applications of the Scorza correspondence:

Rationality of S_3^+. We learned the following by [DK93]. Let V be a 3-dimensional vector space and $\check{V}$ its dual. For a homogeneous form $G \in S^m\check{V}$ of degree m on V, we define the (first) polar $P_a(G)$ of G at $a \in \mathbb{P}(V)$ by $P_a(G) := \frac{1}{m}\sum a_i \frac{\partial G}{\partial x_i}$, where a_i and x_i are coordinates of a, and on V, respectively.

Let $F \in S^4\check{V}$ be a general ternary quartic form on V. Set

$$S^o(F) := \{a \in \mathbb{P}(V) \mid P_a(F) \text{ is projectively equivalent to the Fermat cubic}\}.$$

Then the closure $S(F) := \overline{S}^o(F)$ is again a smooth quartic curve, which is called the Clebsch covariant quartic of F. By taking the second polars of $S(F)$, we have the following correspondence:

$$T(F) := \{(a,b) \in S(F) \times S(F) \mid \mathrm{rank} P_{a,b}(S(F)) \leq 1\}. \tag{2.1}$$

For example, if $P_a(F) = \{x^3 + y^3 + z^3 = 0\}$, then $b = (1:0:0), (0:1:0)$ or $(0:0:1)$, thus $T(F)$ is a $(3,3)$-correspondence. In the end, $T(F)$ turns out to be the Scorza correspondence I_θ defined by a unique theta characteristic θ.

So we have the map $\mathrm{Sc}\colon \mathcal{M}_3^0 \to S_3^+$ such that $\mathrm{Sc}\colon [F=0] \mapsto [S(F), \theta]$ defined over the open set $\mathcal{M}_3^0 \subset \mathcal{M}_3$ where $S(F)$ is nonsingular. This association map was discovered by Scorza and is called the *Scorza map.* Scorza showed it is an injective birational map. Thus S_3^+ is rational since $\mathcal{M}_3$ is known to be rational by [Kat96] (see also [Boh]). The curve F corresponding to a couple $(S(F), \theta)$ is called the *Scorza quartic* of $(S(F), \theta)$. In other words, by setting $\mathcal{H} = S(F)$, F is the unique quartic such that if $(a,b) \in I_\theta (\subset \mathcal{H} \times \mathcal{H})$, then $\mathrm{rk}\, P_{a,b}(F) = 1$ holds.

Mukai's description of a Fano threefold. *A prime Fano threefold of genus* 12 is a smooth projective threefold A_{22} such that $-K_{A_{22}}$ is ample, the class of $-K_{A_{22}}$ generates $\operatorname{Pic} A_{22}$, and such that the genus $g(A_{22}) := \frac{(-K_{A_{22}})^3}{2} + 1 = 12$. Mukai found the description of such a Fano as a variety of power sums.

Definition 2.0.2. Let V be a $(v+1)$-dimensional vector space and let $F \in S^m\check{V}$ be a homogeneous forms of degree m on V. Set

$$\mathrm{VSP}\,(F,n)^o := \{([H_1],\dots,[H_n]) \mid H_1^m + \cdots + H_n^m = F\} \subset \mathrm{Hilb}^n\mathbb{P}(\check{V}).$$

The closure $\mathrm{VSP}\,(F,n) := \overline{\mathrm{VSP}\,(F,n)^o}$ is called the *varieties of power sums* of F.

Theorem 2.0.3 (S. Mukai). *Let $\{F_4 = 0\} \subset \mathbb{P}(V) = \mathbb{P}^2$ be a general plane quartic curve. Then*
(1) $\mathrm{VSP}(F_4, 6) \subset \mathrm{Hilb}^6\check{\mathbb{P}}^2$ *is a general prime Fano threefold of genus* 12*; and conversely,*
(2) *every general prime Fano threefold of genus* 12 *is of this form.*

See [Muk92] and [Muk04]. Mukai observed the following:
(a) The Hilbert scheme of lines on A_{22} is isomorphic to a smooth plane quartic $\mathcal{H}_1$ and the correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on A_{22} gives an ineffective theta characteristic θ on $\mathcal{H}_1$. More precisely, θ is constructed so that the Scorza correspondence I_θ is equal to

$$\{([l],[m]) \in \mathcal{H}_1 \times \mathcal{H}_1 \mid l \cap m \neq \emptyset, l \neq m\}.$$

By the result of Scorza recalled above, the Scorza quartic $\{F_4 = 0\}$ is associated to the pair $(\mathcal{H}_1, \theta)$ in the same ambient plane as the canonically embedded $\mathcal{H}_1$. Theorem 2.0.3 (2) claims that X is recovered as $\mathrm{VSP}\,(F_4, 6)$. (1) follows from (2) since the number of the moduli of prime Fano threefolds of genus 12 is equal to $\dim \mathcal{M}_3 = 6$.
(b) The Hilbert scheme of conics on A_{22} is isomorphic to the plane $\mathcal{H}_2$ and $\mathcal{H}_2$ is naturally considered as the plane $\check{\mathbb{P}}^2$ dual to $\mathbb{P}^2$ since, for a conic q on A_{22}, the lines intersecting q form a hyperplane section of $\mathcal{H}_1$. Further, he showed the six points $[H_1],\dots,[H_6]$ such that $([H_1],\dots,[H_6]) \in \mathrm{VSP}^{\,o}(F_4, 6)$ correspond to six conics through one point of A_{22}.

3. Scorza quartics

Scorza succeeded in associating a unique quartic hypersurface, which is also called *the Scorza quartic*, to a spin curve of any genus g with ineffective theta. In the case $g = 3$, this association turns out to be the inverse of the Scorza map. Dolgachev and Kanev, however, pointed out Scorza overlooked three conditions on spin curves mentioned below to consturct the Scorza quartic.

Let $\mathcal{H} \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g, θ an ineffective theta characteristic on it and $I_\theta \subset \mathcal{H} \times \mathcal{H}$ the Scorza correspondence. Since the linear hull $\langle I_\theta(x) - y\rangle$ for $(x, y) \in I_\theta$ is a hyperplane of $\mathbb{P}^{g-1}$, we can define a morphism $\pi_\theta \colon I_\theta \to |\omega_{\mathcal{H}}| = \check{\mathbb{P}}^{g-1}$ by $(x, y) \mapsto \langle I_\theta(x) - y\rangle$.

The following is a crucial object to construct the Scorza quartic:

Definition 3.0.4. The image $\Gamma(\theta)$ of the above morphism $\pi_\theta \colon I_\theta \to \check{\mathbb{P}}^{g-1}$ (with reduced structure) is called the *discriminant locus* of the pair $(\mathcal{H}, \theta)$.

By Definition 3.0.4, we have the following diagram:

$$\begin{tikzcd} & I_\theta \subset \mathcal{H} \times \mathcal{H} \arrow[dl, "\pi_\theta"'] \arrow[dr, "p"] & \\ \Gamma(\theta) \subset \check{\mathbb{P}}^{g-1} & & \mathcal{H} \subset \mathbb{P}^{g-1}. \end{tikzcd} \tag{3.1}$$

The three conditions mentioned above is the following, which are a kind of generality conditions:

(A1) the degree of the map $I_\theta \to \Gamma(\theta)$ is two, namely, $\langle I_\theta(x') - y'\rangle = \langle I_\theta(x) - y\rangle$ implies $(x', y') = (x, y)$ or (y, x),

(A2) $\Gamma(\theta)$ is not contained in a quadric, and

(A3) I_θ is smooth.

From now on in this section, we assume these conditions.

We can define:

$$\overline{D}_H := \pi_{\theta *} p^*(H \cap \mathcal{H})$$

as a divisor, where H is an hyperplane of $\mathbb{P}^{g-1}$.

By using (A1)–(A3), it is not difficult to see $\deg \Gamma(\theta) = g(g-1)$ and $\deg \overline{D}_H = 2g(g-1)$. Therefore we may expect that $\overline{D}_H$ is a quadric section of $\Gamma(\theta)$. Actually this is true:

Proposition 3.0.5. *$\overline{D}_H$ is cut out by a quadric in $\check{\mathbb{P}}^{g-1}$.*

Now we define the following correspondence:

$$\mathcal{D} := \{(q_1, q_2) \mid q_1 \in \overline{D}_{H_{q_2}}\} \subset \Gamma(\theta) \times \Gamma(\theta),$$

where H_q is the hyperplane of $\mathbb{P}^{g-1}$ corresponding to $q \in \check{\mathbb{P}}^{g-1}$. It is easy to see that $\mathcal{D}$ is symmetric. By Proposition 3.0.5, we see that $\mathcal{D}$ is the restriction of a symmetric $(2, 2)$ divisor $\mathcal{D}'$ of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. Let $\{\check{F}_4 = 0\}$ be the quartic hypersurface obtained by restricting $\mathcal{D}'$ to the diagonal of $\check{\mathbb{P}}^{g-1} \times \check{\mathbb{P}}^{g-1}$. The Scorza quartic is the 'dual' quartic in $\mathbb{P}^{g-1}$ of $\{\check{F}_4 = 0\}$.

To explain this more precisely, we give a quick review of some generality of the theory of polarity. Set $V := H^0(\mathcal{H}, \omega_{\mathcal{H}})\check{}$. Each homogeneous form $F \in S^4\check{V}$ defines a linear map:

$$\begin{array}{rccc} \mathrm{ap}_F \colon & S^2V & \to & S^2\check{V} \\ & G & \mapsto & P_G(F). \end{array}$$

called the *apolarity map*, which is nothing but the linear extension of iterating polar maps. If ap_F is an isomorphism, F is called *non-degenerate* and then the inverse isomorphism is given by a $\check{F} \in S^4V$, that is ${\mathrm{ap}_F}^{-1} = \mathrm{ap}_{\check{F}}$. The form $\check{F} \in S^4V$ is called *the dual form* of F.

It turns out that the constructed $\{\check{F}_4 = 0\}$ is non-degenerate and we can take the dual $\{F_4 = 0\}$, which is the Scorza quartic.

To explain this construction of the Scorza quartic is actually the inverse of the Scorza map in genus 3 case, we remark one of the important properties of the Scorza quartic. By the theory of polarity and the definition of $\check{F}_4$, the fiber of $\mathcal{D} \to \Gamma(\theta)$ over a point $q \in \Gamma(\theta)$ is defined by the second polar $P_{H_q^2}(\check{F}_4)$. Moreover, by definition of $\Gamma(\theta)$, it is easy to derive that $P_{H_q^2}(\check{F}_4) = ab$ for some $a, b \in \mathcal{H}$ such that $(a, b) \in I_\theta$, where $a, b \in \mathbb{P}^{g-1}$ is considered as a linear form on $\check{\mathbb{P}}^{g-1}$. By definition of the dual, we have $P_{a,b}(F_4) = H_q^2$. Thus we have verified the association of the Scorza quartic is the inverse of the Scorza map in the case $g = 3$.

4. Special quartics arising from quintic del Pezzo 3-fold

Now we start explanation of our results.

Trigonal even spin curves of any genus and their Scorza quartics arise from some 3-folds as in Mukai's case.

Let B be the smooth quintic del Pezzo threefold, that is B is a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\mathrm{Pic}\, B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into $\mathbb{P}^6$.

Let d be an arbitrary integer greater than or equal to 6. We consider a general smooth rational curves C of degree d on B obtained inductively from lines, more precisely, smoothings of the union of a degree $d-1$ rational curve and a line intersecting it. Let $f \colon A \to B$ be the blow-up along C and E_C the f-exceptional divisor.

We explain the relation of this with A_{22}. If we take the blow-up $A' \to A_{22}$ along a general line on it, then there is a unique flop $A' \dashrightarrow A$

and birational contraction $A \to B$, which is the blow-up of B along a smooth rational curve of degree 5. Thus the above situation is a generalization of this. Moreover, a general line is mapped to a line on B intersecting C, and a general conic is mapped to a conic on B intersecting C twice or more.

We consider the notions of lines and conics on A, which correspond to lines on B intersecting C, and conics on B intersecting C twice or more.

Definition 4.0.6. A connected and reduced curve $l \subset A$ is called a *line* on A if $-K_A \cdot l = 1$ and $E_C \cdot l = 1$.

By $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$, $f(l)$ is a line on B *intersecting* C.

Proposition 4.0.7. *The Hilbert scheme of lines on A is a smooth trigonal curve $\mathcal{H}_1$ of genus $d-2$.*

Definition 4.0.8. A connected and reduced curve $q \subset A$ is called a *conic* on A if $-K_A \cdot q = 2$ and $E_C \cdot q = 2$.

We showed that the Hilbert scheme of conics on A is an irreducible surface and the normalization morphism is injective, namely, the normalization $\mathcal{H}_2$ parameterizes conics on A in one to one way.

Moreover we have the full description of $\mathcal{H}_2$ as follows. For this, let $D_l \subset \mathcal{H}_2$ be the locus parameterizing conics on A which intersect a fixed line l on A.

Theorem 4.0.9. *$\mathcal{H}_2$ is so-called the* White surface*, namely, the surface obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $s := \binom{d-2}{2}$ points and embedded by $|D_l| = |(d-3)h - \sum_{i=1}^{s} e_i|$, where h is the pull-back of a line, e_i are the exceptional curves of $\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$. Moreover, $\mathcal{H}_2$ is given by intersection of cubics.*

Here we use the notation $\check{\mathbb{P}}^{d-3}$ since the ambient projective space of $\mathcal{H}_2$ and that of the canonical embedding of $\mathcal{H}_1$ can be considered as reciprocally dual as in Mukai's case. We write the ambient of $\mathcal{H}_1$ by $\mathbb{P}^{d-3}$ and that of $\mathcal{H}_2$ by $\check{\mathbb{P}}^{d-3}$.

Set

$$\mathcal{D}_2 := \{([q_1],[q_2]) \in \mathcal{H}_2 \times \mathcal{H}_2 \mid q_1 \cap q_2 \neq \emptyset\}$$

and denote by D_q the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over a point $[q]$. Then $D_q \sim 2D_l$ and $\mathcal{D}_2 \sim p_1^*D_q + p_2^*D_q$. $\mathcal{D}_2$ is obviously symmetric. Thus $\mathcal{D}_2$ is the restriction of a unique symmetric $(2,2)$-divisor $\mathcal{D}_2'$ on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$. The restriction of $\mathcal{D}_2'$ to the diagonal is a quartic hypersurface $\{\check{F}_4' = 0\}$

in $\check{\mathbb{P}}^{d-3}$. We can show that $\check{F}'_4$ is non-degenerate. Then we obtain the unique quartic hypersurface $\{F'_4 = 0\}$ in $\mathbb{P}^{d-3}$ dual to $\check{F}'_4$.

The following is a generalization of Theorem 2.0.3 (2):

Theorem 4.0.10. *Let $f\colon A \to B$ be the blow-up along C, and let $\rho\colon \widetilde{A} \to A$ be the blow-up of A along the strict transforms of $\binom{d-2}{2}$ bi-secant lines of C on B. Then there is an injection from $\widetilde{A}$ to $\mathrm{VSP}\,(F'_4, n)$, where $n := \binom{d-1}{2}$. Moreover the image of $\widetilde{A}$ is uniquely determined by $\mathcal{D}_2$ and is an irreducible component of*

$$\mathrm{VSP}\,(F'_4, n; \mathcal{H}_2) := \overline{\{([H_1], \dots, [H_n]) \mid [H_i] \in \mathcal{H}_2\}} \subset \mathrm{VSP}\,(F'_4, n).$$

To characterize 3-fold $\widetilde{A}$, we need extra deta $\mathcal{H}_2$, which is implicit in Mukai's case. See [TZ08a].

5. Existence of the Scorza quartic

Notice that the construction of F'_4 is quite similar to that of the Scorza quartic. This similarity will be clear once we define a theta characteristic on $\mathcal{H}_1$ and clarify the relation of $\mathcal{H}_1$ and $\mathcal{H}_2$.

For the curve $\mathcal{H}_1$ parameterizing lines on A, we can introduce the incidence correspondence as in Mukai's case:

$$I := \{([l], [m]) \mid l \neq m, l \cap m \neq \emptyset\} \subset \mathcal{H}_1 \times \mathcal{H}_1 \tag{5.1}$$

with reduced structure. We can prove I satisfies the conditions (a)–(e) whence there exists a unique ineffective theta characteristic such that $I = I_\theta$.

Moreover, as we mentioned above, there is a natural duality between the ambient spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$. This gives us a very computable way to describe the discriminant loci $\Gamma(\theta)$ of θ.

Proposition 5.0.11. *For the pair $(\mathcal{H}_1, \theta)$, $\Gamma(\theta)$ is contained in $\mathcal{H}_2$, and the generic point of the curve $\Gamma(\theta)$ parameterizes line pairs on A. Moreover, $\Gamma(\theta) \sim 3(d-2)h - 4\sum_{i=1}^{s} e_i$ on $\mathcal{H}_2$. In particular $\Gamma(\theta)$ is not contained in a cubic section of $\mathcal{H}_2$.*

Moreover, we can consider $\{F'_4 = 0\}$ lives in the same ambient space as canonically embedded $\mathcal{H}_1$.

Proposition 5.0.12. *The special quartic $\{F'_4 = 0\} \subset \mathbb{P}^{d-3}$ of Theorem 4.0.10 coincides with the Scorza quartic of $(\mathcal{H}_1, \theta)$.*

Proof. Noting $\Gamma(\theta) \subset \mathcal{H}_2$, we can show that the restriction of the correspondence defining F'_4 to $\Gamma(\theta) \times \Gamma(\theta)$ coincides with the correspondence defining the Scorza F_4. $\square$

The story goes further. By virtue of the above explicit computation of the discriminant, we can prove that the pair $(\mathcal{H}_1, \theta)$ satisfies the conditions (A1)–(A3). Then, by a standard deformation theoretic argument, we can then verify that the conditions (A1)–(A3) hold also for a general even spin curve, hence we answer affirmatively to the Dolgachev-Kanev Conjecture:

Theorem 5.0.13. *The Scorza quartic exists for a general even spin curve.*

See [TZ08b].

6. Moduli space of trigonal even spin curves

Let $\mathcal{M}_g^{\mathrm{tr}}$ and $\mathcal{S}_g^{+\mathrm{tr}}$ be the moduli space of trigonal curves of genus g and the moduli space of even trigonal spin curves of genus g, respectively. We would like to study $\mathcal{S}_g^{+\mathrm{tr}}$ using the geometry of (B, C). Denote by $\mathcal{H}_d^B$ the Hilbert scheme of general smooth rational curves of degree d as in Section 4. $\mathcal{H}_d^B$ is irreducible. By $\mathrm{Aut}\, B \simeq \mathrm{SL}(2, \mathbb{C})$, we have the natural rational maps $\pi_{\mathcal{S}} \colon \mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \to \mathcal{S}_{d-2}^{+\mathrm{tr}}$ mapping $C_d \mapsto (\mathcal{H}_1, \theta)$, and $\pi_{\mathcal{F}}$ from $\mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C})$ to the moduli space $\mathcal{F}_d$ of $\widetilde{A}_d$ ($= \widetilde{A}$ of degree d) mapping $C_d \mapsto \widetilde{A}_d$.

$$\begin{tikzcd} & \mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \arrow[dl, "\pi_{\mathcal{S}}"'] \arrow[dr, "\pi_{\mathcal{F}}"] & \\ \mathcal{S}_{d-2}^{+\mathrm{tr}} & & \mathcal{F}_d \end{tikzcd}$$

Since $\mathcal{H}_d^B$ is irreducible and $\mathcal{H}_d^B/\mathrm{SL}(2,\mathbb{C}) \to \mathcal{F}_d$ is dominant, we see that $\mathcal{F}_d$ is irreducible.

Proposition 6.0.14. *The map $\pi_{\mathcal{F}}$ is finite. If $d = 6$, then $\deg \pi_{\mathcal{F}} = 2$. If $d \geq 7$, then $\pi_{\mathcal{F}}$ is birational.*

Proof. $\deg \pi_{\mathcal{F}} = 2$ for $d = 6$ follows from the following diagram:

$$\begin{tikzcd} & & \widetilde{A} \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "f"'] \arrow[rr, dashed] & & A' \arrow[dr, "f'"] & \\ B & & & & B, \end{tikzcd}$$

where $A \dashrightarrow A'$ is a flop and $A' \to B$ is also the blow-up along a smooth rational curve C' of degree 6 on B. This reflects the fact $\mathcal{H}_1$ has two different g^1_3's (birationality of $\pi_{\mathcal{F}}$ for $d \geq 7$ will reflect the fact a general trigonal curve of genus ≥ 5 has a unique g^1_3). Indeed, there is one to one correspondence between the sets lines on A and lines on A'. Thus we identify the Hilbert schemes of lines on A and A' and denote it by $\mathcal{H}_1$. $\mathcal{H}_1$ has two triple covers $\mathcal{H}_1 \to C$ and $\mathcal{H}_1 \to C'$. These are defined by two different g^1_3's of $\mathcal{H}_1$. Thus (B, C) and (B, C') are not isomorphic to each other but correspond the same $\widetilde{A}$. □

For genus three curve, the Scorza quartic is useful to prove the rationality of S^+_3. Unfortunately, this is not the case in the higher genus case for the moment since the Scorza quartics are special quartics and there is no description of the loci of them in the space of quartics. Nevertheless, it gives another way to study of S^+_g.

Proposition 6.0.15. *$\pi_{\mathcal{S}}$ factor through $\pi_{\mathcal{F}}$ as $\mathcal{H}^B_d/\mathrm{SL}(2,\mathbb{C}) \to \mathrm{Im}\,\pi_{\mathcal{S}} \to \mathcal{F}_d$. In other words, $\widetilde{A}$ is determined from $(\mathcal{H}_1, \theta)$.*

Proof. From $(\mathcal{H}_1, \theta)$, we can define $\Gamma(\theta)$ and F_4. By Theorem 4.0.9 and Proposition 5.0.11, we obtain $\mathcal{H}_2$ as the intersection of cubics containing $\Gamma(\theta)$. We can define the divisor $\mathcal{D}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$ from the dual $\check{F}_4$. By Theorem 4.0.10, $\widetilde{A}$ is obtained from F_4 and $\mathcal{H}_2$, thus from $(\mathcal{H}_1, \theta)$. □

Corollary 6.0.16. *$\mathrm{Im}\,\pi_{\mathcal{S}}$ is an irreducible component of $\mathcal{S}^{+tr}_{d-2}$ dominating $\mathcal{M}^{\mathrm{tr}}_{d-2}$. In particular a general $\mathcal{H}_1$ is a general trigonal curve of genus $d-2$. $\pi_{\mathcal{S}} \colon \mathcal{H}^B_d/\mathrm{SL}\,(2,\mathbb{C}) \to \mathrm{Im}\,\pi_{\mathcal{S}}$ is finite of degree two if $d = 6$ and birational if $d \geq 7$.*

Proof. Since $\dim \mathcal{H}^B_d = 2d$ and $\dim \mathrm{Aut}\,(B, C_d) \leq \dim \mathrm{Aut}\, B = 3$, we see that $\dim \mathcal{F}_d \geq 2d-3$ by Proposition 6.0.14. By Proposition 6.0.15, $\dim \mathrm{Im}\,\pi_{\mathcal{S}} \geq 2d-3$. Thus by $\dim \mathcal{S}^{+tr}_{d-2} = 2d-3$, the first claim follows.

If $d \geq 7$, then $\pi_{\mathcal{S}}$ is birational by Proposition 6.0.14. If $d = 6$, then, as in the proof of Proposition 6.0.14, two triple covers $\mathcal{H}_1 \to C$ and $\mathcal{H}_1 \to C'$ are defined by two different g^1_3's of $\mathcal{H}_1$, thus (B, C) and (B, C') are not isomorphic to each other. But (B, C) and (B, C') define the same theta characteristic. Thus $\pi_{\mathcal{S}}$ is of degree two. □

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FINITE SUBGROUPS OF THE PLANAR CREMONA GROUPO

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Abstract. In this talk we review some new results about classification of conjugacy classes of finite subgroups of the planar Cremona group.

1. Introduction

Let $\mathrm{Cr}_n(k)$ denote the Cremona group of birational automorphisms of the projective space $\mathbb{P}^n_k$ over a field k. From algebraic point of view

$$\mathrm{Cr}_n(k) = \mathrm{Aut}_k(k(t_1, \ldots, t_n).$$

When $n = 1$, the group of $\mathrm{Cr}_1(k)$ is isomorphic to the linear algebraic group $\mathrm{PGL}_2(k)$. The description of its finite subgroups is well known. There is one conjugacy class of each group, and the groups are isomorphic to either a cyclic, or dihedral, or the group of symmetries of a platonic solid. We will be concerned with the case $n = 2$.

There are three different aspects of the theory depending on the field k.

(i) $k = \mathbb{C}$, the field of complex numbers;
(ii) k is any perfect field and groups are of order prime to the characteristic of k;
(iii) k is algebraically closed of characteristic dividing the order of the group.

In any case the classification of finite subgroups uses the following simple idea. For each finite subgroup $G \subset \mathrm{Cr}_2(k)$ one can find a smooth rational projective algebraic surface X such that G acts biregularly on X inducing the same action on the field of rational functions. Two subgroups are conjugate in $\mathrm{Cr}_2(k)$ if and only if the corresponding surfaces are birationally G-equivariantly isomorphic. Among all surfaces X which "regularize" the subgroup G one can choose minimal one in the sense that it does not allow a non-trivial birational G-equivariant morphism $X \to Y$. It follows from Mori's theory of minimal models that a minimal G-surface X belongs to one of the following classes of surfaces

(i) X has a structure of a conic bundle $f : X \to \mathbb{P}^1_k$ with $m \geq 0$ singular fibres,

(ii) X is a Del Pezzo surface of degree d with $\mathrm{Pic}(X)^G \cong \mathbb{Z}$.

So the problem in each case is reduced to the classification of finite subgroups of the automorphism groups of minimal G-surfaces as above and the classification of birational G-equivariant isomorphisms between minimal surfaces (which are decomposed in elementary links classified by V. Iskovskikh [5]).

2. The case of complex numbers

This is the most classical case, the theory originates from the work of Bertini who classified the conjugacy classes of involutions in $\mathrm{Cr}_2(\mathbb{C})$. We refer to the history and references to the modern work to our paper [4].

Theorem 1. *Each involution in* $\mathrm{Cr}_2(\mathbb{C})$ *is conjugate to either*

(i) *de Jonquières involution;*

(ii) *a Geiser involution;*

(iii) *a Bertini involution.*

Recall the definitions. A de Jonquières involution is defined (in its algebraic form) by the transformation $(x, y) \mapsto (x, \frac{F_{2g+1}(x)}{y})$, where $F(x)$ is a polynomial of degree $2g+1$ without multiple roots.

A Geiser involution is defined geometrically as the deck transformation of the rational map of degree 2 given by the linear system of plane cubic curves through 7 general points in the plane. A Bertini involution is defined similarly by the linear system of plane curves of degree 6 with double points at 8 general points in the plane.

Already in this special case one sees the dramatic difference of conjugacy classes of finite subgroups of the Cremona group and a linear algebraic group. Namely, the set of conjugacy classes is infinite, and in fact can be parametrized by points of algebraic varieties.

One starts the classification from considering subgroups of a conic bundle. They belong to the class of de Jonquières transformations, i.e. transformations which can be algebraically given in the form $(x, y) \mapsto (x, \frac{a(x)y+b(x)}{c(x)y+d(x)})$, where $a(x, b(x), c(x), d(x)$ are rational functions in x. In geometric forms a de Jonquières transformation can be define as a birational transformation of the plane leaving invariant a pencil of lines.

Let $f : X \to \mathbb{P}^1$ be a conic bundle and $p_1, \ldots, p_m$ be the set of points over which the fibres X_{p_i} are reducible conics. Any finite subgroup G of $\mathrm{Aut}(X)$ fits into an exact sequence of groups

$$1 \to H \to G \to \bar{G} \to 1,$$

where $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^1)$ is the image of G in its action on the base of the fibration. The group G also acts on the group $\mathrm{Pic}(X) \otimes \mathbb{Q}$ of X which is generated by K_X and components of fibres, taken one from each. A conic bundle G-surface X is called *exceptional* if the latter action is not trivial. One can describe such conic bundles explicitly. They are isomorphic to a hypersurface in $\mathbb{P}(1,1,g,g)$

$$F_{2g+2}(t_0,t_1) + t_2t_3 = 0,$$

where $F_{2g+2}(t_0,t_1)$ is a binary form of degree $2g$. The group of automorphisms of such a surface is easy to describe. It is isomorphic to the extension $N.P$, where P is the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ leaving the binary form $F_{2g}(t_0,t_1)$ invariant, and $N = \mathbb{C} : 2$ is the group of matrices with determinant ± 1 leaving the binary form t_1t_2 invariant. This allows one to describe all finite subgroups of automorphism of X.

Assume that X is not an exceptional conic bundle. Then the group H leave each fibre F_{x_i} invariant and embeds injectively in the group $2^m := (\mathbb{Z}/2\mathbb{Z})^m$ via switching the components. Since H is a subgroup of the general fibre of $f : X \to \mathbb{P}^1$, it is isomorphic to a subgroup of $\mathrm{PGL}_2(K)$, where K is the field of rational functions on the base. It is known that no finite subgroup of this group is isomorphic to a group 2^s with $s > 2$. This shows that $H \cong 2^s$ with $s = 1$ or 2.

The previous argument shows that G is either isomorphic to an extension $2.P$ or $2^2.P$, where P is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. In the first case, the fixed locus of the non-trivial element in 2 is a hyperelliptic curve of genus g ramified over the set $\Sigma = \{p_1,\ldots,p_m\}, m = 2g+2$, (or a rational, or elliptic curve if $g < 2$)), and P is its group of automorphisms. In the second case, the fixed locus of each non-trivial involution $\tau_i \in 2^2$ is a hyperelliptic curve of some genus p_i such that ramifies over a subset Σ_i of Σ of cardinality n_i such that Σ is partitioned into three subsets A, B, C with $\Sigma_1 = A+B, \Sigma_2 = B+C, \Sigma_3 = A+C$. An example of such surface X is the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ given by the equation

$$a_0(t_0,t_1)z_0^2 + a_1(t_0,t_1)z_0^2 + a_2(t_0,t_1)z_0^2 = 0,$$

where a_0, a_1, a_2 are binary forms of some degree m.

Let us now pass to the case when G is realized on a Del Pezzo surface of degree d. Recall that d takes values between 1 and 9, any surface of degree $d < 8$ is isomorphic to the blow-up of $9-d$ distinct points in an "unnodal position" (e.g. no three points are collinear). When $n = 8$, X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of one point which is not minimal, and can be omitted from consideration. When $n = 8$, the surface X is isomorphic to the projective plane. The classification of the conjugacy classes of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

and $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}_3(\mathbb{C})$ is one-hundred years old. Assume now that $d < 8$. If $d = 7$, the surface X is not G-minimal since the proper inverse transform of the line joining the two points is a G-invariant (-1)-curve. For $d < 7$ one uses the representation of G in the group of orthogonal transformations of the Picard lattice generated by reflections in vectors $v \in \mathrm{Pic}(X)$ such that $v^2 = -2$. This is the notorious Weyl group of a root system of type E_{9-d}, where, by definition, E_3 is equal to type $A_2 \times A_1$, E_4 is equal to A_4, and E_5 is equal to D_5. The representation is always faithful except in the case $d = 6$, where the kernel consists of the group of projective transformations fixing the three points. Dealing with the case by case, it is possible to classify first the conjugacy classes of cyclic subgroups. Then one finds the corresponding surfaces and studies possible additional symmetries. In this way one achieves a complete classification of minimal Del Pezzo G-surfaces. The tables are given in [4].

The final step in the classification is to find out when two minimal Del Pezzo G-surfaces are birationally isomorphic. This is achieved by Iskovskikh's classification of elementary links birationally relating two Del Pezzo surfaces. For example, minimal Del Pezzo G-surfaces with $d \leq 3$ are rigid, in the sense that cannot be birationally isomorphic to other surfaces.

3. The case when k is a perfect field and $(\#G, \mathrm{char}(k)) = 1)$

Here the work has only began since, essentially, only the case of cyclic groups has been studied. Recall that in the case k is algebraically closed any cyclic group of order prime to characteristic can be realized by a subgroup of projective transformations. It is not anymore true if k is not algebraically closed. The relevant useful information about the field k is given by the following two numbers. From now on ℓ is a prime number different from $\mathrm{char}(k)$ and ζ_ℓ be the generator of the group of elements of order ℓ in $\bar{k}$.

Set

$$t_\ell = [k(\zeta_\ell) : k], \quad m_\ell = \sup\{d \geq 1 : \zeta_{\ell^d} \in k(\zeta_\ell)\}.$$

For any group A let $\ell^{\nu_l(A)}$ be the order of its Sylow ℓ-subgroup. The classification of finite subgroups of $\mathrm{PGL}_{n+1}(k) = \mathrm{Aut}(\mathbb{P}^n_k)$ is based on the following result [6].

Theorem 2. *Let A be a finite subgroup of* $\mathrm{PGL}_{n+1}(k)$*. For any $\ell > 2$,*

$$\nu_\ell(A) \leq \sum_{2 \leq s \leq n+1, t_\ell | s} (m_\ell + \nu_\ell(s)).$$

It follows that $\mathrm{PGL}_{n+1}(k)$ does not contain elements of prime order ℓ if $t_\ell \geq n+2$. For example, $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{Q}})$ does not contain elements of order $\ell > 5$.

The first example of an automorphism of order 7 in $\mathrm{Cr}_{\mathbb{Q}}(2)$ was given by J.-P. Serre. It is based on the following idea.

First he uses the fact from [9] that an algebraic 2-dimensional torus over an arbitrary field k is a rational variety (as always over k). Then he proves the following.

Theorem 3. *Let T be an algebraic k-torus and A be a finite subgroup of $T(k)$. Then*

$$\nu_\ell(A) \leq m_\ell \left[\frac{\dim T}{\phi(t_\ell)}\right],$$

where ϕ is the Euler function. Assume $m_\ell < \infty$ (e.g. k is finitely generated over its prime subfield). For any $n \geq 1$ there exists an n-dimensional k-torus T and a finite subgroup A of $T(k)$ such that $\nu_\ell(A) = m_\ell\left[\frac{\dim T}{\phi(t_\ell)}\right]$.

Corollary 4. *A 2-dimensional k-torus T with $T(k)$ containing an element of prime order $\ell > 2$ exists if and only if t_ℓ takes values in the set $\{1, 2, 3, 4, 6\}$.*

We can realize a 2-dimensional k-torus T as an open subset of a Del Pezzo surface of degree 6 that has a structure of a toric k-surface.

The main result of our paper [3] is the following

Theorem 5. *Let k be a perfect field of characteristic $p \geq 0$. Then $\mathrm{Cr}_2(k)$ contains an element of prime order $\ell > 5$ not equal to p if and only if there exists a 2-dimensional algebraic k-torus T such that $T(k)$ contains an element of order ℓ.*

We will also prove the following uniqueness result.

Theorem 6. *Assume that k is of characteristic 0 and does not contain a primitive ℓ-th root of unity. Then $\mathrm{Cr}_2(k)$ does not contain elements of prime order $\ell > 7$ and all elements of order 7 in $\mathrm{Cr}_2(k)$ are conjugate to an automorphism of a Del Pezzo surface of degree 6.*

Using the description of minimal G-surfaces given in the previous section and the fact that a Del Pezzo surface of degree $d \geq 3$ embeds in $\mathbb{P}^d_k$ or admits a canonical double cover of the plane or of quadratic cone for $d = 2$ or $d = 1$, Serre uses his theorem 2 to estimate an order of any finite subgroup of $\mathrm{Cr}_2(k)$ of order prime to $\mathrm{char}(k)$ (see [7].

Theorem 7. *Let* $M(k,\ell) = 2m+3$ *if* $\ell = 2$,

$$M(k,\ell) = \begin{cases} 2m & \text{if } t_\ell = 1,2, \ell > 3, \\ m & \text{if } t_\ell = 3,4, \ell > 3, \\ 0 & \text{if } t_\ell = 5, \ell > 6 \\ 4 & \text{if } t_\ell = 1,2, \ell > 3 \text{or } \ell = 3, t_\ell = m_\ell = 1, \\ 2m+1 & \text{otherwise.} \end{cases}$$

For any finite subgroup A *of* $\mathrm{Cr}_2(k)$

$$\nu_\ell \leq M(k,\ell).$$

Moreover, $M(k,\ell)$ *is the upper bound of the* $\nu_\ell(A)$.

4. Wild subgroups

Here we discuss finite subgroups of $\mathrm{Cr}_2(k)$ of order divisible by $p = \mathrm{char}(k)$. Again the work is still in progress. We study only cyclic groups of order p^s. We will also describe conjugacy classes of elements of order p^2 over algebraically closed field of characteristic $p > 0$.

Using the Jordan form it is easy to prove the following

Lemma 8. *For any element of order* p^s *in* $\mathrm{Aut}(\mathbb{P}^r_k)$ *we have* $s < 1 + \log_p(r+1)$.

For example, when $r = 1$, no elements of order $p^s, s \geq 2$, exist in $\mathrm{Aut}(\mathbb{P}^1_k)$. This easily implies that

Theorem 9. *Let* $f : X \to \mathbb{P}^1_k$ *be a conic bundle and* σ *be an automorphism of* X *of order* p^s *preserving the conic bundle. Then* $s \leq 2$.

A closer look at elements of order p^2 shows that a minimal automorphism of order p^2 of a conic bundle $X \to \mathbb{P}^1_k$ exist only when $p = 2$.

Next we consider the case of Del Pezzo surfaces. For example, if $d = 9$, $X = \mathbb{P}^2_k$, by Lemma 8 we get $s \leq 2$. All elements of order p^2 are conjugate in $\mathrm{Aut}(\mathbb{P}^2_k)$.

If $d = 8$, then $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ because the ruled surface $\mathbf{F}_1$ is not σ-minimal. We know that $\mathrm{Aut}(\mathbf{F}_0)$ contains a subgroup of index 2 isomorphic to $\mathrm{Aut}(\mathbb{P}^1_k) \times \mathrm{Aut}(\mathbb{P}^1_k)$. Applying Lemma 8 we obtain $s = 1$ if $p \neq 2$, and $s \leq 2$ otherwise. The automorphism of X given in affine coordinates by $(x,y) \mapsto (y+1,x)$ is of order 4.

If $d = 7$, as we explained in section 1, the surface is not σ-minimal.

Assume $d = 6$. Then $\mathrm{Aut}(X)$ is isomorphic to the semi-direct product $T \rtimes G$, where $T \cong k^{*2}$ is a 2-dimensional torus and G is a dihedral group $D_{12} \cong (\mathbb{Z}/2\mathbb{Z}) \times S_3$. Since T does not contain elements of order

p and D_{12} does not contain elements of order $p^s, s > 1$, we obtain that the only possibility is $s = 1$ and $p = 2, 3$.

Assume $d = 5$. We know that $\mathrm{Aut}(X)$ acts faithfully on the Picard group of X of a Del Pezzo surface of degree ≤ 5. Via this action it becomes isomorphic to a subgroup of the Weyl group $W(A_4) \cong S_5$. Thus $s = 1$ unless $p = 2$ and $s = 2$. The group $W(A_4)$ acts on $K_X^\perp \cong \mathbb{Z}^4$ via its standard irreducible representation on $\{(a_1, \ldots, a_5) \in \mathbb{Z}^5 : a_1 + \ldots + a_5 = 0\}$. A cyclic permutation of order 4 has a fixed vector. This shows that X is not σ-minimal.

Starting from the cases $d \leq 4$, the arguments become a little more involved. The most difficult case is the case $d = 1$ and $p = 2$. We will give the details.

The linear system $|-2K_X|$ defines a degree 2 map $f : X \to Q$, where Q is a quadratic cone in $\mathbb{P}^3_k$. Again, since $-K_X$ is ample, f is a finite map, and arguing as in the previous case we see that the map is separable. The Galois group of the cover is generated by the Bertini involution. For any divisor D we have

$$D + \gamma^*(D) \sim 2(D \cdot K_X)K_X. \tag{1}$$

This shows that β^* acts as the minus identity on the lattice $K_X^\perp$. The lattice $K_X^\perp$ is isomorphic to the root lattice of type E_8. The involution β^* generates the center of the Weyl group $W(E_8)$.

The automorphism group $\mathrm{Aut}(X)$ is a subgroup of $W(E_8)$. Possible orders $p^s, s > 1$, of minimal automorphisms are 4 and 8 (see [4]).

So we assume $p = 2$. The linear system $|-K_X|$ has one base point p_0. Blowing it up we obtain a fibration $\pi : X' \to \mathbb{P}^1_k$ whose general fibre is an irreducible curve of arithmetic genus 1. Since $-K_X$ is ample, all fibres are irreducible, and this implies that a general fibre is an elliptic curve (see [1]). Let S_0 be the exceptional curve of the blow-up. It is a section of the elliptic fibration. We take it as the zero in the Mordell-Weil group of sections of π. The map $f : X \to Q$ extends to a degree 2 separable finite map $f' : X' \to \mathbf{F}_2$, where $\mathbf{F}_2$ is the minimal ruled surface with the exceptional section E satisfying $E^2 = -2$. Its branch curve is equal to the union of E and a curve B from the divisor class $3f + e$, where f is the class of a fibre and $e = [E]$. We have $f'^*(E) = 2S_0$. The elliptic fibration on X' is the pre-image of the ruling of $\mathbf{F}_2$. We know that $\tau = \sigma^2$ acts identically on the base of the elliptic fibration. Since it also leaves invariant the section S_0, it defines an automorphism of the generic fibre considered as an abelian curve with zero section defined by S_0. If $\tau^2 = 1$, then τ is the negation automorphism, hence defines the Bertini transformation of the projective plane. Its image in the Weyl group $W(E_8)$ generates the center. The group of automorphisms of an abelian curve in characteristic 2 is of order 2 if the absolute invariant of the curve is not equal to 0 or of order 24 otherwise. In the latter case it is isomorphic to $Q_8 \rtimes \mathbb{Z}/3$, where Q_8 is the quaternion group with the center generated by the negation automorphism (see [8], Appendix A). Thus $\tau^4 = 1$ and the Weierstrass model of the generic fibre is

$$y^2 + a_3 y + x^3 + a_4 x + a_6 = 0.$$

In global terms the Weierstrass model of the elliptic fibration $\pi : X' \to \mathbb{P}^1_k$ is a surface in $\mathbb{P}(1,1,2,3)$ given by the equation

$$y^2 + a_3(u,v)y + x^3 + a_4(u,v)x + a_6(u,v),$$

where a_i are binary forms of degree i. It is obtained by blowing down the section S_0 to the point $(u,v,x,y) = (0,0,1,1)$ and is isomorphic to our Del Pezzo surface X. The image of the branch curve B is given by the equation $a_3(u,v) = 0$, i.e. B is equal to the pre-image of an effective divisor of degree 3 on the base plus the section S_0. Since a general point of B is a 2-torsion point of a general fibre, we see that all nonsingular fibres of the elliptic fibration are supersingular elliptic curves (i.e. have no non-trivial 2-torsion points). An automorphism of order 4 of X is defined by

$$(u,v,x,y) \mapsto (u,v,x+s(u,v)^2, y+s(u,v)x+t(u,v)),$$

where s is binary forms of degree 1 and t is a binary form of degree 3 satisfying

$$a_3 = s^3,\ t^2 + a_3 t + s^6 + a_4 s^2 = 0. \tag{2}$$

In particular, it shows that a_3 must be a cube, so we can change the coordinates (u,v) to assume that $s = u, a_3 = u^3$. The second equality in (2) tells that t is divisible by u, so we can write it as $t = uq$ for some binary form q of degree 2 satisfying $q^2 + u^2 q + u^4 + a_4 = 0$. Let α be a root of the equation $x^2 + x + 1 = 0$ and $b = q + \alpha u^2$. Then b satisfies $a_4 = b^2 + u^2 b$ and $t = ub + \alpha u^3$. Conversely, any surface in $\mathbb{P}(1,1,2,3)$ with equation

$$y^2 + u^3 y + x^3 + (b(u,v)^2 + u^2 b(u,v))x + a_6(u,v) = 0 \tag{3}$$

where b is a quadratic form in (u,v) and the coefficient at uv^5 in a_6 is not zero (this is equivalent to that the surface is nonsingular) is a Del Pezzo surface of degree 1 admitting an automorphism of order 4

$$\tau : (u,v,x,y) \mapsto (u,v,x+u^2, y+ux+ub+\alpha u^3).$$

Note that $\tau^2 : (u,v,x,y) \mapsto (u,v,x,y+u^3)$ coincides with the Bertini transformation.

Theorem 10. *Let X be a Del Pezzo surface* (3)*. Then it does not admit an automorphism of order 8.*

Proof. Assume $\tau = \sigma^2$. Since σ leaves invariant $|-K_X|$, it fixes its unique base point, and lifts to an automorphism of the elliptic surface X' preserving the zero section S_0. Since the general fibre of the elliptic fibration $f : X' \to \mathbb{P}^1_k$ has no automorphism of order 8, the transformation σ acts nontrivially on the base of the fibration. Note that the fibration has only one singular fibre F_0 over $(u, v) = (0, 1)$. It is a cuspidal cubic. The transformation σ leaves this fibre invariant and hence acts on $\mathbb{P}^1_k$ by $(u, v) \mapsto (u, u + cv)$ for some $c \in k$. Since the restriction of σ to F_0 has at least two distinct fixed points: the cusp and the origin $F_0 \cap S_0$, it acts identically on F_0 and freely on its complement $X' \setminus F_0$.

Recall that X' is obtained by blowing up 9 points $p_1, \ldots, p_9$ in $\mathbb{P}^2_k$, the base points of a pencil of cubic curves. We may assume that X is the blow-up of the first 8 points, and the exceptional curve over p_9 is the zero section S_0. Let S be the exceptional curve over any other point. We know that $\beta = \sigma^4$ is the Bertini involution of X. Applying formula (1), we obtain that $S \cdot \beta(S) = 3$. Identifying $\beta(S)$ and S with their pre-images in X', we see that $\beta(S) + S = S_0$ in the Mordell-Weil group of sections of $\pi : X' \to \mathbb{P}^1_k$. Thus S and $\beta(S)$ meet at 2-torsion points of fibres. However, all nonsingular fibres of our fibration are supersingular elliptic curves, hence S and $\beta(S)$ can meet only at the singular fibre F_0. Let $Q \in F_0$ be the intersection point. The sections S and $\beta(S)$ are tangent to each other at Q with multiplicity 3. Now consider the orbit of the pair $(S, \beta(S))$ under the cyclic group $\langle\sigma\rangle$. It consists of 4 pairs

$$(S, \sigma^4(S)),\ (\sigma(S), \sigma^5(S)),\ (\sigma^2(S), \sigma^6(S)),\ (\sigma^3(S), \sigma^7(S)).$$

Let $D_i = \sigma^i(S) + \sigma^{i+4}(S)$, $i = 1, 2, 3, 4$. We have $D_1 + \ldots + D_4 \sim -8K_X$,, hence for $i \neq j$ we have $D_i \cdot D_j = (64 - 16)/12 = 4$. Let $Y \to X$ be the blow-up of Q. Since Q is a double point of each D_i, the proper transform $\bar{D}_i$ of each D_i in Y has self-intersection 0 and consists of two smooth rational curves intersecting at one point with multiplicity 2. Moreover, we have $\bar{D}_i \cdot \bar{D}_j = 0$. Applying (1), we get $D_i \in |-2K_X|$. Since Q is a double point of D_i, we obtain $\bar{D}_i \in |-2K_Y|$. The linear system $|-2K_Y|$ defines a fibration $Y \to \mathbb{P}^1_k$ with a curve of arithmetic genus 1 as a general fibre (an elliptic or a quasi-elliptic fibration) and four singular fibres $\bar{D}_i$ of Kodaira's type III. The automorphism σ acts on the base of the fibration and the four special fibres form one orbit. But the action of σ on $\mathbb{P}^1_k$ is of order 2 and this gives us a contradiction. $\square$

Remark 1. A computational proof of Theorem 7 was given by J.-P. Serre.

To summarize one can prove the following result (see [2]).

Theorem 11. *An element of order p^2 not conjugate to a projective transformation exists only if $p = 2$. Assume that k is algebraically closed. An element of order 4 is either conjugate to a projective transformation, or conjugate to an element realized by a minimal automorphism of a conic bundle, or a Del Pezzo surface of degree 1.*

For the completeness sake let us add that elements of order p not conjugate to a projective transformations occur for any p. They can be realized as automorphisms of conic bundles, and if $p = 2, 3, 5$ as automorphisms of Del Pezzo surfaces.

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TOWARD RESOLUTION OF SINGULARITIES FOR ARBITRARY CHARACTERISTICS

HIRAKU KAWANOUE

1. Introduction

We work over an algebraically closed field $k = \overline{k}$.

Our theme is resolution of singularities for a variety X/k. As is well known, resolution of singularities exist in the following cases:

- $\operatorname{char} k = 0$, $\forall d = \dim X$
  - '64 Hironaka,
  - '89 Villamayor (simplification)
  - '91 Bierstone-Milman (simplification)
- $\operatorname{char} k = p > 0$, $\forall d \leq 3$
  - '66 Abhyankar ($\forall p \geq 3! = 6$),
  - '07 Cossart-Piltant ($d = 3,\ p = 2, 3, 5$)

and it is *open* problem for the case of $\operatorname{char} k = p > 0$ and $\forall d \geq 4$.

Our goal is to give constructive proof for the existence of resolution of singularities for the case of $\forall p = \operatorname{char} k$ and $\forall d = \dim X$.

We introduced the **Idealistic Filtration Program (IFP)** for this goal. Here we present the introduction of the idea of IFP.

This is joint work with Kenji Matsuki.

2. Review for $\operatorname{char} k = 0$ approach

We review briefly the known algorithm for resolution in characteristic 0, after Villamayor and Bierstone-Milman. We only deal with the case with *no exceptional divisors*, for simplicity.

One of major approaches for resolution of singularities is to give *embedded resolution*. Namely, for pair $(X \subset M)$, where M is a nonsingular variety and X is a closed subset of M, we construct

$$f\colon M^{\sim} \to M\colon \text{ sequence of blowups along nonsingular centers}$$

such that

(1) f is isomorphic over $M \setminus X$
(2) $f^{-1}(X)$ is a simple normal crossing divisor.

It is well known that the existence of resolution of singularities is deduced from the existence of "canonical" embedded resolution.

In the known approach for resolution in characteristic 0, they construct embedded resolution along the following strategy:

- For each closed point $P \in M$, attach the invariant inv_P.

- Blowup along the maximal locus of invariant, and see the decrease of invariant after blowup.

The invariant inv_P is of the form

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty) \quad (\mu_i \in \mathbb{Q}),$$

and μ_i's are defined inductively as follows:

(0) Put $R_0 = \mathcal{O}_{M,P}$ and $b_0 = 1$. Let $I_0 \subset R_0$ be the defining ideal of X at $P \in M$. We regard the triplet $\boxed{(I_0 \subset R_0, b_0)}$ as the *initial data*. We denote the multiplicity at P by ord_P and define $\boxed{\mu_0 = \mathrm{ord}_P(I_0)/b_0}$.

(1) We denote the set of all derivations $\partial\colon R \to R$ over k as $\mathrm{Der}(R)$. For an ideal $J \subset R$, we define an ideal $D(J) \subset R$ as

$$DJ = J + (\partial g \mid g \in J,\ \partial \in \mathrm{Der}(R))\,.$$

By definition of $\mathrm{ord}_P(I_0)$, we can find $f_0 \in D^{\mathrm{ord}_P(I_0)-1}(I_0)$ with $\mathrm{ord}_P(f_0) = 1$. Fix one such $f_0 \in R$ and put

$$R_1 = R_0/f_0R_0,\ b_1 = \mathrm{ord}_P(I_0)!,\ I_1 = \sum_{0\le i<\mathrm{ord}_P(I_0)} \left(D^{\mathrm{ord}_P(I_0)-i}(I_0)\right)^{b_1/i}.$$

Now we obtain new data $\boxed{(I_1 \subset R_1, b_1)}$. We define $\boxed{\mu_1 = \mathrm{ord}_P(I_1)/b_1}$.

(2) Do the same routine as in (1). Namely, take $f_1 \in D^{\mathrm{ord}_P(I_1)-1}(I_1)$ with $\mathrm{ord}_P(f_1) = 1$, and put

$$R_2 = R_1/f_1R_1,\ b_2 = \mathrm{ord}_P(I_1)!,\ I_2 = \sum_{0\le i<\mathrm{ord}_P(I_1)} \left(D^{\mathrm{ord}_P(I_1)-i}(I_1)\right)^{b_2/i}.$$

We have data $\boxed{(I_2 \subset R_2, b_2)}$ and define $\boxed{\mu_2 = \mathrm{ord}_P(I_2)/b_2}$.

(3) Repeat this procedure until $I_{t+1} = (0)$.

Remark.

(1) The procedure presented above *does* depend on the choice of f_i's. Nevertheless, μ_i's *do not* depend on this ambiguity, and thus invariant is well-defined.

(2) $H = V(f_0)$ is called a *hypersurface of maximal contact* (abbreviated simply as "maximal contact" in the rest of this article) of X at P. A maximal contact is a kind of nonsingular local hypersurface, but we do not give here its precise definition. One of the feature of a maximal contact is the following:

$$\{Q \in M \mid \mathrm{ord}_Q(I_0) \ge \mathrm{ord}_P(I_0)\} \subset H \quad \text{near } P.$$

The data $(I_1 \subset R_1, b_1)$ is regarded as the information of the left hand side in the above equation. In fact, we have

$$\{Q \in M \mid \mathrm{ord}_Q(I_0) \ge \mathrm{ord}_P(I_0)\} = \{Q \in H \mid \mathrm{ord}_Q(I_1) \ge b_1\},$$

which yields the scheme for "induction on dimension" in characteristic 0 case.

In the case of positive characteristic, it is known that a maximal contact *does not* exist in general. This is the main hurdle when we try to apply the known algorithm to the case of positive characteristic.

In IFP, we generalize the object. In the known algorithm, main object is the pair (I, b) with an ideal I and rational number b. We generalize this notion and define idealistic filtration $\mathbb{I}$, introduced in the next section. By analyzing this idealistic filtration algebraically, we can find the substitute of maximal contact, called LGS.

	known case	IFP case
object	pair (I, b)	idealistic filtration $\mathbb{I}$
local hypersurface	maximal contact	LGS of $\mathbb{I}$

3. Idealistic filtration

We introduce the idealistic filtration, the main object of IFP.

Let R be a k-algebra. For a subset $\mathbb{J} \subset R \times \mathbb{R}$ of $R \times \mathbb{R}$, we denote the level a set of $\mathbb{J}$ as $\mathbb{J}_a$. Namely,

$$\mathbb{J}_a = \{f \in R \mid (f, a) \in \mathbb{J}\}.$$

Definition. A subset $\mathbb{I} \subset R \times \mathbb{R}$ is called an *idealistic filtration* on R if $\mathbb{I}$ satisfies the following conditions:

(1) $\mathbb{I}_0 = R$.
(2) $\mathbb{I}_a$ is an ideal of R.
(3) $\mathbb{I}_a \mathbb{I}_b \subset \mathbb{I}_{a+b}$.
(4) $\mathbb{I}_a \supset \mathbb{I}_b \quad (a \leq b)$.

Remark. We interpret "$f \in \mathbb{I}_a$" as the information "$\mathrm{ord}_P(f) \geq a$".

Definition. For a subset $\mathbb{J} \subset R \times \mathbb{R}$, We define the support of $\mathbb{J}$ as

$$\mathrm{Supp}(\mathbb{J}) = \left\{ P \in \mathrm{maxSpec}(R) \mid \inf_{a>0} \frac{\mathrm{ord}_P(\mathbb{J}_a)}{a} \geq 1 \right\}$$

Definition. For a subset $\mathbb{J} \subset R \times \mathbb{R}$, the minimum idealistic filtration containing $\mathbb{J}$ is called the idealistic filtration *generated by* $\mathbb{J}$ and denoted as $\mathrm{G}(\mathbb{J})$.

We introduce the saturation of idealistic filtration to obtain much information for resolution problem.

Definition. Let $\mathbb{I}$ be an idealistic filtration on R. We denote the set of differential operators of degree $\leq t$ on R over k as $\mathrm{Diff}_{\leq t}(R/k)$.

We say $\mathbb{I}$ is $\mathfrak{D}$-*saturated* if the following condition holds:

$$\partial(\mathbb{I}_a) \subset \mathbb{I}_{a-t} \quad \text{for any } t \in \mathbb{Z}_{\geq 0},\ \partial \in \mathrm{Diff}_{\leq t}(R/k) \text{ and } a \in \mathbb{R}.$$

The minimum $\mathfrak{D}$-saturated idealistic filtration containing $\mathbb{I}$ is called the $\mathfrak{D}$-*saturation* of $\mathbb{I}$, and denoted as $\mathfrak{D}(\mathbb{I})$.

Remark. "A differential operator ∂ of degree $\leq t$ on R over k" is a k-linear map $\partial\colon R \to R$ characterized by "generalized Leibnitz rule"

$$\sum_{T \subset S} (-1)^{\#T} F_{S \setminus T} \partial(F_T) = 0,$$

where $S = \{0, 1, \ldots, t\}$, $F_I = \prod_{i \in I} f_i$ and f_i's are arbitrary elements in R. One can find in EGA IV §16 more detailed account.

From now on, replacing $\mathbb{I}$ by $\mathfrak{D}(\mathbb{I})$ if necessary, we always deal with only $\mathfrak{D}$-saturated idealistic filtrations. In the next section, we analyze $\mathfrak{D}$-saturated idealistic filtration and give the definition of LGS.

4. Leading generator system

We introduce the notion of LGS, substitute of maximal contact in any characteristics.

We restrict our attention to the local situation. Thus, in the rest of this article, we assume the following conditions unless specified:

- $R = (R, \mathfrak{m}) = \mathcal{O}_{M,P}$ is a local ring at $P \in M$, where M is a nonsingular variety and $P \in M$ is a closed point.
- $\mathbb{I}$ is a $\mathfrak{D}$-saturated idealistic filtration on R.
- We assume $\mu(\mathbb{I}) \geq 1$, where $\mu(\mathbb{I}) = \inf_{a \in \mathbb{R}_{>0}} \dfrac{\mathrm{ord}_P(\mathbb{I}_a)}{a}$. In other words, we assume $P \in \mathrm{Supp}(\mathbb{I})$.

First we introduce the leading algebra $\mathrm{L}(\mathbb{I})$.

Definition. For $n \in \mathbb{Z}_{\geq 0}$, let $\pi_n \colon R \to R/\mathfrak{m}^{n+1}$ be the natural projection. Since $\mu(\mathbb{I}) \geq 1$, we have $\pi_n(\mathbb{I}_n) \subset \mathfrak{m}^n$. We define the *leading algebra* $\mathrm{L}(\mathbb{I})$ of $\mathbb{I}$ as

$$\mathrm{L}(\mathbb{I}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \pi_n(\mathbb{I}_n) \subset \mathrm{Gr}(R) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

In our setting, it is clear that the graded ring $\mathrm{Gr}(\mathbb{I})$ is isomorphic to a polynomial ring over k, i.e. $\mathrm{Gr}(R) \cong k[Y]$. Therefore $\mathrm{L}(\mathbb{I})$ corresponds to a graded k-subalgebra $L \subset k[Y]$. Since $\mathbb{I}$ is $\mathfrak{D}$-saturated, we also see that L is stable under differential operations. That is, we see

$$\partial_{Y^I} L \subset L \quad (\forall I\text{: multi-index}),$$

where ∂_{Y^I} is defined by the formula $\partial_{Y^I} Y^J = \binom{J}{I} Y^{J-I}$.

Now we face the following question:

> Let $L \subset k[Y]$ be a graded k-subalgebra which is stable under differential operations. What can we say on such L?

In fact, in characteristic 0, L is generated by homogeneous part $S_1 \subset S$ of degree 1. Moreover, if $P \in X \subset M$ and $\mathbb{I} = \mathfrak{D}(\mathrm{G}(I_X \times \{\mathrm{ord}_P(I_X)\}))$, non-zero elements of S_1 correspond to maximal contact of X. It is important nature of maximal contact if one try to define invariant, since it is necessary to eliminate $\mathrm{ord}_P(I_X)$ part to detect higher order $\mu_1, \mu_2, \ldots$. Therefore, the substitute of maximal contact should correspond to generators of $\mathrm{L}(\mathbb{I})$ as k-algebra.

In positive characteristic case, we have the following result:

Proposition (Hironaka-Oda)**.** *Assume* $\mathrm{char}\, k = p > 0$. *Let* $L \subset k[Y]$ *be a graded* k*-subalgebra stable under differential operations. Then, there exist integers* $N, e_1, \ldots, e_N \in \mathbb{Z}_{\geq 0}$ *and homogeneous elements* $f_1, \ldots, f_N \in k[Y]_1$ *of degree* 1 *such that* $f_1, \ldots, f_N$ *are* k*-linearly independent and* $\{f_1^{p^{e_1}}, \ldots, f_N^{p^{e_N}}\}$ *generates* L *as a* k*-algebra.*

Remark. As stated above, generators of L live in several degree p-th power parts. In characteristic 0 case, we should regard $p = \infty$. Then, all elements of degree p^i with $i > 0$ disappear, and generators only appear at degree $p^0 = 1$ part.

By virtue of the above proposition, we define LGS.

Definition. The representative $\mathbb{H} \subset \mathbb{I}$ of generators of $\mathrm{L}(\mathbb{I})$, in the shape of above proposition, is called a *leading generator system (LGS)* of $\mathbb{I}$. By definition,

$$\mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \leq i \leq N\} \subset \mathbb{I}, \quad h_i = f_i^{p^{e_i}} + (\text{higher order part})$$

Remark.
(1) Note that LGS is not unique.
(2) If $e_i > 0$, h_i defines a singular local hypersurface. This is big difference to the original maximal contact.

5. Paired invariants

We introduce invariants σ and $\mu^{\sim}$, which are necessary to translate the known algorithm to IFP.

5.1. **Framework.** In the known algorithm for characteristic 0 case, we repeat the procedure of "restrict data to maximal contact H (= low dimensional ambient space)" and "estimating order on H".

object	(I_0, b_0)	(I_1, b_1)	$\cdots$	(I_t, b_t)	$(0, b_{t+1})$
ambient space	M	$\supset H_1$	$\cdots$	$\supset H_t$	$\supset H_{t+1}$
(higher) order	μ_0	μ_1	$\cdots$	μ_t	∞

Initial data is a pair $(I_0, b_0) = (I_X, 1)$ on M, and invariant is

$$\mathrm{inv}_P = (\mu_0, \mu_1, \ldots, \mu_t, \infty).$$

In IFP, we cannot restrict data to low dimensional space since LGS may give singular local hypersurfaces. Therefore, we continue to stay the *same* (original) ambient space. Instead of restriction, we

- *enlarge* $\mathbb{I}$ and *enlarge* its LGS, and
- estimate order *modulo LGS*.

object	$\mathbb{I}_0$	$\subset \mathbb{I}_1$	$\cdots$	$\subset \mathbb{I}_t$	$\subset \mathbb{I}_{t+1}$
ambient space	M	M	$\cdots$	M	M
(higher) order	$(\sigma_0, \mu_0^{\sim})$	$\cdots$	$\cdots$	$(\sigma_t, \mu_t^{\sim})$	(σ_{t+1}, ∞)

Initial data is an idealistic filtration $\mathbb{I}_0 = \mathrm{G}(I_X \times \{1\})$ on M, and invariant is

$$\mathrm{inv}_P = ((\sigma_0, \mu_0^{\sim}), \ldots, (\sigma_t, \mu_t^{\sim}), (\sigma_{t+1}, \infty)).$$

By this translation, we need 2 new invariants σ and $\mu^{\sim}$ introduced below:

5.2. **Definitions.** Settings and notations are same to the ones in §4. We denote an LGS of $\mathbb{I}$ as

$$\mathbb{H} = \{(h_i, p^{e_i}) \mid 1 \leq i \leq N\}.$$

Definition ($\sigma \leftrightarrow$ "dimension of ambient space".)**.**
$\sigma(\mathbb{I})$ is defined as an infinite sequence

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_1, \ldots) \in \mathbb{Z}_{\geq 0}^{\infty},$$

where each σ_i is defined by the formula $\sigma_e = \dim R - \#\{i \mid e_i \leq e\}$.

In characteristic 0 case, $\sigma(\mathbb{I})$ is automatically a constant sequence. Namely,

$$\sigma(\mathbb{I}) = (\sigma_0, \sigma_0, \ldots), \quad \sigma_0 = \dim R - \text{"\# of maximal contact"},$$

Definition ($\mu^{\sim}$ ↔"order on low dimensional ambient space".)**.**

We define the order modulo $\mathbb{H}$ of an ideal $J \subset R$ as

$$\mathrm{ord}_{\mathbb{H}}(J) = \sup \left\{ n \geq 0 \mid J \subset \mathfrak{m}^n + \sum_{i=1}^{N} Rh_i \right\}.$$

$\mu^{\sim}(\mathbb{I})$ is defined by

$$\mu^{\sim}(\mathbb{I}) = \inf_{a>0} \frac{\mathrm{ord}_{\mathbb{H}}(\mathbb{I}_a)}{a}.$$

In characteristic 0 case, all $H_i = V(h_i)$ and $\bigcap_i H_i$ are nonsingular. Thus $\mathrm{ord}_{\mathbb{H}}(J)$ is the order of J, estimated on $\bigcap_i H_i$.

Proposition. *$\sigma(\mathbb{I})$ and $\mu^{\sim}(\mathbb{I})$ are independent of the choice of $\mathbb{H}$.*

6. Results

As is already repeated, LGS may define singular local hypersurface, and it causes several serious problems. We present 2 results to overcome such problems.

In the known algorithm for characteristic 0 case, the maximum locus of invariant defines the center of next blowup. It is given by the intersection of maximal contact, which is automatically nonsingular due to the nonsingularity of maximal contact. In IFP case, the maximum locus of invariant corresponds to the support of last enlarged idealistic filtration $\mathbb{I}_{t+1}$, where $\mu^{\sim}(\mathbb{I}_{t+1}) = \infty$. The following theorem guarantees the nonsingularity of maximum locus of invariant in IFP.

Theorem (Nonsingulaity principle)**.** *Settings and notations are same to the ones in §4. Assume $\mu^{\sim}(\mathbb{I}) = \infty$. Then, the following holds:*

(1) *$\mathbb{I}$ is generated by any LGS $\mathbb{H}$, i.e.* $\mathrm{G}(\mathbb{H}) = \mathbb{I}$.

(2) *There exist a part of regular system of parameters $\{g_i \mid 1 \leq i \leq N\} \subset R$ and non-negative integers $\{e_i \mid 1 \leq i \leq N\} \subset \mathbb{Z}_{\geq 0}$ such that $\{(g_i^{p^{e_i}}, p^{e_i}) \mid 1 \leq i \leq N\}$ is an LGS of $\mathbb{I}$.*

Especially, $\mathrm{Supp}(\mathbb{I})$ *is a germ of nonsingular variety at $P \in M$.*

We explain only the last statement. By (1), we have $\mathrm{Supp}(\mathbb{I}) = \mathrm{Supp}(\mathbb{H})$. Choose $\mathbb{H}$ given in (2). Then, $\mathrm{Supp}(\mathbb{H}) = V(g_i \mid 1 \leq i \leq N)$, which is nonsingular at P.

On the nonsingular ambient space, multiplicity is upper semi-continuous. Therefore, so is the invariant in known characteristic 0 case. In IFP, as we use *the order modulo LGS*, we have to verify the upper semi-continuity of the invariant.

Theorem. *Let $M = \mathrm{Spec}\, R$ be a nonsingular affine variety over k, and $\mathbb{I}$ a $\mathfrak{D}$-saturated idealistic filtration on R. We denote $\mathbb{I}_P$ as the localization of $\mathbb{I}$ at $P \in M$. Then, the pair $(\sigma(\mathbb{I}_P), \mu^{\sim}(\mathbb{I}_P))$ with lexicographical order defines an upper semi-continuous function on* $\mathrm{maxSpec}\, R$.

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Construction of surfaces of general type with $p_g = 0$ via $\mathbb{Q}$-Gorenstein smoothings

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For nonsingular projective curve C, genus C=0 implies C is isomorphic to the projective line $\mathbb{P}^1$.

The projective plane $\mathbb{P}^2$ has $p_g(\mathbb{P}^2) = 0$ and $q(\mathbb{P}^2) = 0$.
so naturally

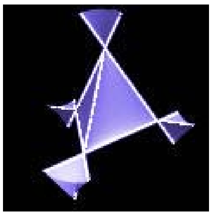
Question (by Max Noether)
Is every nonsingular projective surface S with $p_g(S) = q(S) = 0$ a rational surface?

$p_g(S) := h^0(K_S)$ geometric genus of S, K_S canonical divisor
$q(S) := h^0(\Omega_S)$ irregularity of S

Answer: There are counterexamples:

- Enriques surfaces (Enriques, 1894): normalization of singular sextic in $\mathbb{P}^3$, $\kappa = 0$.

$$(x_0x_1x_2)^2 + (x_0x_1x_3)^2 + (x_0x_2x_3)^2 + (x_1x_2x_3)^2$$
$$+x_0x_1x_2x_3(x_0^2 + x_1^2 + x_2^2 + x_3^2) = 0 \quad \text{in } \mathbb{P}^3.$$



- Rationality criterion was proved by Castelnouvo:

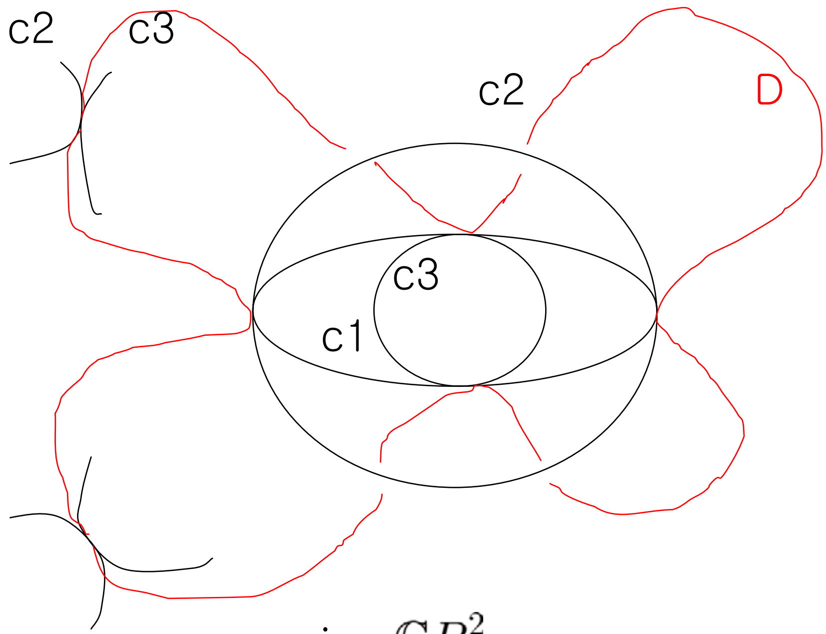
$$P_2(X) = h^0(X, 2K_X) = 0, q(X) = 0$$

$\Rightarrow$ X is a rational surface.

- Godeaux surface (Godeaux, 1931): minimal surface of general type with $p_g = 0, K^2 = 1$ obtained as the quotient of a smooth quintic surface in $\mathbb{P}^3$ by a free $\mathbb{Z}_5$ -action.

- The first example of a surface general type with $p_g = 0, K^2 = 2$ was constructed by Campedelli in the 30's as a ramified double cover of $\mathbb{P}^2$;
more precisely as the desingularization of a double cover of $\mathbb{P}^2$ branched along a reducible curve of degree 10 with 6 [3, 3] points not lying on a conic.
Nowdays minimal surface of general type with $p_g = 0, K^2 = 2$ are called (numerical) Campedelli surfaces.

c1, c2, c3 conics, D quartic



in $\mathbb{C}P^2$

Severi conjectured (1949)

$p_g(X) = 0, H_1(X, \mathbb{Z}) = 0 \;\; \Rightarrow \;\; X$ rational surface ?

Dolgachev constructed elliptic surfaces with

$p_g = 0, \pi_1 = 1, K^2 = 0, \;\; \kappa = 1.$

Nowdays these surfaces are called Dolgachev surfaces.

Question

Is there a minimal surface of general type with

$p_g(X) = 0, \pi_1(X) = 1 \, (or \; H_1(X, \mathbb{Z}) = 0)?$

Surfaces of general type X with $p_g = 0$ (and so $q = 0$) in principle can be classified, since the moduli space has finitely many components by Giesker's theorem.
By the Miyaoka-Yau inequality,
X minimal $\Rightarrow$ $1 \leq K_X^2 \leq 9.$

In practice not much is known. Surfaces of general type with geometric genus zero have been studied by algebraic geometers for a long time and plenty of examples have been constructed, but at present a classification seems still out of reach.

Surfaces of general type with $p_g = 0$ are important to classify surfaces of general type, and to study threefolds with a fibration to a curve.

How does one construct examples?
Mainly two appoaches (classical methods)

- Godeaux - taking quotients by group actions of known surfaces (finite quotient methods)
- Campedelli – constructing suitable covers of known surfaces (covering methods)

Barlow [Invent. Math. 1985] constructed a simply connected minimal surface of general type with $p_g = 0, K^2 = 1$ obtained by a variation of the Godeaux construction, in which the group has some isolated fixed points.

It was the first and up to 2006 was the only known example of a simply connected surface of general type with vanishing geometric genus .

Recently, Bauer, Catanese, and others construct many examples of surfaces of general type with $p_g = 0 (\pi_1 \neq 1)$ and gave a classification that admit an unramified covering which is isomorphic to a product of curves.

(generalization Beauville's construction)

It is the first systematic way to find many examples of surfaces of general type with $p_g = 0$.

A interesting and hard question concerning these surfaces is the construction of simply connected examples, which are of great interest also in the study of differentiable four-manifolds.

X topological 4-manifold

$Q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z})$

(unimodular symmetric bilinear form)

Freedman's theorem: If $\pi_1(X) = 1$ then X is uniquely determined up to homeomorphism by Q.

A simply connected surface of general type with $p_g = 0$ is homeomorphic (not diffeomorphic) to a rational surface.

Barlow surface is homeomorphic to $\mathbb{P}^2 \# 8\bar{\mathbb{P}}^2$.

All known methods seem to be not useful in producing new simply connected examples and it has long been an open question whether there exist simply connected surfaces of general type with $p_g = 0, K^2 > 1$.

Y.Lee, J. Park [Invent. Math. 2007] constructed a minimal surface of general type with $p_g = 0, K^2 = 2, \pi_1 = 1$ by using a new method.

- Idea from a moduli space

Assume that there is a surface of general type X satisfying the given numerical invariants $\chi(\mathcal{O}_X) = p_g - q + 1, K^2$.

Gieseker proved that there is a quasi-projective moduli space $\mathcal{M}$ of X by Geometric Invariant Theory.

$$\dim \mathcal{M} \geq h^1(X, T_X) - h^2(X, T_X)$$

Compactify $\mathcal{M}$ by adding points corresponding to singular surfaces at boundary. There is a natural way to do this using Minimal Model Program of threefolds.

- Idea from Park's symplectic construction [Invent, 2005]

- Examples are constructed by a new method ($\mathbb{Q}$-Gorenstein smoothings of singular rational surfaces).

- The main example construction goes as follows:

Step 1: choose a special pencil of cubics in $\mathbb{P}^2$

$$\lambda(3line) + \mu(conic + line)$$

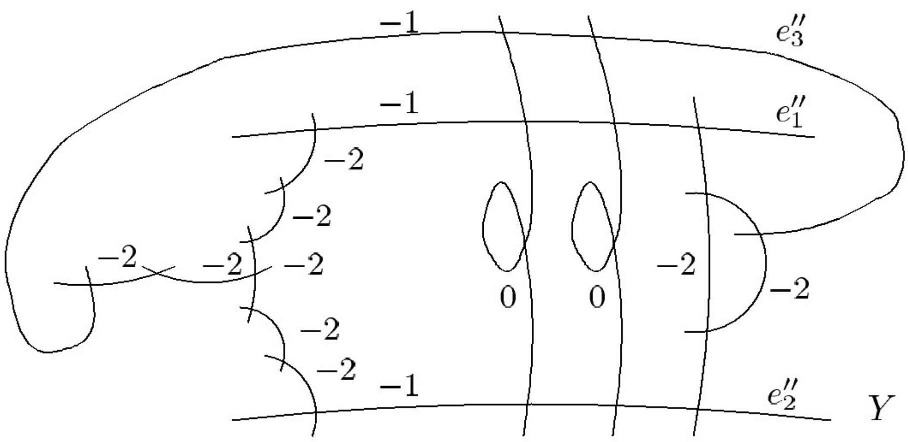
blows up its base locus —> a elliptic rational surface

Step 2: blows up further

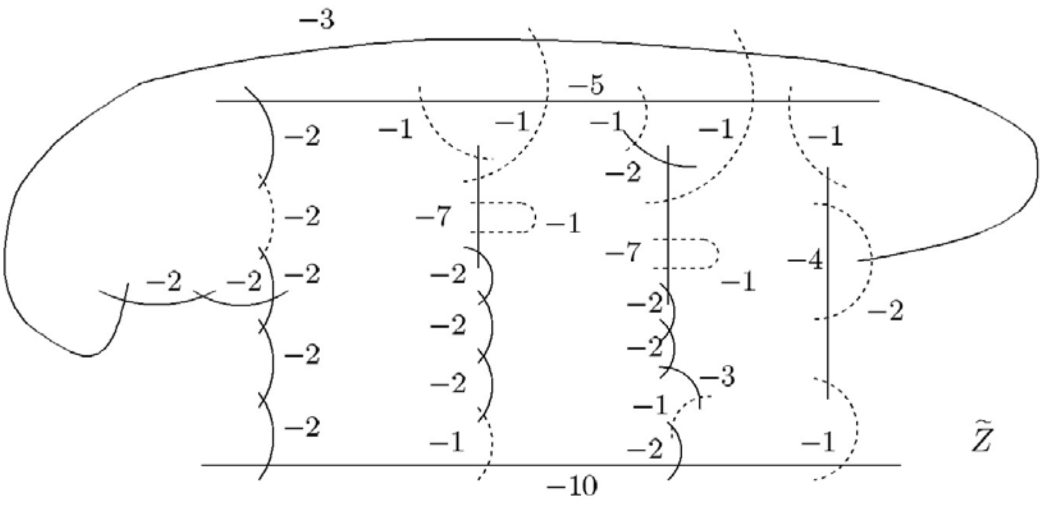
—> 5 disjoints chains of rational curves

—> blown down them to get a singular rational surface X.

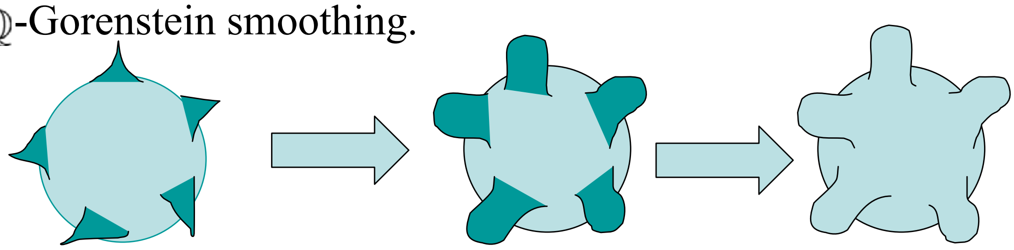
Step 1



Step 2



Every singularity of X is of class T, namely it admits a local $\mathbb{Q}$-Gorenstein smoothing.



Step 3: Using deformation theory, there is indeed a global $\mathbb{Q}$ -Gorenstein smoothing of X,

a one parameter family $\mathcal{X} \to \Delta$ of projective surfaces s.t.
the central fiber is X ;
the general fibre X_t is smooth and projective;
the relative canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$ -Cartier.

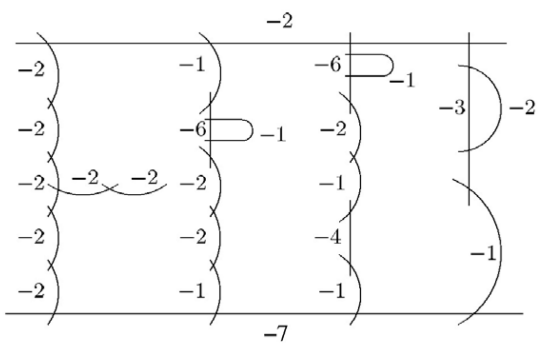
X_0(five singularities of class T) $\rightsquigarrow$ (deformation to) X_t

Step 4:

What properties does X_t have?

- By deformation, $K^2_{X_t} = 2$, $p_g(X_t) = q(X_t) = 0$.
- By configuration of the construction, X_t is minimal.
- By configuration of the construction and by using standard argument of Minor fiber, $\pi_1(X_t) = 1$

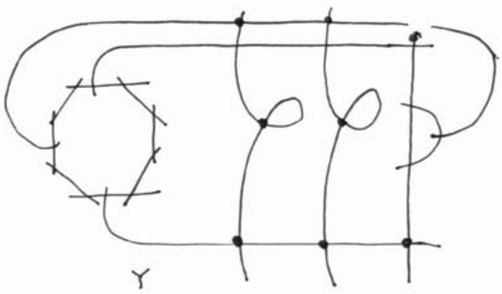
Example. $(p_g = 0, K^2 = 1, \pi_1 = 1)$



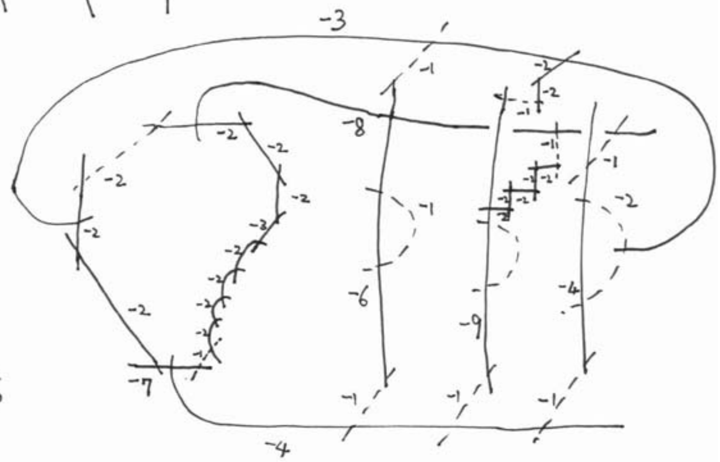
(-7) – (-2) – (-2) – (-2), (-6) – (-2) – (-2),
(-2) – (-6) – (-2) – (-3), (-4)

$K^2 =$ -11+12=1

$K^2 = 3$, $p_g = 0$. $\pi_1 = 1$ (H. Park, J. Park, D. Shim)



Y



$C_{2,1}$: $\overset{-4}{\circ}$

$C_{9,1}$: $\overset{-9}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$

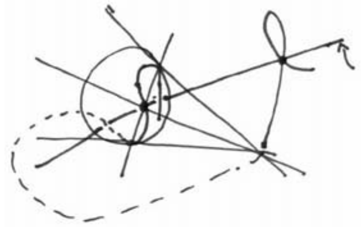
$C_{19,5}$: $\overset{-4}{\circ}$—$\overset{-7}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-3}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$

$C_{35,6}$: $\overset{-6}{\circ}$—$\overset{-8}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-3}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$

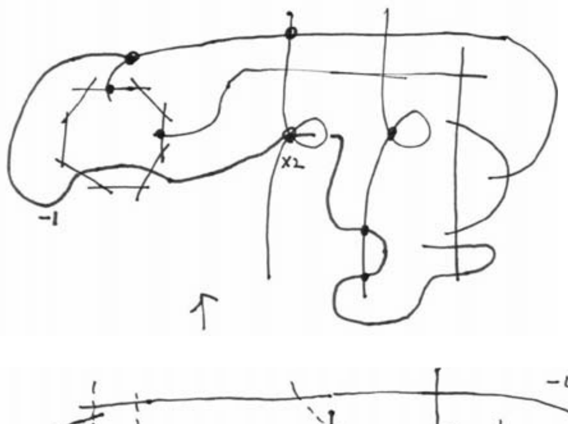
$K^2 = -21 + 24 = 3$

$Y \# 20\,\overline{\mathbb{P}}^2$

$K^2 = 4$, $p_g = 0$. $\pi_1 = 1$ (H. Park, J. Park. D. Shim)



bisection



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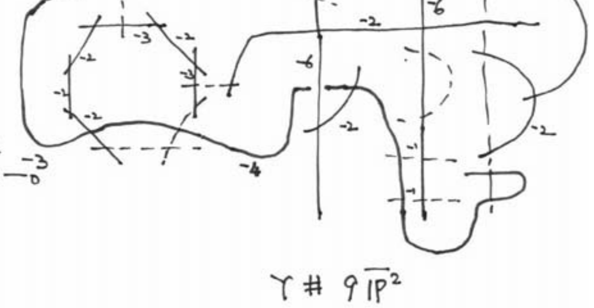


$C_{252,145}$:

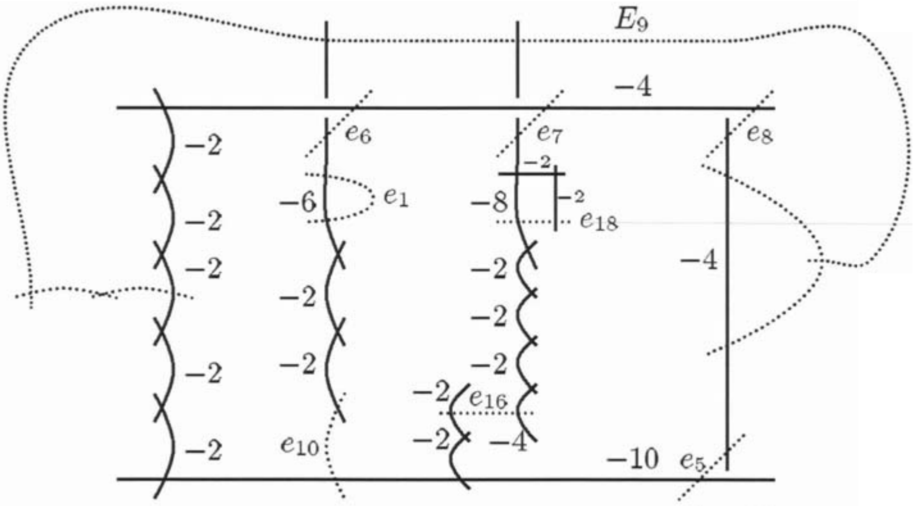
$\overset{-2}{\circ}$—$\overset{-4}{\circ}$—$\overset{-6}{\circ}$—$\overset{-2}{\circ}$—$\overset{-6}{\circ}$—$\overset{-2}{\circ}$—$\overset{-4}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-2}{\circ}$—$\overset{-3}{\circ}$—$\overset{-2}{\circ}$—$\overset{-3}{\circ}$

$K^2 = -9 + 13 = 4$

$Y \# 9\,\overline{\mathbb{P}}^2$

Example $(p_g = 0, K^2 = 2, H_1(X,\mathbb{Z}) = \mathbb{Z}_2)$

Lee-Park



$C_{22,15}$ $\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-10}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ}$ $\quad Y \# 18\overline{\mathbb{P}}^2$

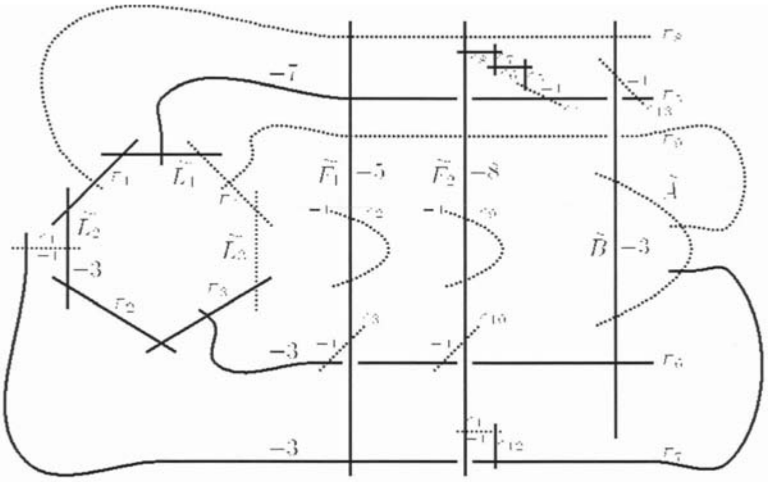
$C_{4,1}$: $\overset{-6}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}$

$C_{16,11}$: $\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-8}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-4}{\circ}$ $\quad K^2 = -18 + 20 = 2$

$C_{2,1}$ $\overset{-4}{\circ}$

Example $(p_g = 0, K^2 = 3, H_1(X,\mathbb{Z}) = \mathbb{Z}_2)$

H.Park - J. Park - D. Shin

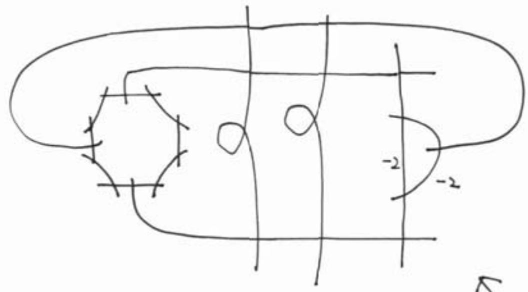


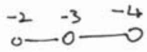
$C_{110,67}$: $\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-5}{\circ}-\overset{-7}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-3}{\circ}-\overset{-3}{\circ}$ $\quad Y \# 13\overline{\mathbb{P}}^2$

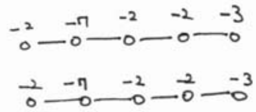
$C_{6,1}$: $\overset{-8}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}-\overset{-2}{\circ}$

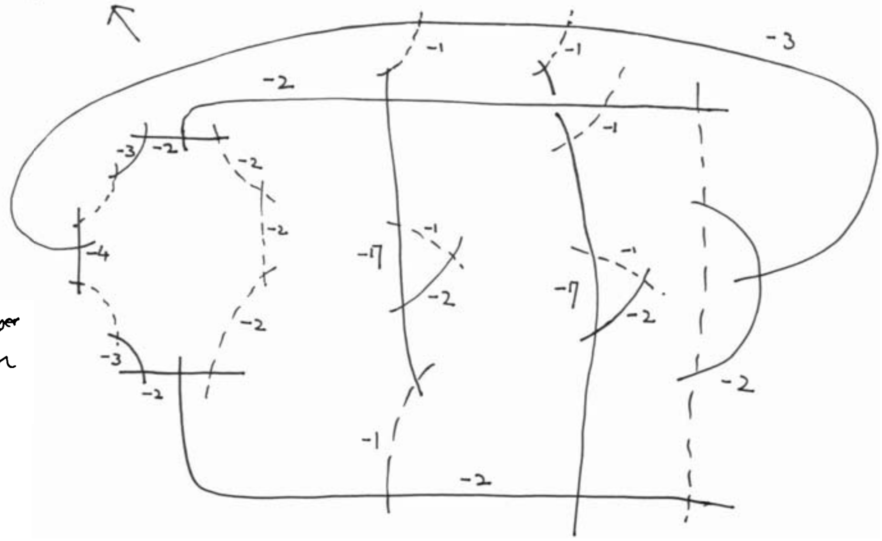
$K^2 = -13 + 16 = 3$

$K^2 = 2,\ p_g = 0,\ H_1 = \mathbb{Z}_3$ (Lee - Park)



$\frac{1}{18}(1,5)$ 

$\frac{1}{81}(1,44)$ 



$K^2 = -10 + 13 - 1 = 2$

-10: # of blow-ups

$13 = 3+5+5$

1: 2nd betti number of milnor fiber

The construction problem,

to find a simply connected surface of general type

with $p_g = 0$ and given $1 \leq K^2 \leq 4$, is solved.

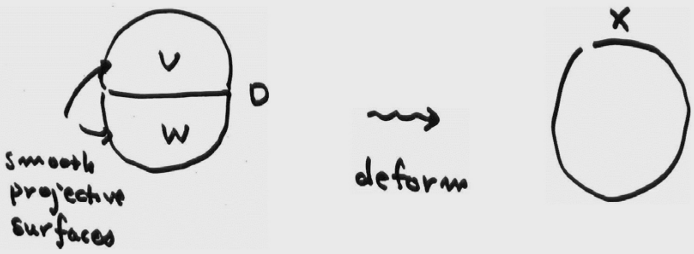
Question: Is there a minimal simply connected surface of general type with $p_g = 0$ and $5 \leq K^2 \leq 8$?

Remark: Similar construction does not work for $5 \leq K^2 \leq 8$.

In all constructions, $H^2(X_t, T_{X_t}) = 0$.

$H^1(X_t, T_{X_t}) = 10 - 2K^2_{X_t}$.

Symplectic construction should be modified.

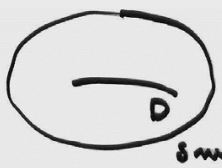


$$N_{D|V} \otimes N_{D|W} = \mathcal{O}_D$$

$\Rightarrow$ $\exists$ symplectic 4-manifold X

complex structure on X ?

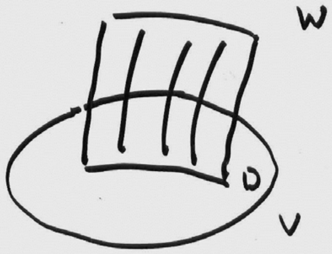
need some complicated conditions



V smooth projective surface

D smooth projective curve in V

Assume $\eta \neq 0 \in H^1(N_{D|V}) \Rightarrow 0 \to N_{D|V} \to \mathcal{E} \to \mathcal{O}_D \to 0$ ($\mathcal{E}$: rank 2 vector bundle)



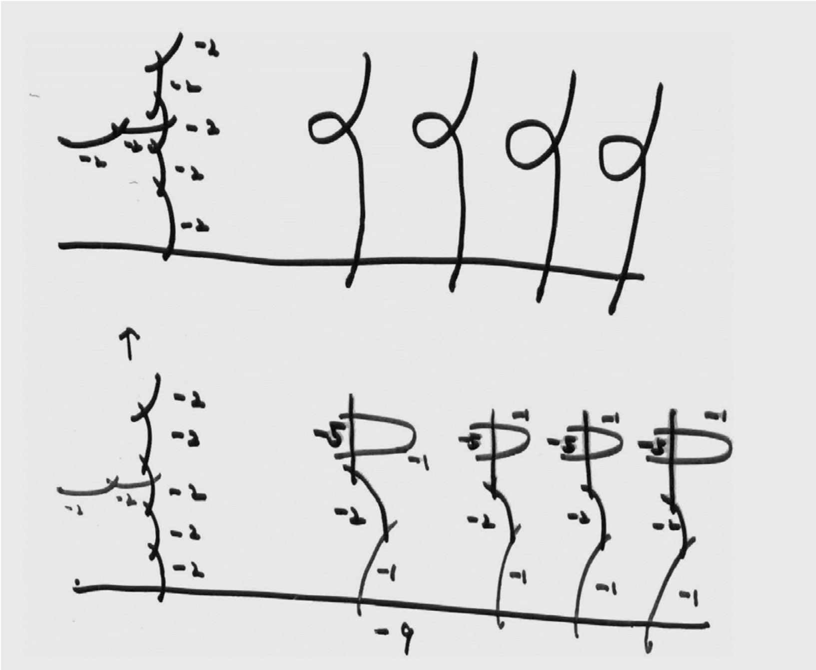
$\Rightarrow$ Let $W = \mathbb{P}_D(\mathcal{E})$

$$N_{D|V} \otimes N_{D|W} = \mathcal{O}_D$$

But no complex structure on X

if η is not image $H^1(T_V) \to H^1(N_{D|V})$

Modified version of Park's symplectic construction.



$\mathbb{Q}$-Gorenstein deformations

$(X_0, 0)$ germ of two-dimension quotient singularity

1st order deformation (local) $\leftrightarrow$ $T^1_{X_0} = Ext^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$

Obstruction space lies in $T^2_{X_0} = Ext^2_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$

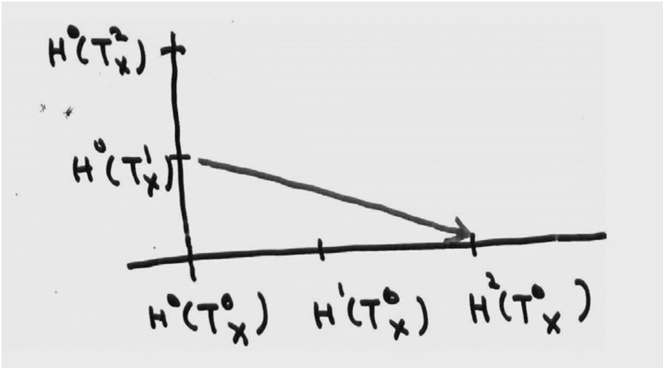
X normal projective surface with quotient singularities

1st order deformation (global) $\leftrightarrow$ $\mathbf{T}^1_X = \mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$

Obstruction space lies in $\mathbf{T}^2_X = \mathbf{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$

Spectral sequence $E_2^{p,q} = H^p(X, T^q_X) \Rightarrow \mathbf{T}^{p,q}_X$

$$H^i(X, T^j_X) = 0 \text{ if } i, j \geq 1$$



$$0 \to H^1(T_X^0) \to \mathbf{T}_X^1 \to H^0(T_X^1) \to H^2(T_X^0) \to T_0^2(X) \to 0$$

$$0 \to T_0^2(X) \to \mathbf{T}_X^2 \to H^0(T_X^2) \to 0$$

Key Part: $\mathrm{Ker}[H^0(T_X^1) \to H^2(T_X^0)]$

[Wahl], [Manetti] If $H^2(T_X^0) = 0$ then every local deformation $(X_0, 0)$ of the singularities may be globalized.

What condition implies $H^2(T_X^0) = 0$?

Main two technical Lemmas

1. Let V be the minimal resolution of X and E be the reduced Exceptional divisors.

$$H^2(T_V(-log\, E)) = 0 \Rightarrow H^2(T_X^0) = 0$$

2. From the constructions, our examples satisfy

$$H^2(T_V(-log\, E)) = 0.$$

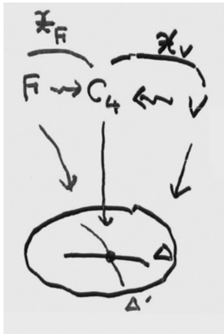
Consider special quotient singularities (singularity of class T)

1) To have a nice smoothing part in the 1st order deformation which has no obstruction (local smoothing),
2) To control numerical invariants and to use topological properties (Milnor fiber) of a general fiber of smoothing .

Example) $V \subset \mathbb{P}^5$ Vernose surface $C_V \subset \mathbb{P}^6$ cone of V

$F = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2)) \subset \mathbb{P}^5$ $C_F \subset \mathbb{P}^6$ cone of F

$C_4 \subset \mathbb{P}^5$ cone of the rotional normal curve of degree 4



$$K_{C_4}^2 = 9 = K_V^2, \; K_F^2 = 8$$

$K_{\mathcal{X}_V/\Delta}$ $\mathbb{Q}$-Cartier, $K_{\mathcal{X}_F/\Delta'}$ not $\mathbb{Q}$ -Cartier

If X_0 is a quotient singularity of type $\frac{1}{dr^2}(1, dra-1)$.
then $X_0 = Y_0/ < \sigma >, \; Y_0 : xy - z^{dr} = 0.$
$\sigma : (x, y, z) \to (\xi x, \xi^{-1} y, \xi^a z)$ ξ is a primitive r-th root of unity

And there is a $\mathbb{Q}$-Gorenstein smoothing.

$$X = Y/ < \sigma > \to \Delta, \; Y : xy - z^{dr} + t = 0$$

σ acts on Y via $\sigma : (x, y, z, t) \to (\xi x, \xi^{-1} y, \xi^a z, t)$
K_Y Cartier => K_X $\mathbb{Q}$-Cartier.

There is a d-dimensional $\mathbb{Q}$ -Gorenstein smoothing.
Conversely, if there is a $\mathbb{Q}$-Gorenstein smoothing then it is a RDP or a cyclic quotient singularity of type $\frac{1}{dr^2}(1, dra-1)$.

[Kollár, Shepherd-Barron], [Wahl]

Let $(X_0, 0)$ be a two-dimensional quotient singularity. If $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then either $(X_0, 0)$ is a RDP or it is a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$ where $\gcd(n, a)=1$.

Moreover, every such cyclic quotient singularity admits a $\mathbb{Q}$-Gorenstein smoothing.

i) $\overset{-4}{\bullet}$ $\qquad \overset{-3}{\bullet}-\overset{-2}{\bullet}-\overset{-2}{\bullet}\cdots\overset{-2}{\bullet}-\overset{-3}{\bullet}$ class T

ii) If $\overset{-b_1}{\bullet}-\overset{-b_2}{\bullet}-\cdots\overset{-b_{r-1}}{\bullet}-\overset{-b_r}{\bullet}$ class T

then $\overset{-2}{\bullet}-\overset{-b_1}{\bullet}-\overset{-b_2}{\bullet}\cdots\overset{-b_{r-1}}{\bullet}-\overset{-b_r-1}{\bullet}$

$\overset{-b_1-1}{\bullet}-\overset{-b_2}{\bullet}-\cdots\overset{-b_r}{\bullet}-\overset{-2}{\bullet}$ class T

iii) Every singularity of class T that is not RDP can be obtained by starting with one of (i) and iterating to steps in (ii)

e.g. $\overset{-4}{\bullet}$ $\qquad \overset{-5}{\bullet}-\overset{-2}{\bullet}$

$\overset{-6}{\bullet}-\overset{-2}{\bullet}-\overset{-2}{\bullet}$ $\qquad \overset{-2}{\bullet}-\overset{-5}{\bullet}-\overset{-3}{\bullet}$ $\qquad \cdots$

Main Lemma 1

Let V be the minimal resolution of X and E be the reduced exceptional divisors.

Then $H^2(T_V(-log\, E)) = 0 \Rightarrow H^2(T_X^0) = 0.$

Idea (suggested by Manetti)

Let $\pi : V \to X.$ Then $\pi_* T_V = T_X^0$ [Burns-Wahl].

$0 \to \pi_* T_V(-log\, E) \to T_X^0 \to \Delta \to 0$, Δ supported in the Sing X.

$$H^2(T_X^0) = H^2(\pi_* T_V(-log\, E)).$$

$$R^1\pi_* T_V(-log\, E) = R^2\pi_* T_V(-log\, E) = 0 :$$

We may assume that X is affine. $0 \to T_V(-Z) \to T_V(-log\, E) \to T_Z^0 \to 0$

Z is effective divisor supported in E [Burns-Wahl].

Z sufficiently big ($-Z$ is π-ample) $H^i(T_V(-Z)) = 0, i > 0.$

$$H^1(T_V(-log\, E)) = H^1(T_Z^0),\ H^2(T_V(-log\, E)) = 0.$$

$H^1(T_Z^0) = 0$ [Laufer] Two dimensional quotient singularity is taut.

Main Lemma 2

From the constructions, our examples satisfy

$$H^2(T_{\tilde{Z}}(-log\, E)) = 0.$$

Key Lemma 1: V nonsingular surface, D s.n.c. divisor ir$V$.

$f : V' \to V$ blow-up of V at a point in D. Set $D' = f^{-1}(D)_{red}.$

Then $h^2(T_{V'}(-log\, D')) = h^2(T_V(-log\, D)).$

Key Lemma 2: Let Z be a blow-up at two singular points of two nodal curves in special fibers.

It is enough to prove that

$H^2(Z, T_Z(-log\, D_Z)) = 0.$ $D_Z = F_1 + F_2 + F + D + S_1 + S_2 + S_3.$

F_i (-4)-curve in special fibers, F proper transform of conic,

D $\tilde{E}_6$- one (-2)-curve, S_i section

Key Lemma 3: It is enough to prove $H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) = 0.$

$$H^0(Z, \Omega_Z(K_Z + F_1 + F_2 + F + D)) = 0 \subseteq H^0(Y, \Omega_Y(C + F + D))$$

C general fiber

$$H^0(Y, \Omega_Y(C + F + D)) = H^0(Y, \Omega_Y(3C - E - D'))$$

E line, $D + D' = \tilde{E}_6$ fiber

Key Lemma 4: Y rational elliptic surface. Assume that the elliptic fibration $g : Y \to \mathbb{P}^1$ is relatively minimal without multiple fibers.

C general fiber of $g : Y \to \mathbb{P}^1$. Then

$$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(k)) = H^0(Y, \Omega_Y(kC)), k \geq 1.$$

Thank you for listening.

POLARIZED ENDOMORPHISMS ON NORMAL PROJECTIVE VARIETIES

DE-QI ZHANG

Abstract. This is the summary of the paper [14]. We show that polarized endomorphisms of rationally connected threefolds with at worst terminal singularities are equivariantly built up from those on $\mathbb{Q}$-Fano threefolds, Gorenstein log del Pezzo surfaces and $\mathbb{P}^1$. Similar results are obtained for polarized endomorphisms of uniruled threefolds and fourfolds. As a consequence, we show conceptually that every smooth Fano threefold with a polarized endomorphism of degree > 1, is rational.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. We study *polarized* endomorphisms $f : X \to X$ of varieties X, i.e., those f with $f^*H \sim qH$ for some $q > 0$ and some ample line bundle H. Every surjective endomorphism of a projective variety of Picard number one, is polarized. If $f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \to \mathbb{P}^n$ is a surjective morphism and $X \subset \mathbb{P}^n$ a f-stable subvariety, then $f^*H \sim qH$ and hence $f|X : X \to X$ is polarized; here $H \subset X$ is a hyperplane and $q = \deg(F_i)$. If A is an abelian variety and $m_A : A \to A$ the multiplication map by an integer $m \neq 0$, then $m_A^*H \sim m^2H$ and hence m_A is polarized; here $H = L + (-1)^*L$ with L an ample divisor, or H is any ample divisor with $(-1)^*H \sim H$. One can also construct polarized endomorphisms on quotients of $\mathbb{P}^n$ or A. So there are many examples of polarized endomorphisms f. See [16] for the many conjectures on such f.

In [11], it is proved that a normal variety X with a non-isomorphic polarized endomorphism f either has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$ (and further is a quotient of an abelian variety when $\dim X \leq 3$), or is uniruled so that f descends to a polarized endomorphism f_Y of the non-uniruled base variety Y (so $K_Y \sim_{\mathbb{Q}} 0$) of a specially chosen maximal rationally connected fibration $X \dashrightarrow Y$. By the induction on dimension and since Y has a dense set of f_Y-periodic points $y_0, y_1, \ldots$ (cf. [2, Theorem 5.1]), the study of polarized endomorphisms is then reduced to that of rationally connected varieties Γ_{y_i} as fibres of the graph $\Gamma = \Gamma(X/Y)$ (cf. [11, Remark 4.3]).

The study of non-isomorphic endomorphisms of singular varieties (like Γ_{y_i} above) is very important from the dynamics point of view, but is very hard

1991 *Mathematics Subject Classification.* 14E20, 14J45, 14E08, 32H50.
Key words and phrases. polarized endomorphism, uniruled variety, rationality of variety.

even in dimension two and especially for rational surfaces; see [9] (about 150 pages).

We consider polarized endomorphisms of rationally connected varieties (or more generally of uniruled varieties) of dimension ≥ 3. Theorem 1.1 – 1.8 below are our main results.

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial threefold having only terminal singularities and a polarized endomorphism of degree $q^3 > 1$. Suppose that X is rationally connected. Then we have :*

(1) *There is an $s > 0$ such that $(f^s)^*_{|N^1(X)} = q^s \operatorname{id}$.*
(2) *Either X is rational, or $-K_X$ is big.*
(3) *There are only finitely many irreducible divisors $M_i \subset X$ with the Iitaka D-dimension $\kappa(X, M_i) = 0$.*

Theorem 1.1 (3) apparently does not hold on an abelian variety A with a subtorus of codimension one, though the multiplication map m_A is polarized as mentioned above. Neither it holds for $X = S \times \mathbb{P}^1$, where S is a rational surface with infinitely many (-1)-curves (the blowup of nine general points of $\mathbb{P}^2$ is such S as observed by Nagata).

Theorem 1.1 (1) above strengthens (in our situation) Serre's result [12] on a conjecture of Weil (in the projective case): (**Serre**) If f is a polarized endomorphism of degree $q^{\dim X} > 1$ of a smooth variety X then every eigenvalue of $f^*|N^1(X)$ has the same modulus q.

The proof of Theorem 1.2 below is conceptually done. In a recent paper [15], we have removed the polarizedness assumption in Theorem 1.2.

Theorem 1.2. *Let X be a smooth Fano threefold with a polarized endomorphism of degree > 1. Then X is rational.*

A klt $\mathbb{Q}$-Fano variety has only finitely many extremal rays. A similar phenomenon occurs in the quasi-polarized case.

Theorem 1.3. *Let X be a $\mathbb{Q}$-factorial rationally connected threefold having only Gorenstein terminal singularities and a quasi-polarized endomorphism of degree > 1. Then X has only finitely many K_X-negative extremal rays.*

We expect a possible application of Theorem 1.4 below (see Theorem 1.7 for a more detailed version) to the Dynamic Manin-Mumford conjecture for (X, f) formulated by S. -W. Zhang in [16, Conjecture 1.2.1]. This conjecture for (X, f) is essentially equivalent to that for (X_r, g_r) because f^{-1}, as seen in Theorem 1.7, preserves the maximal subset of X where the birational map $X \cdots\to X_r$ is not holomorphic.

Further, X_r is better to be dealt with because it has a fibration structure preserved by g_r. *The existence of such a fibration $\pi : X_r \to Y$ is guaranteed* when X is uniruled by the recent development in MMP.

Theorem 1.4. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$.*

Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power, we have:

(1) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic.*

(2) *Let $\pi : X_r \to Y$ be an extremal contraction with $\dim Y \le 2$. Then g_r is polarized and it descends to a polarized endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ with $\pi \circ g_r = h \circ \pi$.*

The claim in the abstract about the building blocks of polarized endomorphisms, is justified by the remark below.

Remark 1.5.

(1) The Y in Theorem 1.4 is $\mathbb{Q}$-factorial and has at worst log terminal singularities.

(2) Suppose that the X in Theorem 1.4 is rationally connected. Then Y is also rationally connected. Suppose further that X has at worst terminal singularities and $(\dim X,\ \dim Y) = (3, 2)$. Then Y has at worst Du Val singularities by [8, Theorem 1.2.7]. So there is a composition $Y \to \hat{Y}$ of divisorial contractions and an extremal contraction $\hat{Y} \to B$ such that either $\dim B = 0$ and $\hat{Y}$ is a Du Val del Pezzo surface of Picard number 1, or $\dim B = 1$ and $\hat{Y} \to B \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with all fibres irreducible. After replacing f by its power, h descends to polarized endomorphisms $\hat{h} : \hat{Y} \to \hat{Y}$, and $k : B \to B$ (of degree $q^{\dim B}$); see Theorems 1.6.

(3) By [2, Theorem 5.1], there are dense subsets $Y_0 \subset Y$ (for the Y in Theorem 1.4) and $B_0 \subset B$ (when $\dim B = 1$) such that for every $y \in Y_0$ (resp. $b \in B_0$) and for some $r(y) > 0$ (resp. $r(b) > 0$), $g^{r(y)}|W_y$ (resp. $\hat{h}^{r(b)}|\hat{Y}_b$) is a well-defined polarized endomorphism of the Fano fibre.

We remark that Noboru Nakayama has produced many examples of polarized f on abelian surfaces which are not scalar. The result below shows that this happens only on abelian surfaces and their quotients.

Theorem 1.6. *Let X be a normal projective surface. Suppose that $f : X \to X$ is an endomorphism such that $f^*P \equiv qP$ for some $q > 1$ and some big Weil $\mathbb{Q}$-divisor P. Then we have:*

(1) *f is polarized of degree q^2.*

(2) *There is an $s > 0$ such that $(f^s)^*|\mathrm{Weil}(X) = q^s \,\mathrm{id}$ unless X is Q-abelian with $\mathrm{rankWeil}(X) \in \{3, 4\}$.*

More generally, we prove the two theorems below. Theorem 1.7 below includes Theorem 1.4 as a special case.

Theorem 1.7. *Let X be a $\mathbb{Q}$-factorial n-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism f of degree $q^n > 1$. Let $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_r$ be a composition of divisorial contractions and flips. Replacing f by its positive power,* (I) *and* (II) *hold:*

(I) *The dominant rational maps $g_i : X_i \dashrightarrow X_i$ $(0 \le i \le r)$ (with $g_0 = f$) induced from f, are all holomorphic. Further, g_i^{-1} preserves* *each irreducible component of the exceptional locus of* $X_i \to X_{i+1}$ (*when it is divisorial*) *or of the flipping contraction* $X_i \to Z_i$ (*when* $X_i \dashrightarrow X_{i+1} = X_i^+$ *is a flip*).

(II) *Let* $\pi : W = X_r \to Y$ *be the contraction of a* K_W*-negative extremal ray* $\mathbb{R}_{\ge 0}[C]$, *with* $\dim Y \le n-1$. *Then* $g := g_r$ *descends to a surjective endomorphism* $h : Y \to Y$ *of degree* $q^{\dim Y}$ *such that*

$$\pi \circ g = h \circ \pi.$$

For all $0 \le i \le r$, *all eigenvalues of* $g_i^*|N^1(X_i)$ *and* $h^*|N^1(Y)$ *are of modulus* q; *there are big line bundles* H_{X_i} *and* H_Y *satisfying*

$$g_i^* H_{X_i} \sim q H_{X_i}, \quad h^* H_Y \sim q H_Y.$$

Suppose further that either $\dim Y \le 2$ *or* $\rho(Y) = 1$. *Then* H_W *and* H_Y *can be chosen to be ample and* g *and* h *are polarized.*

The contraction π below exists by the MMP for threefolds.

Theorem 1.8. *Let* X *be a* $\mathbb{Q}$*-factorial rationally connected threefold having at worst terminal singularities and a polarized endomorphism of degree* >1. *Let* $X \dashrightarrow W$ *be a composition of divisorial contractions and flips, and* $\pi : W \to Y$ *an extremal contraction of non-birational type. Suppose either* $\dim Y \ge 1$, *or* $\dim Y = 0$ *and* W *is smooth. Then* X *is rational.*

The difficulty 1.9. In Theorem 1.4, if $X \to X_1$ is a divisorial contraction, one can descend a polarized endomorphism f on X to an one on X_1, but the latter may not be polarized any more because the pushfoward of a nef divisor may not be nef in dimension ≥ 3 (the first difficulty). If $X \dashrightarrow X_1$ is a flip, then in order to descend f on X to some holomorphic f_1 on X_1, one has to show that a power of f preserves the centre of the flipping contraction (the second difficulty). The second difficulty is taken care by a key lemma where the polarizedness is essentially used.

The question below is the generalization of Theorem 1.2 and the famous conjecture: every smooth Fano n-fold of *Picard number one* with a non-isomorphic surjective endomorophism, is $\mathbb{P}^n$ (for its affirmative solution when $n = 3$, see Amerik-Rovinsky-Van de Ven [1] and Hwang-Mok [4]).

Question 1.10. *Let* X *be a smooth Fano* n*-fold with a non-isomorphic polarized endomorphism. Is* X *rational ?*

For the recent development on endomorphisms of algebraic varieties, we refer to Amerik-Rovinsky-Van de Ven [1], Fujimoto-Nakayama [3], Hwang-Mok[4], S. -W. Zhang [16], as well as [10], [13].

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STABLE POINTS ON STACKS

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1. Approximating algebraic stacks to schemes or algebraic spaces

A coarse moduli space for an algebraic stack[1] is an algebraic space that is the closest to the algebraic stack. First let us recall the definition of a coarse moduli space:

Definition 1.1. Let $\mathcal{X}$ be an algebraic (Artin) stack over a scheme S. A coarse moduli map for $\mathcal{X}$ is a morphism

$$\pi : \mathcal{X} \to X$$

over S such that

(1) X is an algebraic space over S,
(2) π is universal among maps to algebraic spaces,
(3) for any algebraically closed S-field K, π gives rise to a bijective map from the set of the isomorphism classes of $\mathcal{X}(K)$ to the set of K-valued points $X(K)$.

Informally speaking, we have the rough slogan:

$$\text{Algebraic stack} = \text{Groupoid valued sheaf} + \text{Algebraically Geometric structures},$$

$$\text{Scheme or Algebraic space} = \text{Set values sheaf} + \text{Algebraically Geometric structures}.$$

From this point of view, it is clear that the coarse moduli space X for an algebraic stack $\mathcal{X}$ loses the information arising from the non-trivial morphisms which belong to groupoids. For example, in general, the category of sheaves on $\mathcal{X}$ is quite different from that of X. However, in the treatment of algebraic stacks we often need the existence of a coarse moduli space. Namely, the proof sometimes relies on the existence of a coarse moduli space. The typical use can be found in the proof of Riemann-Roch theorem for Deligne-Mumford stacks due to Toën ([13]). Thus, coarse moduli spaces provide useful bridges between the geometry of stacks and schemes and algebraic spaces.

Now we will try to construct a coarse moduli space for a given stack $\mathcal{X}$. If we ignore "Algebraically Geometric structures" in the above slogan, we easily find the way: Take a connected component of groupoids. Namely, view $\mathcal{X}$ as a functor

$$\mathcal{X} : (\text{Schemes})_{\text{ét}} \longrightarrow (\text{Groupoids})$$

and define $\pi_0(\mathcal{X})(S) = \pi_0(\mathcal{X}(S))$ for any scheme S. In other words, $\pi_0(\mathcal{X})(S)$ is the set of isomorphism classes of groupoids $\mathcal{X}(S)$. A sheafification after this truncating procedure gives rise to a sheaf $\bar{\mathcal{X}}$ on the site $(\text{Schemes})_{\text{ét}}$. This sheaf $\bar{\mathcal{X}}$ is the best approximation of $\mathcal{X}$ in the category of sheaves on $(\text{Schemes})_{\text{ét}}$. The picture of this

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[1] In this note, by an algebraic stack we mean an Artin stack. We refer to [10] as the general reference.

construction is clear and easy to understand since it is nothing but a truncation. However, if one takes account into "Algebraically Geometric structures", then the problem becomes very subtle and difficult. To understand this, let us recall the known class of algebraic stacks which have their coarse moduli spaces, and examples which do not. The theorem we first recall is a well-known result due to Keel and Mori ([9]).

Theorem 1.2 (Keel-Mori). *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a noetherian base scheme S. Let $I\mathcal{X} \to \mathcal{X}$ be the first (or second) projection in the diagram*

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X}.
\end{tikzcd}$$

Suppose that $I\mathcal{X} \to \mathcal{X}$ is a finite morphism. Then there exists a coarse moduli map

$$\pi : \mathcal{X} \to X$$

such that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, and π is proper and quasi-finite. Moreover $\mathcal{X}$ has finite diagonal, then $\mathcal{X}$ is separated.

The stack $I\mathcal{X} \to \mathcal{X}$ is called the inertia stack of $\mathcal{X}$. This stack parametrizes the automorphisms of objects in $\mathcal{X}$. Namely, for any $\alpha : T \to \mathcal{X}$, the fiber product $\mathrm{pr}_2 : I\mathcal{X} \times_{\mathcal{X}} T \to T$ represents the functor

$$\mathrm{Aut}_T(\alpha) : (T\text{-schemes}) \to (\text{groups})$$

which to any $f : T' \to T$ associates the group $\mathrm{Aut}_T(\alpha)(T') := \{\text{automorphisms of } f^*\alpha\}$. The inertia stack can be viewed as a kind of the free loop space for $\mathcal{X}$. The condition $I\mathcal{X} \to \mathcal{X}$ is a finite morphism, is equivalent to imposing that every object in $\mathcal{X}$ has a finite automorphism group scheme. In characteristic zero, algebraic stacks whose inertia are finite, are always Deligne-Mumford.

Next we consider examples of stacks which do not admit coarse moduli spaces. Examples we will keep in mind are the moduli stack of vector bundles on an algebraic variety, and more generally, the moduli stack of G-bundles on the algebraic variety, where G is an algebraic group. Another example is the moduli stack of objects of derived category of coherent cohomology on a scheme. What happen on an algebraic stack which does not admit a coarse moduli space? In order to make an observation, consider the open immersion

$$\mathbb{G}_m \hookrightarrow \mathbb{A}^1$$

of a torus into an affine line over the complex number field. It gives rise to the natural action of $\mathbb{G}_m$ on $\mathbb{A}^1$. Take the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$. It is the moduli stack of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf $\mathbb{A}^1$, and s is a section on $\mathbb{A}^1$. Since we have two $\mathbb{G}_m$-orbits on $\mathbb{A}^1$, there are two closed points on $[\mathbb{A}^1/\mathbb{G}_m]$. On the other hand, $[\mathbb{A}^1/\mathbb{G}_m]$ is connected, thus if we assume that $[\mathbb{A}^1/\mathbb{G}_m]$ has a coarse moduli space, then it is connected and has exactly two closed points. But such a complex analytic space does not exists, and we conclude that $[\mathbb{A}^1/\mathbb{G}_m]$ does not have a coarse moduli space. Put another way, notice that the dimension of the stabilizer at the origin on $\mathbb{A}^1$ is positive whereas the other points have 0-dimensional stabilizer groups. This collapses the "Algebraic Geometric structures" (see [11, page 6]). Therefore if one hope that an algebraic stack $\mathcal{X}$ has a coarse moduli space, then objects in $\mathcal{X}$ should have the equidimensional automorphisms, that is, $I\mathcal{X} \to \mathcal{X}$ is equidimensional. We can ask the converse: if $I\mathcal{X} \to \mathcal{X}$ is equidimesional, then does $\mathcal{X}$ have a coarse moduli space? Unfortunately, this problem is quite subtle. Even in the case where $I\mathcal{X} \to \mathcal{X}$ is quasi-finite, we (at least the author) do not know whether or not $\mathcal{X}$ has a coarse moduli space. Another point we should note concerns the problem of the finite generation of invariant rings.

2. Intrinsic Stability on Algebraic stacks

In the proceeding section, we discuss coarse moduli spaces for algebraic stacks, especially the example of an algebraic stack that does not admit a coarse moduli space. The theory dealing with the last problem was essentially proposed by Mumford in the case $\mathcal{X} = [X/G]$ where X is an algebraic scheme, and G is a reductive group acting on X, that is, Geometric Invariant Theory (GIT) ([11]). Suppose that a reductive group G acts on an algebraic scheme X. Mumford defined pre-stable points on X with respect to the action of G, and proved that G-orbit space of pre-stable points has a structure of a scheme called the geometric quotient Y (see [11]). It is rephrased that the quotient stack $[X(\mathrm{Pre})/G]$ has a "coarse moduli scheme"

$$[X(\mathrm{Pre})/G] \to Y.$$

Thus, from our point of view, Mumford' GIT provides a machinery that chooses an open substack of $[X/G]$ which admits a coarse moduli (if we further take a suitable line bundle on $[X/G]$, then we have a polarized coarse moduli of the open substack of stable points). Inspired from Mumford's theory and Keel-Mori theorem, we want to propose the idea:

> Introduce intrinsically "stability" on a general Artin stack $\mathcal{X}$ so that stable points $\mathcal{X}^s$ form an open substack which admits a coarse moduli map $\mathcal{X}^s \to X$.

We first remark that we want to define "intrinsically stable points" on $\mathcal{X}$ by using local properties on $\mathcal{X}$, and thus we do not take account into the global flavour. At this point, the reader might begin to object that if we do not use the global aspects on $\mathcal{X}$ (such as linearized line bundle in GIT), the resulting coarse moduli space is not a good space, for example, often not separated. Here we would like to call the reader's attention to the observation: Keel-Mori theorem, which we want to take a position to generalize, tells us no global information of the coarse moduli space. Recall that Theorem 1.2 says that if $\mathcal{X}$ has finite diagonal, then the coarse moduli space X for $\mathcal{X}$ is separated. Nevertheless, if we assume that the existence of a (not necessarily separated) coarse moduli space X, then the proof of the separatedness of X is quite formal. Of course, the price is that Keel-Mori theorem tells us very little about how to prove that X is separated. In my opinion, one of the reasons why Keel-Mori theorem is useful, is that the finiteness of $I\mathcal{X} \to \mathcal{X}$ is a local condition on $\mathcal{X}$, and the global aspect should be treated in the next step by case-by-case approaches.

In [8], we introduced some stabilities which have relations described as follows:

$$(\text{GIT-like p-stable}) \ \subset \ (\text{p-stable}) \ \supset \ (\text{strong p-stable})$$

In this note, we discuss and focus on GIT-like stability, which has some remarkable properties. Also, we briefly mention strong p-stability.

Definition 2.1 (GIT-like p-stable point)**.** Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k. Let p be a closed point on $\mathcal{X}$. The point p is GIT-like p-stable if there exists an effective versal deformation $\xi \in \mathcal{X}(A)$ for p (see Remark 2.2), which has the following properties:

(a) The special fiber of the automorphism group $\mathrm{Aut}_{\mathrm{Spec}\,A}(\xi) \to \mathrm{Spec}\,A$ is linearly reductive.

(b) If I denotes the ideal generated by nilpotent elements in A, then there exists a normal subgroup scheme $\mathcal{F}$ of $\mathrm{Aut}(\xi) \times_A (A/I) \to \mathrm{Spec}\,A/I$ such that the following conditions hold: (i) $\mathcal{F}$ is smooth and affine over $\mathrm{Spec}\,A/I$, and whose geometric fibers are connected, (ii) the quotient $\mathrm{Aut}(\xi) \times_A (A/I)/\mathcal{F}$ is finite over $\mathrm{Spec}\,A/I$, and (iii) for any two morphisms $\alpha, \beta : T \rightrightarrows \mathrm{Spec}\,A/I$ such that $\alpha^*\xi \cong \beta^*\xi$, we have $\alpha^*\mathcal{F} \cong \beta^*\mathcal{F}$ in $\mathrm{Aut}_T(\alpha^*\xi) \cong \mathrm{Aut}_T(\beta^*\xi)$.

Remark 2.2. (i) The letter "p" in the terms GIT-like p-stable, p-stable.. is the initial of *pointwise*.

(ii) We say that $\xi \in \mathcal{X}(A)$ is an effective versal deformation for a closed point p if
  (a) A is a complete noetherian local k-ring whose residue field is of finite type over k,
  (b) the special fiber of $\xi : \mathrm{Spec}\,A \to \mathcal{X}$ lies over p,
  (c) the corresponding morphism $\mathrm{Spec}\,A \to \mathcal{X}$ is formally smooth, i.e., it satisfies the usual lifting property (cf. [8], [2]).

(iii) Recall the definition of linearly reductivity. An algebraic group G over k is linearly reductive if the functor

$$(G\text{-vector spaces over } k) \to (k\text{-vector spaces}) \ M \mapsto M^G$$

is exact. In characteristic zero, an algebraic group is linearly reducitve if and only if it is a reductive group.

(iv) GIT-like stability depends only on the reduced algebraic stack $\mathcal{X}_{\mathrm{red}}$ associated to $\mathcal{X}$.

(v) To verify that a given group scheme $\mathcal{G}$ over a reduced scheme S is smooth (over S), it is enough to prove that $\mathcal{G} \to S$ is equidimensional, and all fibers are smooth.

(vi) The condition (iii) in (b) in Definition 2.1 is a natural compatibility condition.

(vii) Our definition fits in with Artin's representability criterion ([3]) which is desribed in terms of deformation theory. Of course, our formulation is influenced by Artin's works.

(viii) A closed point on $\mathcal{X}$ is said to be a strong p-stable point if there exists an effective versal deformation $\xi \in \mathcal{X}(A)$ such that there exists a flat normal subgroup scheme $\mathcal{F} \subset \mathrm{Aut}(\xi)$ such that $\mathrm{Aut}(\xi)/\mathcal{F} \to \mathrm{Spec}\,A$ is a finite morphism, and the compatibility condition as in (iii) in Definition 2.1 holds.

To give a feeling for GIT-like p-stability defined above, we will consider the following example. The relationship with GIT will be discussed in the next section. Let G be a connected reductive group over $\mathbb{C}$ and C a connected smooth projective curve over $\mathbb{C}$. Let $\mathcal{M}$ be the moduli algebraic stack of Higgs G-bundles on C. The automorphism of every Higgs G-bundle $(E, \phi \in \Gamma(C, \mathfrak{G}_E \times_{\mathbb{C}} \Omega_X))$ contains the center of G. The center $\mathrm{Cent}(G)$ is a reductive group and for any family $(\tilde{E}, \tilde{\phi})$ of Higgs G-bundle over $C \times_{\mathbb{C}} T$, $\mathrm{Cent}(G) \times_{\mathbb{C}} T$ is a normal subgroup in $\mathrm{Aut}_T((\tilde{E}, \tilde{\phi}))$. A Higgs bundle (E, ϕ) is GIT-like p-stable (in other words, the corresponding point on $\mathcal{M}$ is GIT-like p-stable) if and only if an effective versal deformation $(\mathcal{E}, \Phi) \in \mathcal{M}(A)$ for (E, ϕ) has a finite automorphism group scheme modulo $\mathrm{Cent}(G) \times_{\mathbb{C}} \operatorname{Spec} A$.

Now we are ready to state the existence theorem of coarse moduli spaces for GIT-like p-stable points ([8]).

Theorem 2.3. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field. Then the open substack $\mathcal{X}^{gs}$ of GIT-like p-stable points has a coarse moduli map*

$$\pi : \mathcal{X}^{gs} \to X.$$

Moreover π is universally closed morphism and of finite type.

The construction takes three steps:

• *First Step.* Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Applying the algebraization, we may assume that the inertia stack $I\mathcal{X}_0 \to \mathcal{X}_0$ contains a smooth and affine subgroup stack $\mathcal{F} \subset I\mathcal{X}_0$, whose geometric fibers are connected. Namely, $\mathcal{X} = \mathcal{X}^{gs}$. Then the rigidification technique removes the automorphisms in $\mathcal{F}$, and we obtain the "rigidified" stack $\mathcal{X}_0^{\mathrm{rig}}$.

• *Second Step.* By our assumption, $\mathcal{X}_0^{\mathrm{rig}}$ has a finite inertia stack $I\mathcal{X}_0^{\mathrm{rig}} \to \mathcal{X}_0^{\mathrm{rig}}$. Then by Keel-Mori theorem, there exists a coarse moduli space X_0 for $\mathcal{X}_0^{\mathrm{rig}}$. The composite $\mathcal{X}_0 \to \mathcal{X}_0^{\mathrm{rig}} \to X_0$ is also a coarse moduli map for $\mathcal{X}_0$. (The first and second steps are rather formal parts in our strategy.)

• *Third Step.* Now we want to construct a coarse moduli space X for $\mathcal{X}$ by deforming X_0 as follows:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, dotted] \\ X_0 \arrow[r, dotted] & X \end{tikzcd}$$

At this point, there are some points we should note. Even if an algebraic stack $\mathcal{Y}$ and the associated reduced stack $\mathcal{Y}_0$ have coarse moduli spaces Y and Y_0, the natural morphism $Y_0 \to Y$ is not necessarily a deformation. To make things simple, assume that $\mathcal{Y} = [\operatorname{Spec} A/G]$, $\mathcal{Y}_0 = [\operatorname{Spec}(A/I)/G]$, $Y_0 = \operatorname{Spec}(A/I)^G$ and $Y = \operatorname{Spec} A^G$, where G is an algebraic group and I is a nilpotent ideal of A. If G is linearly reductive, then $A^G \to (A/I)^G$ is surjective, thus $Y_0 \to Y$ is a deformation as expected. However, if G is unipotent, then it happens that $A^G \to (A/I)^G$ is not surjective. Note $(A/I)^G = \Gamma(\mathcal{Y}_0, \mathcal{O}_{\mathcal{Y}_0})$ and $A^G = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Thus, we need to verify that after étale localization on X_0, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is surjective. It is accomplished by constructing the "étale local quotient structure" of $\mathcal{X}$. In this part, we essentially use the linearly redutivity of automorphism groups. Finally, developing the deformation theory of coarse moduli spaces, we construct the desired deformation $X_0 \hookrightarrow X$.

3. Comparing with Geometric Invariant Theory

We assume that the base field k is algebraically closed of characteristic zero. In this section, we discuss the relationship between our GIT-like p-stability and Geometric

Invariant Theory due to Mumford ([11]). Let X be an algebraic scheme over k. Let G be a reductive group scheme over k. Let $\sigma : G \times_k X \to X$ be an action on X. Let $X(\mathrm{Pre}) \subset X$ be the open subset of X, consisting of pre-stable points in the sense of [11, Definition 1.7]. The relation is described by

Theorem 3.1. *Let $[X(\mathrm{Pre})/G]$ be the open substack of $[X/G]$. Let*

$$[X/G]^{gs}$$

be the open substack consisting of GIT-like p-stable points on $[X/G]$. Let $\mathcal{S}$ be the maximal open substack of $[X/G]$, admitting a coarse moduli space that is a scheme. (The open substack $\mathcal{S} \subset \mathcal{X}$ is characterized by the following universality: If $\mathcal{U} \subset \mathcal{X}$ has a coarse moduli space which is a scheme, then $\mathcal{U} \subset \mathcal{S}$.) Then we have

$$[X(\mathrm{Pre})/G] = [X/G]^{gs} \cap \mathcal{S}.$$

From this evidence, we can say that GIT-like p-stability is an intrinsic generalization of the local part of Mumford's GIT. (Pre-stability in GIT is a local part of GIT.)

Let us briefly explain how one can view pre-stable points in the sense of GIT as GIT-like p-stable points. Let $x \in X$ be a closed pre-stable point. By the definition, there exists a G-invariant affine neighborhood U of x, such that the action of G on U is closed. That is to say, every orbit is a closed set in U. Notice that G acts also on the reduced scheme U_{red} associated to U (because the base field is perfect). Clearly, the action of G on U_{red} is closed. Let

$$\mathsf{Stab} \to U_{\mathrm{red}}$$

be the stabilizer group scheme defined to be the top horizontal arrow in the cartesian diagram

$$\begin{tikzcd}
(G \times U_{\mathrm{red}}) \times_{U_{\mathrm{red}} \times U_{\mathrm{red}}} U_{\mathrm{red}} \arrow[r] \arrow[d] & U_{\mathrm{red}} \arrow[d, "\text{diagonal}"] \\
G \times U_{\mathrm{red}} \arrow[r, "(\sigma, \mathrm{pr}_2)"] & U_{\mathrm{red}} \times U_{\mathrm{red}}
\end{tikzcd}$$

where $\sigma : G \times U_{\mathrm{red}} \to U_{\mathrm{red}}$ is the action. The group scheme $\mathsf{Stab} \to U_{\mathrm{red}}$ is a (non-flat) equidimensional group scheme over U_{red}. According to Matsushima's theorem, we see that each fiber of $\mathsf{Stab} \to U_{\mathrm{red}}$ is a reductive algebraic group. Let $\mathcal{F}$ be the identity component of Stab. Then by SGA3 ([4]), $\mathcal{F}$ is smooth and affine over U_{red}, whose geometric fibers are connected. Moreover it can be shown that $\mathsf{Stab}/\mathcal{F}$ is a finite scheme over U_{red}. Since the completion of the local ring $\mathcal{O}_{X,x}$ gives rise to a versal deformation of the corresponding point on $[X/G]$, thus we see that the filtration $\mathcal{F} \subset \mathsf{Stab}$ over U_{red} yields the structure of a GIT-like p-stable point.

Remark 3.2. In Mumford's GIT, it is essential to have the quotient of a scheme by a reductive group. However, an algebraic Artin stack is not necessarily of the quotient form $[X/G]$, where X is a scheme (or more generally algebraic space), and G is a group scheme. In practice, it is quite hard to prove that a given algebraic stack is a quotient stack even if it has (cf. [7]). Moreover, it is hopeless to control the quotient structure. (In a sense, a quotient form should be viewed as a good coordinate.) On the other hand, our stability is defined in the intirisic way, thus it seems to be flexible and convenient, especially in the case where stacks have modular interpretations.

4. Hidden Properness of algebraic stacks: An application

In the final section, we will discuss the finiteness of coherent cohomology. In particular, we will propose "hidden properness" of algebraic (Artin) satcks. First we would like to remind the definition of proper morphisms between algebraic stacks.

Definition 4.1 ([10])**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The morphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be proper if the following conditions hold:

(i) f is universally closed map,
(ii) f is of finite type,
(iii) f is separated, i.e., the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

We have a finiteness of coherent cohomology for algebraic stacks:

Theorem 4.2 (Laumon, Moret-Bailly, Faltings, Olsson, Gabber)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of locally noetherian algebraic stacks over locally noetherian base scheme. Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{X}$. Then for any $i \geq 0$, the sheaf $Rf^i_*\mathcal{E}$ is coherent on $\mathcal{Y}$.*

The finiteness theorem of coherent cohomology for proper algebraic stacks has been proved by Laumon and Moret-Bailly under some restrictive hypotheses (cf. [10, (15.6)]). Later, Faltings proved the finiteness theorem for general proper morphisms via a surprising method of rigid geometry (cf. [5]). Recently, Olsson-Gabber proved Chow's lemma for algebraic stacks and reproved the finiteness theorem (cf. [12]).

Now we would like to reader's attention to:

> The separatedness for algebraic (Artin) stacks is a quite strong assumption.

To understand it, let $f : \mathcal{X} \to \mathcal{Y}$ be a separated morphism. For simplicity, suppose that $\mathcal{Y}$ is a noetherian affine scheme $\operatorname{Spec} A$. (The proof of the finiteness can be reduced to the case $\mathcal{Y} = \operatorname{Spec} A$.) Let $\alpha, \beta : \operatorname{Spec} K \to \mathcal{X}$ be morphisms where K is an algebraically closed field. Then the fiber product of

$$\begin{tikzcd} & \operatorname{Spec} K \arrow[d, "(\alpha,\beta)"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_A \mathcal{X} \end{tikzcd}$$

is the algebraic space $\operatorname{Isom}(\alpha, \beta)$, which represents the functor

$$(K\text{-schemes}) \longrightarrow (\text{sets})$$

sending $h : T \to \operatorname{Spec} K$ to the set $\operatorname{Hom}_{\mathcal{X}(T)}(h^*\alpha, h^*\beta)$. The algebraic space $\operatorname{Isom}(\alpha, \beta)$ is empty or isomorphic to the proper algebraic group $\operatorname{Aut}(\alpha)_{/K}$. (Note that $\operatorname{Isom}(\alpha, \beta) \to \operatorname{Spec} K$ is proper.) The identity component of the reduced (smooth) algebraic group associated to $\operatorname{Aut}(\alpha)_{/K}$ is a (possibly 0-dimensional) abelian variety. Thus, if $\mathcal{X}$ has an object whose automorphism is a positive-dimensional affine group scheme, then $\mathcal{X}$ is not separated, in particular, not proper. This causes one of main drawbacks of algebraic stacks. Also, this observation tells us that if $\mathcal{X}$ is separated over A, then $\mathcal{X}$ practically has finite diagonal. Consequently, in such a situation, if $\mathcal{X}$ is proper over A, then it has a proper coarse moduli space (by Keel-Mori theorem).

We are now in the position to state our finiteness.

Theorem 4.3. *Let $\mathcal{X}$ be an algebraic stack of finite type over a field k. Suppose that all closed points are GIT-like p-stable, and a coarse moduli space for $\mathcal{X}$ is proper k. Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{X}$. Then for any $i \geq 0$, the cohomology $H^i(\mathcal{X}, \mathcal{E})$ is finite dimensional. Moreover, (of course) the relative version of this statement holds.*

Clearly, our finiteness does not contain Theorem 4 because $\mathcal{X}$ in Theorem 4.3 is supposed to have linearly reductive automorphisms (and we work only over a field). But, nevertheless, we would like to stress that our finiteness is applicable to a certain class of non-proper algebraic stacks (in particular, our assumption is fairly weak in characteristic zero, and it can be applied to algebraic stacks having positive dimensional affine automorphisms groups). We should think that such algebraic stacks behave like proper, and have "hidden properness" (although here we ignore the finiteness concerning constructible sheaves). The proof is different from Falting's one and Olsson-Gabber's one. Our proof is done by showing that the coarse moduli map $\mathcal{X} \to X$ has the "hidden properness". Given an algebraic stack of finite type and all closed points are GIT-like p-stable, we have a version of valuative criterion for the properness of a coarse moduli space for $\mathcal{X}$ ([8]). Using it, we can state our finiteness without making reference to the coarse moduli space.

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HOMOLOGICAL MIRROR SYMMETRY FOR CUSP SINGULARITIES

ATSUSHI TAKAHASHI

1. Statement and the result

We associate two triangulated categories to a triple $A := (\alpha_1, \alpha_2, \alpha_3)$ of positive integers called a *signature*: the bounded derived category $D^b\mathrm{coh}(X_A)$ of coherent sheaves on a weighted projective line $X_A := \mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}$ and the bounded derived category $D^b\mathrm{Fuk}^{\to}(f_A)$ of the directed Fukaya category for a "cusp singularity" $f_A := x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3} + q^{-1}xyz$, $(q \in \mathbb{C}^*)$. Here, we consider f_A as a *tame polynomial* if $\chi_A := 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 1 > 0$ and as a *germ* of a holomorphic function if $\chi_A \leq 0$.

Then, the *Homological Mirror Symmetry (HMS) conjecture* for cusp singularities can be formulated as follows:

Conjecture 1.1 ([T1]). *There should exist an equivalence of triangulated categories*

$$D^b\mathrm{coh}(X_A) \simeq D^b\mathrm{Fuk}^{\to}(f_A).$$

□

Combining results in [GL] with known results in singularity theory, one can easily see that the HMS conjecture holds at the Grothendieck group level, i.e., there is an isomorphism

$$(K_0(D^b\mathrm{coh}(X_A)), \chi + {}^t\chi) \simeq (H_2(Y_A, \mathbb{Z}), -I),$$

where Y_A denotes the Milnor fiber of f_A.

The HMS conjecture is shown if $\alpha_3 = 1$ (Auroux-Katzarkov-Orlov [AKO], Seidel [Se1], van Straten, Ueda, ...). Also the cases $A = (4,4,2), (6,3,2)$, which correspond to two of three simple elliptic hypersurface singularities, are known ([AKO], [U], [T2], ...).

The following is our main theorem:

Theorem 1.2. *Assume that $\alpha_3 = 2$ or $A = (3,3,3)$. Then the HMS conjecture holds.* □

The keys in our proof are; the reduction of surface singularities to curve singularities (the stable equivalence of Fukaya categories given in [Se2] section 17), the use of A'Campo's divide [A1][A2] in order to describe the Fukaya category, and mutations of

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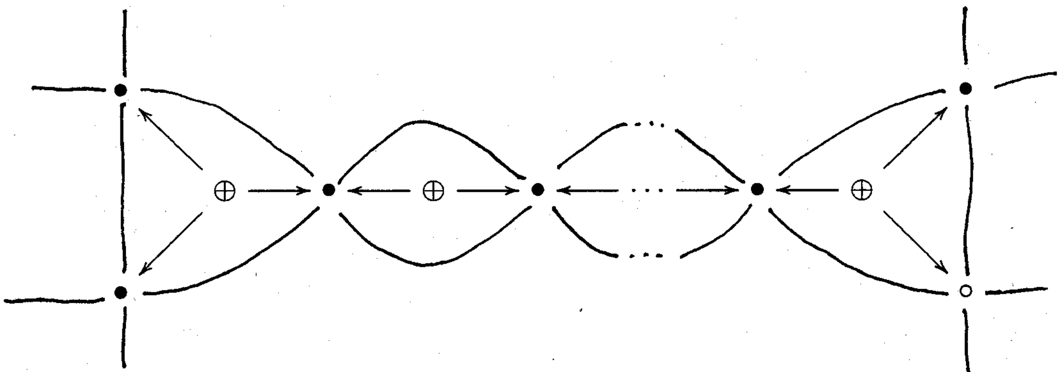
exceptional collections (distinguished basis of vanishing Lagrangian cycles). We shall give devides for cusp singularities with $\alpha_3 = 2$ and also quivers with relations associated to them.

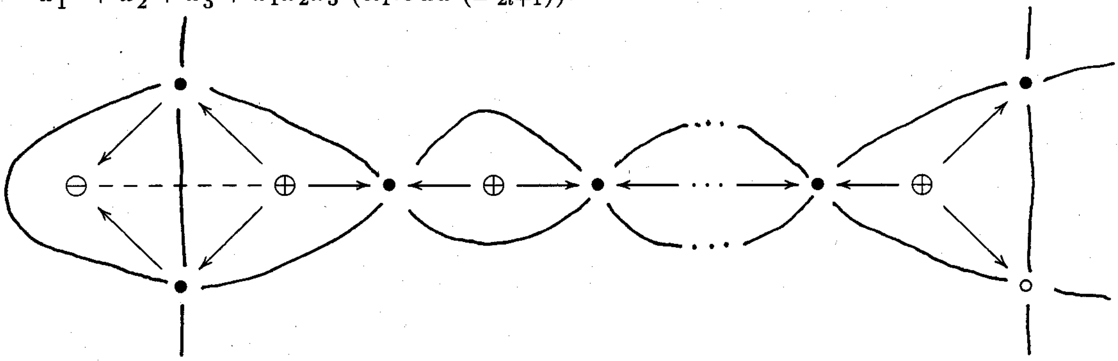
2. Devides and quivers with relations

2.1. $\chi_A > 0$. After applying suitable mutations, we shall obtain the *extended Dynkin quiver* of type $A = (\alpha_1, \alpha_2, \alpha_3)$ ($\circ$ denotes the vertex to remove in order to get the Dynkin quiver of the same type). It is known by [GL] that $D^b\mathrm{coh}(X_A)$ is equivalent to the derived category of extended Dynkin quiver of type A.

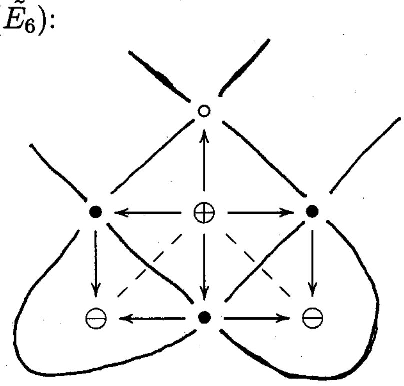
$x_1^{\alpha_1} + x_2^2 + x_3^2 + x_1x_2x_3$ (α_1:even ($\tilde{D_{2l}}$)):



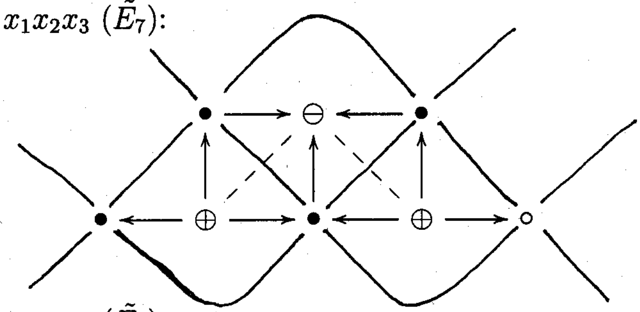
$x_1^{\alpha_1} + x_2^2 + x_3^2 + x_1x_2x_3$ (α_1:odd ($\tilde{D_{2l+1}}$)):



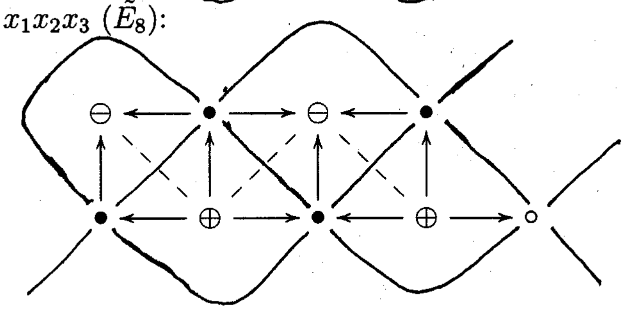
$x_1^3 + x_2^3 + x_3^2 + x_1x_2x_3$ ($\tilde{E_6}$):



$x_1^4 + x_2^3 + x_3^2 - x_1x_2x_3$ ($\tilde{E}_7$):

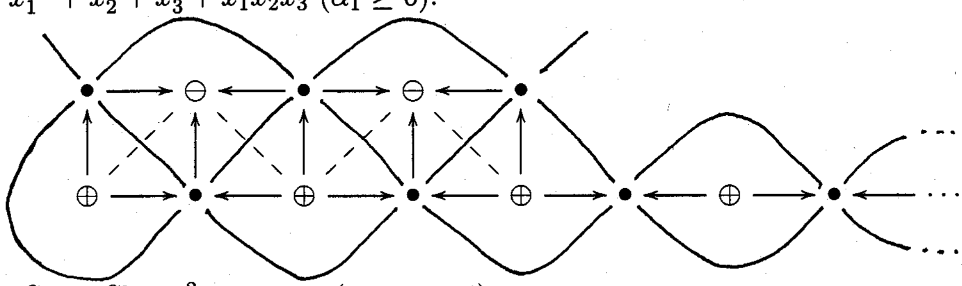


$x_1^5 + x_2^3 + x_3^2 - x_1x_2x_3$ ($\tilde{E}_8$):

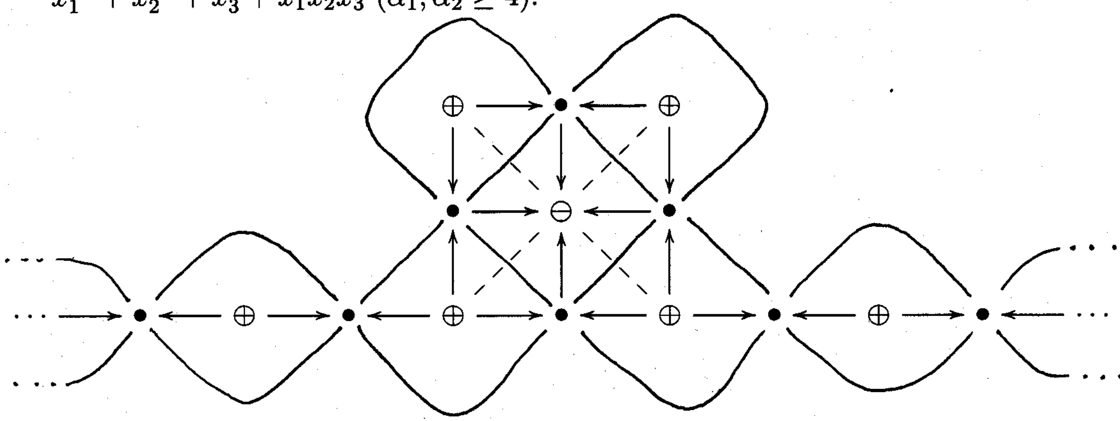


2.2. $\chi_A \leq 0$. Note that the number of vertices (= *Milnor number* of the singularity) is given by $\alpha_1 + \alpha_2 + \alpha_3 - 1$.

$x_1^{\alpha_1} + x_2^3 + x_3^2 + x_1x_2x_3$ ($\alpha_1 \geq 6$):



$x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^2 + x_1x_2x_3$ ($\alpha_1, \alpha_2 \geq 4$):



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Pandharipande-Thomas theory and wall-crossings in derived categories

Yukinobu Toda

Abstract

In [18], Pandharipande and Thomas introduced the notion of stable pairs on Calabi-Yau 3-folds and constructed the counting invariant of them. Conjecturally such invariant is equivalent to Donaldson-Thomas invariants and Gromov-Witten invariants via generating functions. In this article, we give a transformation formula of generating series of invariants counting stable pairs under flops. We use wall-crossing formula in the derived category.

1 Curve counting on Calabi-Yau 3-folds

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. there is a nowhere vanishing holomorphic 3-form on X. We are interested in the curve counting theories on X. There are three such theories, called Gromov-Witten (GW) theory, Donaldson-Thomas (DT) theory, and Pandharipande-Thomas (PT) theory. Conjecturally these theories are equivalent in terms of generating functions. Let us recall these theories.

For $g \ge 0$ and $\beta \in H_2(X,\mathbb{Z})$, the *GW-invariant* $N_{g,\beta}$ is defined by the integration of the virtual class,

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q},$$

where $\overline{M}_g(X,\beta)$ is the moduli stack of stable maps $f\colon C \to X$ with $g(C) = g$ and $f_*[C] = \beta$. We consider the following generating series,

$$\mathrm{GW}(X) = \exp\left(\sum_{g,\beta\neq 0} N_{g,\beta}\lambda^{2g-2}v^{\beta}\right).$$

For $n \in \mathbb{Z}$ and $\beta \in H_2(X,\mathbb{Z})$, let $I_n(X,\beta)$ be the Hilbert scheme of 1-dimensional subschemes $Z \subset X$ satisfying

$$[Z] = \beta, \quad \chi(\mathcal{O}_Z) = n.$$

The obstruction theory on $I_n(X,\beta)$ is obtained by viewing it as a moduli space of ideal sheaves, (cf. [21],) and the *DT-invariant* $I_{n,\beta}$ is defined by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The generating function of the *reduced DT-theory* is

$$\mathrm{DT}(X) = \sum_{n,\beta} I_{n,\beta} q^n v^\beta / \sum_n I_{n,0} q^n.$$

The theory of stable pairs and their counting invariants are introduced and studied by Pandharipande and Thomas [18] to give a geometric interpretation of the reduced DT-theory. By definition, a *stable pair* is data (F, s),

$$s \colon \mathcal{O}_X \longrightarrow F,$$

where F is a pure one dimensional sheaf on X, and s is a morphism with a zero dimensional cokernel. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of stable pairs (F, s) with

$$[F] = \beta, \quad \chi(F) = n,$$

is constructed in [18], denoted by $P_n(X, \beta)$. The obstruction theory on $P_n(X, \beta)$ is obtained by viewing stable pairs (F, s) as two term complexes,

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow 0 \longrightarrow \cdots . \tag{1}$$

Here the degree of $\mathcal{O}_X$ is -1 and the degree of F is 0. The *PT-invariant* $P_{n,\beta}$ is defined by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

The corresponding generating function is

$$\mathrm{PT}(X) = \sum_{n,\beta} P_{n,\beta} q^v v^\beta.$$

The series $\mathrm{GW}(X)$, $\mathrm{DT}(X)$ and $\mathrm{PT}(X)$ are conjecturally equal after suitable variable change. In order to state this, we need the following conjecture, called *rationality conjecture.*

Conjecture 1.1. [16, Conjecture 2], [18, Conjecture 3.2] *For a fixed β, the generating series*

$$\mathrm{DT}_\beta(X) = \sum_{n\in\mathbb{Z}} I_{n,\beta} q^n / \sum_{n\in\mathbb{Z}} I_{n,0} q^n, \quad \mathrm{PT}_\beta(X) = \sum_{n\in\mathbb{Z}} P_{n,\beta} q^n,$$

are Laurent expansions of rational functions of q, invariant under $q \leftrightarrow 1/q$.

The above conjecture is solved for $\mathrm{DT}_\beta(X)$ when X is a toric local Calabi-Yau 3-fold [16], and for $\mathrm{PT}_\beta(X)$ when β is an irreducible curve class [19]. Now we can state the conjectural GW-DT-PT-correspondences.

Conjecture 1.2. [16, Conjecture 3], [18, Conjecture 3.3] *After the variable change $q = -e^{i\lambda}$, we have*

$$\mathrm{GW}(X) = \mathrm{DT}(X) = \mathrm{PT}(X).$$

The variable change $q = -e^{i\lambda}$ is well-defined by Conjecture 1.1.

Note that ideal sheaves $I \subset \mathcal{O}_X$ are objects in $D^b(X)$, where $D^b(X)$ is the bounded derived category of coherent sheaves on X. We can also interpret stable pairs (F, s) as objects in $D^b(X)$ by viewing them as two term complexes (1). As discussed in [18, Secction 3], the equality $\mathrm{DT}(X) = \mathrm{PT}(X)$ should be interpreted as a wall-crossing formula for counting invariants in the category $D^b(X)$. Our purpose is to interpret PT-invariant as counting "stable" objects in the derived category with respect to some stability condition on $D^b(X)$, and study $\mathrm{PT}(X)$ via wall-crossing phenomena in the derived category.

2 Motivations

Before stating our result, we give a rough sketch of our motivation. Let $\mathcal{D}$ be a triangulated category, e.g. bounded derived category of coherent sheaves $D^b(X)$ on an algebraic variety X. Its objects consist of bounded complexes of coherent sheaves,

$$\cdots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots \to \mathcal{F}^j \to 0 \to \cdots,$$

where $\mathcal{F}^i \in \mathrm{Coh}(X)$. Historically such a class of categories was introduced to formulate the generalization of several duality theories, such as Poincaré duality, Serre duality. (cf. [2], [6].) On the other hand, the notion of triangulated categories draw much attention recently from the viewpoint of string theory. In terms of string theory, an object in the derived category of coherent sheaves is considered to represent a D-brane of type B, and a conjectural symmetry (Homological mirror symmetry) between the category of A-branes (Fukaya category) and B-branes (derived category) is proposed by Kontsevich [13].

In 2002, an important notion of stability conditions on triangulated categories was introduced by Bridgeland [4]. For a triangulated category $\mathcal{D}$, he associates a space $\mathrm{Stab}(\mathcal{D})$, which has a structure of complex manifold. So we have the following correspondence,

$$\text{triangulated category} \longrightarrow \text{complex manifold}$$

There are several motivations to introduce the complex manifold $\mathrm{Stab}(\mathcal{D})$.

- Classically there is a notion of stability condition on vector bundles on curves. (cf. [17].) We want to generalize this notion to objects in derived categories. For each $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the associated notion of σ-semistable objects in $\mathcal{D}$. So each point $\sigma \in \mathrm{Stab}(\mathcal{D})$ gives a generalization of the classical notion of stability condition. In terms of string theory, σ-semistable objects are considered to be the D-branes of BPS-state.

- The space $\mathrm{Stab}(\mathcal{D})$ is considered to describe the (extended) stringy Kähler moduli space, which should be isomorphic to the moduli space of complex structures on the mirror side. Thus it is an interesting problem to compare the space $\mathrm{Stab}(\mathcal{D})$ with the moduli space of the complex structures under mirror symmetry.

Since the theory of stability conditions on triangulated categories has been proposed recently, the theory is not so developed yet. One of the big issues is the existence problem of stability conditions, especially on the derived category of coherent sheaves on projective Calabi-Yau 3-folds. We will address this problem later.

Conjecturally the objects (1) are stable with respect to a certain stability condition on $D^b(X)$. Note that an object E given in (1) satisfies the following condition,

$$\mathrm{ch}(E) = (-1, 0, \beta, n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6, \quad \det E = \mathcal{O}_X, \tag{2}$$

Under the above background, we suggest the following story.

- For a projective Calabi-Yau 3-fold X, let $\mathcal{D} = D^b(X)$. We expect that there are stability conditions $\sigma, \tau \in \mathrm{Stab}(\mathcal{D})$ such that ideal sheaves $I_C[1]$ and objects (1) become stable with respect to σ, τ respectively.

- We expect that for any $\sigma \in \mathrm{Stab}(\mathcal{D})$, there is the algebraic moduli stack of σ-semistable objects $E \in \mathcal{D}$ with fixed phase and satisfy (2). We denote that moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$. For a particular choice of σ, the stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ should be the gerb over $I_n(X, \beta)$ or $P_n(X, \beta)$.

- We expect that there is the generalized Donaldson-Thomas invariant,

$$DT_{n,\beta}(\sigma) \in \mathbb{Q},$$

  counting σ-semistable objects $E \in \mathcal{D}$ satisfying (2). $DT_{n,\beta}(\sigma)$ should be defined as the integration of the "logarithm" of the moduli stack $\mathcal{M}^{(-1,0,\beta,n)}(\sigma)$ in the Hall algebra of $\mathcal{D}$, after multiplying Behrend's weight function [1]. This procedure (expect multiplication of weight function) follows from Joyce's sequent works [9], [10], [11], [11], [12]. It should be possible to use the motivic milnor fiber idea of Kontsevich-Soibelman [14] to involve weight function into Joyce's invariants. A particular choice of σ yields $DT_{n,\beta}(\sigma) = I_{n,\beta}$ or $P_{n,\beta}$. However $DT_{n,\beta}(\sigma)$ give new invariants by deforming σ.

- We want to know how $DT_{n,\beta}(\sigma)$ varies under change of σ. In principle, there is a wall and chamber structure on $\mathrm{Stab}(\mathcal{D})$ so that $DT_{n,\beta}(\sigma)$ does not change if σ deforms in a chamber. However if σ crosses a wall, then the invariant $DT_{n,\beta}(\sigma)$ jumps and its difference should be expressed in terms of the structure of the Ringel-Hall Lie algebra associated to $\mathcal{D}$. Thus we should have the wall-crossing formula of the invariants $DT_{n,\beta}(\sigma)$.

- Applying the wall-crossing formula of $DT_{n,\beta}(\sigma)$, we expect that several properties or equalities of the generating functions of sheaf counting are realized. For instance, DT-PT correspondence [18], DT-NCDT correspondence [20], flop formula of DT-invariants [7], and the rationality conjecture of the generating functions of DT or PT-invariants should be explained by wall-crossing formula. (cf. [22].)

At this moment, there are several technical difficulties to realize the above story. One of them is to find stability conditions, which will be discussed in the next section.

3 Stability conditions

First let us give the definition of stability conditions introduced in [4].

Definition 3.1. A stability condition on a triangulated category $\mathcal{D}$ consists of data $\sigma = (Z, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$, and Z is a group homomorphism,

$$Z\colon K(\mathcal{D}) \longrightarrow \mathbb{C},$$

which satisfies the following axiom.

- For a non-zero object $0 \neq E \in \mathcal{A}$, we have

  $$Z(E) \in \mathbb{H} := \{r \exp(i\pi\phi) \mid 0 < \phi \leq 1, r > 0\}.$$

  Especially one can choose the argument $\arg Z(E) \in (0, \pi]$ uniquely. An object $E \in \mathcal{A}$ is said to be Z-(semi)stable if for any non-zero object $F \subset E$, one has

  $$\arg Z(F) \leq (<) \arg Z(E).$$

- There is a Harder-Narasimhan property, i.e. any $E \in \mathcal{A}$ admits a filtration

  $$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

  such that each $F_i = E_i/E_{i-1}$ is Z-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here we give some examples.

Example 3.2. (i) Let $\mathcal{D} = D^b(C)$ for a smooth projective curve C, and $Z\colon K(C) \to \mathbb{C}$ be

$$Z(E) = -\deg(E) + \mathrm{rk}(E) \cdot i.$$

Then the pair $(Z, \mathrm{Coh}(C))$ determines a stability condition on $\mathcal{D}$. In this case, an object $E \in \mathrm{Coh}(C)$ is Z-semistable if and only if it is a semistable sheaf in the sense of [17].

(ii) Let A be a finite dimensional k-algebra with k a field, and $\mathcal{D} = D^b(\mathcal{A})$ where $\mathcal{A} = \mathrm{mod}\, A$ is the abelian category of finitely generated right A-modules. Then there is a finite number of simple objects $S_1, \cdots, S_N \in \mathcal{A}$ which generates $\mathcal{A}$. One can choose $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that $Z(S_i) \in \mathbb{H}$ for all $1 \leq i \leq N$. Then the pair $(Z, \mathcal{A})$ determines a stability condition on $\mathcal{D}$.

So far, the spaces $\mathrm{Stab}(\mathcal{D})$ for several $\mathcal{D}$ have been studied in detail. For instance, see [5], [15], [8], [23]. On the other hand, the following problem has been a big issue in studying stability conditions.

Problem 3.3. *Given a triangulated category $\mathcal{D}$, do we have an example of a stability condition on $\mathcal{D}$, i.e.* $\mathrm{Stab}(\mathcal{D}) \neq \emptyset$*?*

The above problem is non-trivial especially for the case $\mathcal{D} = D^b(X)$, where X is a smooth projective variety with $\dim X \geq 2$. In this case, one can show that there is no stability condition $(Z, \mathcal{A})$ with $\mathcal{A} = \mathrm{Coh}(X)$. As an analogue of Example 3.2 (i), one might try to construct Z to be the group homomorphism

$$Z(E) = -c_1(E) \cdot \omega + \mathrm{rk}(E) \cdot i,$$

for a fixed ample divisor ω. However the pair $(Z, \mathrm{Coh}(X))$ does not give a stability condition since $Z([\mathcal{O}_x]) = 0$ for a closed point $x \in X$. When $\dim X = 2$, the examples of stability conditions are constructed by tilting the abelian category $\mathrm{Coh}(X)$, (cf. [5].) However we do not know any example of stability conditions when $\dim X \geq 3$, except the case that there is a derived equivalence $D^b(X) \cong D^b(A)$ for a finite dimensional algebra A. (e.g. $X = \mathbb{P}^3$.)

From the viewpoint of mirror symmetry, the most important case is when X is a projective Calabi-Yau 3-fold. In this case, there are some ideas coming from string theory. Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is ample } \}.$$

Let $Z_{(B,\omega)} \colon K(X) \to \mathbb{C}$ be

$$Z_{(B,\omega)}(E) = -\int e^{-(B+i\omega)} \,\mathrm{ch}(E)\sqrt{\mathrm{td}_X}.$$

We can state the following conjecture.

Conjecture 3.4. *For $\omega \gg 0$, there should exist the heart of a bounded t-structure $\mathcal{A}_{(B,\omega)} \subset D^b(X)$ such that the pair $\sigma_{(B,\omega)} = (Z_{(B,\omega)}, \mathcal{A}_{(B,\omega)})$ is a stability condition on $D^b(X)$.*

The above conjecture holds true if $\dim X \leq 2$.

4 Stability conditions on D0-D2-D6 bound states

Let $\mathrm{Coh}_{\leq 1}(X)$ be

$$\mathrm{Coh}_{\leq 1}(X) := \{E \in \mathrm{Coh}(X) \mid \dim \mathrm{Supp}(E) \leq 1\}.$$

Instead of working with $D^b(X)$, we study stability conditions on $\mathcal{D}_X$,

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b(X).$$

Here for a set of objects $S \subset D^b(X)$, we denote by $\langle S \rangle_{\mathrm{tr}}$ the smallest triangulated subcategory of $D^b(X)$, which contains S. We also denote by $\langle S \rangle_{\mathrm{ex}}$ the smallest extension closed subcategory of $D^b(X)$, which contains S. We have the following lemma, whose proof will be appear in [24].

Lemma 4.1. *There is a bounded t-structure on $\mathcal{D}_X$, whose heart $\mathcal{A}_X$ satisfies*

$$\mathcal{A}_X = \langle \mathcal{O}_X[1], \mathrm{Coh}_{\leq 1}(X) \rangle.$$

Let $A(X)_{\mathbb{C}}$ be the complexified ample cone,

$$A(X)_{\mathbb{C}} := \{B + i\omega \in H^2(X, \mathbb{C}) \mid \omega \text{ is ample. } \}.$$

For a following data,

$$s \in \mathbb{R}_{<0}, \quad t \in A(X)_{\mathbb{C}}, \quad u \in \mathbb{H},$$

we define a map

$$Z_{(s,t,u)} \colon K(\mathcal{D}_X) \longrightarrow \mathbb{C},$$

as

$$Z_{(s,t,u)}(E) = s\,\mathrm{ch}_3(E) + t\,\mathrm{ch}_2(E) - u\,\mathrm{ch}_0(E).$$

We have the following, which will be appear in [24].

Lemma 4.2. *The pair $(Z_{(s,t,u)}, \mathcal{A}_X)$ determines points in $\mathrm{Stab}(\mathcal{D}_X)$. In particular $\mathrm{Stab}(\mathcal{D}_X) \neq \emptyset$.*

We have the following embedding,

$$\mathcal{U}_X := \mathbb{R}_{<0} \times A(X)_{\mathbb{C}} \times \mathbb{H} \subset \mathrm{Stab}(\mathcal{D}_X).$$

The following result will be proved in [24].

Theorem 4.3. *(i) For $\sigma \in \mathcal{U}_X$, there is the algebraic moduli stack of finite type*

$$\mathcal{M}^{(-1,0,\beta,n)}(\sigma),$$

which parameterizes σ-semistable objects $E \in \mathcal{A}_X$ with

$$\mathrm{ch}(E) = (-1, 0, \beta, n), \quad \det(E) = \mathcal{O}_X.$$

(ii) Suppose that $u \in \mathbb{R}_{<0}$ for $\sigma = (s, t, u)$. For $u \ll 0$, we have

$$\mathcal{M}^{(-1,0,\beta,n)}(\sigma) = [P_n(X, \beta)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts on $P_n(X, \beta)$ trivially.

5 Flop formula

Applying Theorem 4.3 and wall-crossing formula developed by Joyce [12], Kontsevich-Soibelman [14], we can study how generating series of invariants counting stable pairs transform under flops. Instead of working with $\mathrm{PT}(X)$, let us consider the generating series,

$$\hat{\mathrm{PT}}(X) = \sum_{n,\beta} \chi(P_n(X, \beta)) q^n v^\beta.$$

The series $\hat{\mathrm{PT}}(X)$ is closely related to $\mathrm{PT}(X)$ in the following sense.

- Suppose that $P_n(X, \beta)$ is smooth and connected. Then we have

$$P_{n,\beta} = (-1)^{\dim P_n(X,\beta)} \chi(P_n(X, \beta)).$$

- In general, there is a constructible function $\nu\colon P_n(X,\beta) \to \mathbb{Z}$, constructed by Behrend [1], such that
$$P_{n,\beta} = \sum_{n\in\mathbb{Z}} n\chi(\nu^{-1}(n)).$$

Let us consider a diagram of flop of Calabi-Yau 3-folds,

$$\begin{tikzcd}
X^+ \arrow[rr, dashed, "\phi"] \arrow[dr, "f^+"'] & & X \arrow[dl, "f"] \\
& Y. &
\end{tikzcd}$$

In this situation, Bridgeland [3] showed the equivalence of derived categories,

$$\Phi\colon D^b(X^+) \xrightarrow{\sim} D^b(X).$$

It is easy to see that Φ restricts to the equivalence,

$$\Phi\colon \mathcal{D}_{X^+} \xrightarrow{\sim} \mathcal{D}_X,$$

hence we have the isomorphism,

$$\Phi_*\colon \operatorname{Stab}(\mathcal{D}_{X^+}) \xrightarrow{\cong} \operatorname{Stab}(\mathcal{D}_X).$$

We have the following. (cf. [24].)

Lemma 5.1. *We have*
$$\Phi_*\overline{\mathcal{U}}_{X^+} \cap \overline{\mathcal{U}}_X \neq \emptyset.$$

The above lemma implies that we can relate stability conditions relevant to stable pairs on X to those on X^+. Let $\hat{\mathrm{PT}}(X/Y)$ be the subseries

$$\hat{\mathrm{PT}}(X/Y) = \sum_{n, f_*\beta=0} \chi(P_n(X,\beta))q^n v^\beta.$$

Applying wall-crossing formula by Joyce [12], from a point in $\mathcal{U}_X$ to $\Phi_*\mathcal{U}_{X^+}$, we obtain the following. (cf. [24].)

Theorem 5.2. *Under the above situation, we have*

$$\frac{\hat{\mathrm{PT}}(X)}{\hat{\mathrm{PT}}(X/Y)} = \phi_* \frac{\hat{\mathrm{PT}}(X^+)}{\hat{\mathrm{PT}}(X^+/Y)},$$
$$\hat{\mathrm{PT}}(X/Y) = i\circ\phi_*\hat{\mathrm{PT}}(X^+/Y).$$

Here the variable change is $\phi_*(\beta,n) = (\phi_*\beta,n)$ and $i(\beta,n) = (-\beta,n)$.

We can apply Joyce's wall-crossing formula of counting invariants. Unfortunately we are unable to involve Behrend's constructible function into Joyce's work, so our application is restricted to Euler number of version of the relevant moduli spaces at this moment.

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Induced nilpotent orbits and birational geometry

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This exposition is based on two lectures in the conferences at Kinosaki (Oct. 2008), and at Tokyo (Dec. 2008).

Introduction.

Let G be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. A nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is an orbit of a nilpotent element of $\mathfrak{g}$ by the adjoint action of G on $\mathfrak{g}$. Then $\mathcal{O}$ admits a natural symplectic 2-form ω and the nilpotent orbit closure $\bar{\mathcal{O}}$ has symplectic singularities in the sense of [Be] and [Na3] (cf. [Pa], [Hi]). In [Ri], Richardson introduced the notion of so-called the *Richardson orbit*. A nilpotent orbit $\mathcal{O}$ is called Richardson if there is a parabolic subgroup Q of G such that $\mathcal{O} \cap n(\mathfrak{q})$ is an open dense subset of $n(\mathfrak{q})$, where $n(\mathfrak{q})$ is the nil-radical of $\mathfrak{q}$. Later, Lusztig and Spaltenstein [L-S] generalized this notion to the *induced orbit*. A nilpotent orbit $\mathcal{O}$ is an induced orbit if there are a parabolic subgroup Q of G and a nilpotent orbit $\mathcal{O}'$ in the Levi subalgebra $\mathfrak{l}(\mathfrak{q})$ of $\mathfrak{q} := \mathrm{Lie}(Q)$ such that $\mathcal{O}$ meets $n(\mathfrak{q}) + \mathcal{O}'$ in an open dense subset. If $\mathcal{O}$ is an induced orbit, one has a natural map (cf. (1.2))

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

The map ν is a generically finite, projective, surjective map. This map is called the *generalized Springer map*. In this paper, we shall study the induced orbits from the view point of *birational geometry*. For a Richardson orbit $\mathcal{O}$, the Springer map ν is a map from the cotangent bundle $T^*(G/Q)$ of the flag variety G/Q to $\bar{\mathcal{O}}$. In [Fu], Fu proved that, if $\bar{\mathcal{O}}$ has a crepant (projective) resolution, it is a Springer map. Note that Q is not unique (even up to the conjugate) for a Richardson orbit $\mathcal{O}$. This means that $\bar{\mathcal{O}}$ has many different crepant resolutions. In [Na], the author has given a description of all crepant resolutions of $\bar{\mathcal{O}}$ and proved that any two different crepant resolutions are connected by *Mukai flops*. The purpose of this paper is to generalize these to *all* nilpotent orbits $\mathcal{O}$. If $\mathcal{O}$ is not Richardson, $\bar{\mathcal{O}}$ has no crepant resolution. The substitute of a crepant resolution, is a $\mathbf{Q}$-*factorial terminalization*. Let X be a complex algebraic variety with rational Gorenstein singularities. A partial resolution $f : Y \to X$ of X is said to be a $\mathbf{Q}$-*factorial terminalization* of X if Y has only $\mathbf{Q}$-factorial terminal singularities and f is a birational projective morphism such that $K_Y = f^*K_X$. A $\mathbf{Q}$-factorial terminalization is a crepant resolution exactly when Y is smooth. Recently, Birkar-Cascini-Hacon-McKernan [B-C-H-M] have established the existence of minimal models of complex algebraic varieties of general type. As a corollary of this, we know that X always has a $\mathbf{Q}$-factorial terminalization. In particular, $\bar{\mathcal{O}}$ should have a $\mathbf{Q}$-factorial terminalization. The author would like to pose the following conjecture.

Conjecture. *Let $\mathcal{O}$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Let $\tilde{\mathcal{O}}$ be the normalization of $\bar{\mathcal{O}}$. Then one of the following holds:*

(1) *$\tilde{\mathcal{O}}$ has $\mathbf{Q}$-factorial terminal singularities.*

(2) *There are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a nilpotent orbit $\mathcal{O}'$ of $\mathfrak{l}$ such that (a): $\mathcal{O} = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}}(\mathcal{O}')$ and (b): the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}')$ is a $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ via the generalized Springer map.*

Moreover, if $\tilde{\mathcal{O}}$ does not have $\mathbf{Q}$-factorial terminal singularities, then every $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is of the form (2). Two $\mathbf{Q}$-factorial terminalizations are connected by Mukai flops (cf. [Na], p.91).

The main result of this report is that Conjecture is true when $\mathfrak{g}$ is classical. Recently, Fu checked Conjecture for $\mathfrak{g}$ exceptional by a case-by-case method using the computer program GAP 4 (arxiv: 0809.5109, version 2). Combining this with our result, Conjecture holds true in full generality. However, a conceptual proof without the classification of nilpotent orbits, is still missing. This is a summary of [Na -1]. For details on proofs, see the original paper [Na -1].

§1. Preliminaries

(1.1) *Nilpotent orbits and resolutions*: Let G be a complex simple algebraic group and let $\mathfrak{g}$ be its Lie algebra. G has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. By the Jacobson-Morozov theorem, one can find a semi-simple element $h \in \mathfrak{g}$, and a nilpotent element $y \in \mathfrak{g}$ in such a way that $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. For $i \in \mathbf{Z}$, let

$$\mathfrak{g}_i := \{z \in \mathfrak{g} \ [h, z] = iz\}.$$

Then one can write

$$\mathfrak{g} = \oplus_{i \in \mathbf{Z}} \mathfrak{g}_i.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $h \in \mathfrak{h}$. Let Φ be the corresponding root system and let Δ be a base of simple roots such that h is Δ-dominant, i.e. $\alpha(h) \geq 0$ for all $\alpha \in \Delta$. In this situation,

$$\alpha(h) \in \{0, 1, 2\}.$$

The weighted Dynkin diagram of $\mathcal{O}_x$ is the Dynkin diagram of $\mathfrak{g}$ where each vertex α is labeled with $\alpha(h)$. A nilpotent orbit $\mathcal{O}_x$ is completely determined by its weighted Dynkin diagram. A Jacobson-Morozov parabolic subalgebra for x is the parabolic subalgebra $\mathfrak{p}$ defined by

$$\mathfrak{p} := \oplus_{i \geq 0} \mathfrak{g}_i.$$

Let P be the parabolic subgroup of G determined by $\mathfrak{p}$. We put

$$\mathfrak{n}_2 := \oplus_{i \geq 2} \mathfrak{g}_i.$$

Then $\mathfrak{n}_2$ is an ideal of $\mathfrak{p}$; hence, P has the adjoint action on $\mathfrak{n}_2$. Let us consider the vector bundle $G \times^P \mathfrak{n}_2$ over G/P and the map

$$\mu : G \times^P \mathfrak{n}_2 \to \mathfrak{g}$$

defined by $\mu([g, z]) := Ad_g(z)$. Then the image of μ coincides with the closure $\bar{\mathcal{O}}_x$ of $\mathcal{O}_x$ and μ gives a resolution of $\bar{\mathcal{O}}_x$(cf. [K-P], Proposition 7.4). We call μ the *Jacobson-Morozov resolution* of $\bar{\mathcal{O}}_x$. The orbit $\mathcal{O}_x$ has a natural closed non-degenerate 2-form ω (cf. [C-G], Prop. 1.1.5., [C-M], 1.3). By μ, ω is regarded as a 2-form on a Zariski open subset of $G \times^P \mathfrak{n}_2$. By [Pa], [Hi], it extends to a 2-form on $G \times^P \mathfrak{n}_2$. In other words, $\bar{\mathcal{O}}_x$ has symplectic singularity. Let $\tilde{\mathcal{O}}_x$ be the normalization of $\bar{\mathcal{O}}_x$. In many cases, one can check the $\mathbf{Q}$-factoriality of $\tilde{\mathcal{O}}_x$ by applying the following lemma to the Jacobson-Morozov resolution:

Lemma (1.1.1). *Let $\pi : Y \to X$ be a projective resolution of an affine variety X with rational singularities. Let ρ be the relative Picard number for π. If* $\mathrm{Exc}(\pi)$ *contains ρ different prime divisors, then X is $\mathbf{Q}$-factorial.*

(1.2) *Induced orbits*

(1.2.1). Let G and $\mathfrak{g}$ be the same as in (1.1). Let Q be a parabolic subgroup of G and let $\mathfrak{q}$ be its Lie algebra with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$. Here n is the nil-radical of $\mathfrak{q}$ and $\mathfrak{l}$ is a Levi-part of $\mathfrak{q}$. Fix a nilpotent orbit $\mathcal{O}'$ in $\mathfrak{l}$. Then there is a unique nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ meeting $n + \mathcal{O}'$ in an open dense subset ([L-S]). Such an orbit $\mathcal{O}$ is called the nilpotent orbit induced from $\mathcal{O}'$ and we write

$$\mathcal{O} = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l}}(\mathcal{O}').$$

Note that when $\mathcal{O}' = 0$, $\mathcal{O}$ is the Richardson orbit for Q. Since the adjoint action of Q on $\mathfrak{q}$ stabilizes $n + \bar{\mathcal{O}}'$, one can consider the variety $G \times^Q (n + \bar{\mathcal{O}}')$. There is a map

$$\nu : G \times^Q (n + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}$$

defined by $\nu([g, z]) := Ad_g(z)$. Since $\mathrm{Codim}_{\mathfrak{l}}(\mathcal{O}') = \mathrm{Codim}_{\mathfrak{g}}(\mathcal{O})$ (cf. [C-M], Prop. 7.1.4), ν is a generically finite dominating map. Moreover, ν is factorized as

$$G \times^Q (n + \bar{\mathcal{O}}') \to G/Q \times \bar{\mathcal{O}} \to \bar{\mathcal{O}}$$

where the first map is a closed embedding and the second map is the 2-nd projection; this implies that ν is a projective map. In the remainder, we call ν the generalized Springer map for $(Q, \mathcal{O}')$.

(1.2.2). Assume that Q is contained in another parabolic subgroup $\bar{Q}$ of G. Let $\bar{L}$ be the Levi part of $\bar{Q}$ which contains the Levi part L of Q. Let $\bar{\mathfrak{q}} = \bar{\mathfrak{l}} \oplus \bar{\mathfrak{n}}$ be the Levi decomposition. Note that $\bar{L} \cap Q$ is a parabolic subgroup of $\bar{L}$ and $\mathfrak{l}(\bar{L} \cap Q) = \mathfrak{l}$. Let $\mathcal{O}_1 \subset \bar{\mathfrak{l}}$ be the nilpotent orbit induced from $(\bar{L} \cap Q, \mathcal{O}')$. Then there is a natural map

$$\pi : G \times^Q (n + \bar{\mathcal{O}}') \to G \times^{\bar{Q}} (\bar{n} + \bar{\mathcal{O}}_1)$$

which factorizes ν as $\bar{\nu} \circ \pi = \nu$. Here $\bar{\nu}$ is the generalized Springer map for $(\bar{Q}, \mathcal{O}_1)$.

(1.2.3). Assume that there are a parabolic subgroup Q_L of L and a nilpotent orbit $\mathcal{O}_2$ in the Levi subalgebra $\mathfrak{l}(Q_L)$ such that $\mathcal{O}'$ is the nilpotent orbit induced from $(Q_L, \mathcal{O}_2)$. Then there is a parabolic subgroup Q' of G

such that $Q' \subset Q$, $\mathfrak{l}(Q') = \mathfrak{l}(Q_L)$ and $\mathcal{O}$ is the nilpotent orbit induced from $(Q', \mathcal{O}_2)$. The generalized Springer map ν' for $(Q', \mathcal{O}_2)$ is factorized as

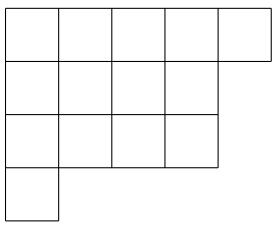
$$G \times^{Q'} (\mathfrak{n}' + \bar{\mathcal{O}}_2) \to G \times^{Q} (\mathfrak{n} + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

Lemma (1.2.4). *Let*

$$\nu : G \times^{Q} (n + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}$$

be a generalized Springer map defined in (1.2.1). Then the normalization of $G \times^{Q} (n + \bar{\mathcal{O}}')$ *is a symplectic variety.*

(1.3) *Nilpotent orbits in classical Lie algebras*: When $\mathfrak{g}$ is a classical Lie algebra, $\mathfrak{g}$ is naturally a Lie subalgebra of $\mathrm{End}(V)$ for a complex vector space V. Then we can attach a partition $\mathbf{d}$ of $n := \dim V$ to each orbit as the Jordan type of an element contained in the orbit. Here a partition $\mathbf{d} := [d_1, d_2, ..., d_k]$ of n is a set of positive integers with $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. Another way of writing $\mathbf{d}$ is $[d_1^{s_1}, ..., d_k^{s_k}]$ with $d_1 > d_2 ... > d_k > 0$. Here $d_i^{s_i}$ is an s_i times d_i's: $d_i, d_i, ..., d_i$. The partition $\mathbf{d}$ corresponds to a Young diagram. For example, $[5, 4^2, 1]$ corresponds to



When an integer e appears in the partition $\mathbf{d}$, we say that e is a *member* of $\mathbf{d}$. We call $\mathbf{d}$ *very even* when $\mathbf{d}$ consists with only even members, each having even multiplicity.

Let us denote by ϵ the number 1 or -1. Then a partition $\mathbf{d}$ is ϵ-admissible if all even (resp. odd) members of $\mathbf{d}$ have even multiplicities when $\epsilon = 1$ (resp. $\epsilon = -1$). The following result can be found, for example, in [C-M, §5].

Proposition (1.3.1) *Let* $\mathcal{N}o(\mathfrak{g})$ *be the set of nilpotent orbits of* $\mathfrak{g}$.

$(1)(A_{n-1})$: *When* $\mathfrak{g} = \mathfrak{sl}(n)$, *there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of partitions* $\mathbf{d}$ *of* n.

$(2)(B_n)$: *When* $\mathfrak{g} = \mathfrak{so}(2n+1)$, *there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n+1$ *with* $\epsilon = 1$.

(3)(C_n): *When* $\mathfrak{g} = \mathfrak{sp}(2n)$*, there is a bijection between* $\mathcal{N}o(\mathfrak{g})$ *and the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n$ *with* $\epsilon = -1$*.*

(4)(D_n): *When* $\mathfrak{g} = \mathfrak{so}(2n)$*, there is a surjection* f *from* $\mathcal{N}o(\mathfrak{g})$ *to the set of* ϵ*-admissible partitions* $\mathbf{d}$ *of* $2n$ *with* $\epsilon = 1$*. For a partition* $\mathbf{d}$ *which is not very even,* $f^{-1}(\mathbf{d})$ *consists of exactly one orbit, but, for very even* $\mathbf{d}$*,* $f^{-1}(\mathbf{d})$ *consists of exactly two different orbits.*

Take an ϵ-admissible partition $\mathbf{d}$ of a positive integer m. If $\epsilon = 1$, we put $\mathfrak{g} = so(m)$ and if $\epsilon = -1$, we put $\mathfrak{g} = sp(m)$. We denote by $\mathcal{O}_{\mathbf{d}}$ a nilpotent orbit in $\mathfrak{g}$ with Jordan type $\mathbf{d}$. Note that, except when $\epsilon = 1$ and $\mathbf{d}$ is very even, $\mathcal{O}_{\mathbf{d}}$ is uniquely determined. When $\epsilon = 1$ and $\mathbf{d}$ is very even, there are two possibilities for $\mathcal{O}_{\mathbf{d}}$. If necessary, we distinguish the two orbits by the labelling: $\mathcal{O}^I_{\mathbf{d}}$ and $\mathcal{O}^{II}_{\mathbf{d}}$. Let us fix a classical Lie algebra $\mathfrak{g}$ and study the relationship among nilpotent orbits in $\mathfrak{g}$. When $\mathfrak{g}$ is of type B or D (resp. C), we only consider the ϵ-admissible partitions with $\epsilon = 1$ (resp. $\epsilon = -1$). We introduce a partial order in the set of the partitions of (the same number): for two partitions $\mathbf{d}$ and $\mathbf{f}$, $\mathbf{d} \geq \mathbf{f}$ if $\Sigma_{i \leq k} d_i \geq \Sigma_{i \leq k} f_i$ for all $k \geq 1$. On the other hand, for two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$ in $\mathfrak{g}$, we write $\mathcal{O} \geq \mathcal{O}'$ if $\mathcal{O}' \subset \bar{\mathcal{O}}$. Then, $\mathcal{O}_{\mathbf{d}} \geq \mathcal{O}_{\mathbf{f}}$ if and only if $\mathbf{d} \geq \mathbf{f}$. When $\mathbf{d}$ and $\mathbf{f}$ are ϵ-admissible partitions with $\mathbf{f} \geq \mathbf{g}$, we call this pair an ϵ*-degeneration* or simply a *degeneration*.

Now let us consider the case $\mathfrak{g}$ is of type B, C or D.

Assume that an ϵ- degeneration $\mathbf{d} \geq \mathbf{f}$ is *minimal* in the sense that there is no ϵ-admissible partition $\mathbf{d}'$ (except $\mathbf{d}$ and $\mathbf{f}$) such that $\mathbf{d} \geq \mathbf{d}' \geq \mathbf{f}$. Kraft and Procesi [K-P] have studied the *normal slice* $N_{\mathbf{d},\mathbf{f}}$ of $\mathcal{O}_{\mathbf{f}} \subset \bar{\mathcal{O}}_{\mathbf{d}}$ in such cases. If, for two integers r and s, the first r rows and the first s columns of $\mathbf{d}$ and $\mathbf{f}$ coincide and the partition $(d_1, ..., d_r)$ is ϵ-admissible, then one can erase these rows and columns from $\mathbf{d}$ and $\mathbf{f}$ respectively to get new partitions $\mathbf{d}'$ and $\mathbf{f}'$ with $\mathbf{d}' \geq \mathbf{f}'$. If we put $\epsilon' := (-1)^s\epsilon$, then $\mathbf{d}'$ and $\mathbf{f}'$ are both ϵ'-admissible. The pair $(\mathbf{d}', \mathbf{f}')$ is also minimal. Repeating such process, one can reach a degeneration $\mathbf{d}_{irr} \geq \mathbf{f}_{irr}$ which is *irreducible* in the sense that there are no rows and columns to be erased. By [K-P], Theorem 2, $N_{\mathbf{d},\mathbf{f}}$ is analytically isomorphic to $N_{\mathbf{d}_{irr},\mathbf{f}_{irr}}$ around the origin. According to [K-P], a minimal and irreducible degeneration $\mathbf{d} \geq \mathbf{f}$ is one of the following:

a: $\mathfrak{g} = sp(2)$, $\mathbf{d} = (2)$, and $\mathbf{f} = (1^2)$.

b: $\mathfrak{g} = sp(2n)$ $(n > 1)$, $\mathbf{d} = (2n)$, and $\mathbf{f} = (2n-2, 2)$.

c: $\mathfrak{g} = so(2n+1)$ $(n > 0)$, $\mathbf{d} = (2n+1)$, and $\mathbf{f} = (2n-1, 1^2)$.

d: $\mathfrak{g} = sp(4n+2)$ $(n > 0)$, $\mathbf{d} = (2n+1, 2n+1)$, and $\mathbf{f} = (2n, 2n, 2)$.

e: $\mathfrak{g} = so(4n)$ $(n > 0)$, $\mathbf{d} = (2n, 2n)$, and $\mathbf{f} = (2n-1, 2n-1, 1^2)$.
f: $\mathfrak{g} = so(2n+1)$ $(n > 1)$, $\mathbf{d} = (2^2, 1^{2n-3})$, and $\mathbf{f} = (1^{2n+1})$.
g: $\mathfrak{g} = sp(2n)$ $(n > 1)$, $\mathbf{d} = (2, 1^{2n-2})$, and $\mathbf{f} = (1^{2n})$.
h: $\mathfrak{g} = so(2n)$ $(n > 2)$, $\mathbf{d} = (2^2, 1^{2n-4})$, and $\mathbf{f} = (1^{2n})$.

In the first 4 cases (a,b,c,d,e), $\mathcal{O}_{\mathbf{f}}$ have codimension 2 in $\bar{\mathcal{O}}_{\mathbf{d}}$. In the last 3 cases (f,g,h), $\mathcal{O}_{\mathbf{f}}$ have codimension ≥ 4 in $\bar{\mathcal{O}}_{\mathbf{d}}$.

Proposition (1.3.2) *Let $\mathcal{O}$ be a nilpotent orbit in a classical Lie algebra $\mathfrak{g}$ of type B, C or D with Jordan type $\mathbf{d} := [(d_1)^{s_1}, ..., (d_k)^{s_k}]$ $(d_1 > d_2 > ... > d_k)$. Let Σ be the singular locus of $\bar{\mathcal{O}}$. Then $\mathrm{Codim}_{\bar{\mathcal{O}}}(\Sigma) \geq 4$ if and only if the partition $\mathbf{d}$ has full members, that is, any integer j with $1 \leq j \leq d_1$ is a member of $\mathbf{d}$. Otherwise, $\mathrm{Codim}_{\bar{\mathcal{O}}}(\Sigma) = 2$.*

(1.4.1) *Jacobson-Morozov resolutions in the case of classical Lie algebras*(cf. [CM], 5.3): Let V be a complex vector space of dimension m with a non-degenerate symmetric (or skew-symmetric) form $< , >$. In the symmetric case, we take a basis $\{e_i\}_{1 \leq i \leq m}$ of V in such a way that $< e_j, e_k >= 1$ if $j + k = m + 1$ and otherwise $< e_j, e_k >= 0$. In the skew-symmetric case, we take a basis $\{e_i\}_{1 \leq i \leq m}$ of V in such a way that $< e_j, e_k >= 1$ if $j < k$ and $j + k = m + 1$, and $< e_j, e_k >= 0$ if $j + k \neq m + 1$. When $(V, < , >)$ is a symmetric vector space, $\mathfrak{g} := so(V)$ is the Lie algebra of type $B_{(m-1)/2}$ (resp. $D_{m/2}$) if m is odd (resp. even). When $(V, < , >)$ is a skew-symmetric vector space, $\mathfrak{g} := sp(V)$ is the Lie algebra of type $C_{m/2}$. In the remainder of this paragraph, $\mathfrak{g}$ is one of these Lie algebra contained in $\mathrm{End}(V)$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra consisting of all diagonal matrices, and let Δ be the standard base of simple roots. Let $x \in \mathfrak{g}$ be a nilpotent element. As in (1.1), one can choose h, $y \in \mathfrak{g}$ in such a way that $\{x, y, h\}$ is a $sl(2)$-triple. If necessary, by replacing x by its conjugate element, one may assume that $h \in \mathfrak{h}$ and h is Δ-dominant. Assume that x has Jordan type $\mathbf{d} = [d_1, ..., d_k]$. The diagonal matrix h is described as follows. Let us consider the sequence of integers of length m:

$d_1 - 1, d_1 - 3, ..., -d_1 + 3, -d_1 + 1, d_2 - 1, d_2 - 3, ..., -d_2 + 3, -d_2 + 1, ..., d_k - 1, d_k - 3, ..., -d_k + 3, -d_k + 1$.

Rearrange this sequence in the non-increasing order and get a new sequence $p_1^{t_1}, ..., p_l^{t_l}$ with $p_1 > p_2 ... > p_l$ and $\Sigma t_i = m$. Then

$$h = \mathrm{diag}(p_1^{t_1}, ..., p_l^{t_l}).$$

Here $p_i^{t_i}$ means the t_i times of p_i's: $p_i, p_i, ..., p_i$. It is then easy to describe explicitly the Jacobson-Morozov parabolic subalgebra $\mathfrak{p}$ of x and its ideal $\mathfrak{n}_2$ (cf. (1.1)). The Jacobson-Morozov parabolic subgroup P is the stabilizer group of certain isotropic flag $\{F_i\}_{1\leq i\leq r}$ of V. Here, an isotropic flag of V (of length r) is a increasing filtration $0 \subset F_1 \subset F_2 \subset ... \subset F_r \subset V$ such that $F_{r+1-i} = F_i^{\perp}$ for all i. The flag type of P is $(t_1, ..., t_l)$. The nilradical $\mathfrak{n} := \oplus_{i>0}\mathfrak{g}_i$ of $\mathfrak{p}$ consists of the elements z of $\mathfrak{g}$ such that $z(F_i) \subset F_{i-1}$ for all i. On the other hand, it depends on the weighted Dynkin diagram for x how $\mathfrak{n}_2$ takes its place in $\mathfrak{n}$.

Lemma (1.4.2) *Assume that* $\mathbf{d}$ *has full members. For each minimal* ϵ*-degeneration* $\mathbf{d} \geq \mathbf{f}$*, the fiber* $\mu^{-1}(\mathcal{O}_{\mathbf{f}})$ *has codimension* 1 *in* $G \times^P \mathfrak{n}_2$*.*

Corollary (1.4.3) *Assume that* $\mathbf{d}$ *is an* ϵ*-admissible partition and it has full members. Let* $\tilde{\mathcal{O}}_{\mathbf{d}}$ *be the normalization of* $\bar{\mathcal{O}}_{\mathbf{d}}$*. Then,* $\tilde{\mathcal{O}}_{\mathbf{d}}$ *has only* $\mathbf{Q}$*-factorial termainal singularities except when* $\mathfrak{g} = so(4n+2)$*,* $n \geq 1$ *and* $\mathbf{d} = [2^{2n}, 1^2]$*.*

Proof. Let k be the maximal member of $\mathbf{d}$. Then there are $k-1$ minimal degenerations $\mathbf{d} \geq \mathbf{f}$. By Lemma (1.4.2), $\mathrm{Exc}(\mu)$ contains at least $k-1$ irreducible divisors. When $\epsilon = 1$ (i.e, $\mathfrak{g} = so(V)$) and there is a minimal degeneration $\mathbf{d} \geq \mathbf{f}$ with $\mathbf{f}$ very even, there are two nilpotent orbits with Jordan type $\mathbf{f}$. Thus, in this case, $\mathrm{Exc}(\mu)$ contains at least k irreducible divisors. On the other hand, for the Jacobson-Morozov parabolic subgroup P, $b_2(G/P) = k-1$ when $\mathfrak{g} = sp(V)$, or $\mathfrak{g} = so(V)$ with $\dim V$ odd. When $\mathfrak{g} = so(V)$ and $\dim V$ is even, we must be careful; if the flag type of P is of the form $(p_1, ..., p_{k-1}; 2; p_{k-1}, ..., p_1)$, $b_2(G/P) = k$. This happens when $\dim V = 4n+2$ and $\mathbf{d} = [2^{2n}, 1^2]$ or when $\dim V = 8m + 4n + 4$ and $\mathbf{d} = [4^{2m}, 3, 2^{2n}, 1]$. In the latter case, $\mathbf{d}$ has a minimal degeneration $\mathbf{d} \geq \mathbf{f}$ with $\mathbf{f} = [4^{2m}, 2^{2n+2}]$, which is very even. Note that $b_2(G/P)$ coincides with the relative Picard number ρ of the Jacobson-Morozov resolution. By these observations, we know that μ has at least ρ exceptional divisors except when $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$. Therefore, $\tilde{\mathcal{O}}_{\mathbf{d}}$ are $\mathbf{Q}$-factorial in these cases. By (1.3.2) they have terminal singularities. When $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$, $\mathcal{O}_{\mathbf{d}}$ is a Richardson orbit and the Springer map gives a small resolution of $\bar{\mathcal{O}}_{\mathbf{d}}$. Therefore, $\tilde{\mathcal{O}}_{\mathbf{d}}$ has non-$\mathbf{Q}$-factorial terminal singularities.

(1.5) *Induced orbits in classical Lie algebras*: Let $\mathbf{d} = [d_1^{s_1}, ..., d_k^{s_k}]$ be an ϵ-admissible partition of m. According as $\epsilon = 1$ or $\epsilon = -1$, we put $G = SO(m)$ or $G = Sp(m)$ respectively. Assume that $\mathbf{d}$ does not have full members. In other words, for some p, $d_p \geq d_{p+1}+2$ or $d_k \geq 2$. We put $r = \Sigma_{1\leq j\leq p} s_j$. Then $\mathcal{O}_{\mathbf{d}}$ is an induced orbit (cf. [C-M], 7.3). More explicitly, there are a parabolic subgroup Q of G with (isotropic) flag type $(r, m-2r, r)$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, and a nilpotent orbit $\mathcal{O}'$ of $\mathfrak{l}$ such that $\mathcal{O}_{\mathbf{d}} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$. Here, $\mathfrak{l}$ has a direct sum decomposition $\mathfrak{l} = gl(r) \oplus \mathfrak{g}'$, where $\mathfrak{g}'$ is a simple Lie algebra of type $B_{(m-2r-1)/2}$ (resp. $D_{(m-2r)/2}$, resp. $C_{(m-2r)/2}$) when $\epsilon = 1$ and m is odd (resp. $\epsilon = 1$ and m is even, resp. $\epsilon = -1$). Moreover, $\mathcal{O}'$ is a nilpotent orbit of $\mathfrak{g}'$ with Jordan type $[(d_1-2)^{s_1}, ..., (d_p-2)^{s_p}, d_{p+1}^{s_{p+1}}, ..., d_k^{s_k}]$. Let us consider the generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}_{\mathbf{d}}$$

(cf. (1.2)).

Lemma (1.5.1). *The map ν is birational. In other words,* $\deg(\nu) = 1$.

§2. Main Results

(2.1) Let X be a complex algebraic variety with rational Gorenstein singularities. A partial resolution $f : Y \to X$ of X is said to be a **Q**-*factorial terminalization* of X if Y has only **Q**-factorial terminal singularities and f is a birational projective morphism such that $K_Y = f^*K_X$. In particular, when Y is smooth, f is called a crepant resolution. In general, X has no crepant resolution; however, by [B-C-H-M], X always has a **Q**-factorial terminalization. But, in our case, the **Q**-factorial terminalization can be constructed very explicitly without using the general theory in [B-C-H-M].

Proposition (2.1.1). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Let $\tilde{\mathcal{O}}$ be the normalization of $\bar{\mathcal{O}}$. Then one of the following holds:*
(1) *$\tilde{\mathcal{O}}$ has **Q**-factorial terminal singularities.*

(2) *There are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ and a nilpotent orbit $\mathcal{O}'$ of $\mathfrak{l}$ such that (a): $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$ and (b): the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}')$ is a **Q**-factorial terminalization of $\tilde{\mathcal{O}}$ via the generalized Springer map.*

Proof. When $\mathfrak{g}$ is of type A, every $\tilde{\mathcal{O}}$ has a Springer resolution; hence (2) always holds. Let us consider the case $\mathfrak{g}$ is of B, C or D. Assume that (1) does not hold. Then, by (1.4.3), the Jordan type $\mathbf{d}$ of $\mathcal{O}$ does not have full members except when $\mathfrak{g} = so(4n+2)$, $n \geq 1$ and $\mathbf{d} = [2^{2n}, 1^2]$. In the exceptional case, $\mathcal{O}$ is a Richardson orbit and the Springer map gives a crepant resolution of $\tilde{\mathcal{O}}$; hence (2) holds. Now assume that $\mathbf{d}$ does not have full members. Then, by (1.5), $\mathcal{O}$ is an induced nilpotent orbit and there is a generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

This map is birational by (1.5.1). Let us consider the orbit $\mathcal{O}'$ instead of $\mathcal{O}$. If (1) holds for $\mathcal{O}'$, then ν induces a $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$. If (1) does not hold for $\mathcal{O}'$, then $\mathcal{O}'$ is an induced orbit. By (1.2.3), one can replace Q with a smaller parabolic subgroup Q' in such a way that $\mathcal{O}$ is induced from $(Q', \mathcal{O}_2)$ for some nilpotent orbit $\mathcal{O}_2 \subset \mathfrak{l}(Q')$. The generalized Springer map ν' for $(Q', \mathcal{O}_2)$ is factorized as

$$G \times^{Q'} (\mathfrak{n}' + \bar{\mathcal{O}}_2) \to G \times^Q (\mathfrak{n} + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

The second map is birational as explained above. The first map is locally obtained by a base change of the generalized Springer map

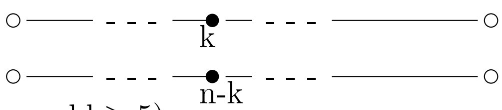
$$L(Q) \times^{L(Q) \cap Q'} (\mathfrak{n}(L(Q) \cap Q') + \bar{\mathcal{O}}_2) \to \bar{\mathcal{O}}'.$$

This map is birational by (1.5.1). Therefore, the first map is also birational, and ν' is birational. This induction step terminates and (2) finally holds.

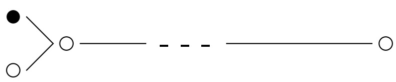
(2.2) We shall next show that *every* $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is of the form in Proposition (2.1.1) except when $\tilde{\mathcal{O}}$ itself has $\mathbf{Q}$-factorial terminal singularities. In order to do that, we need the following Proposition.

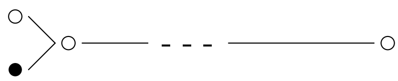
Proposition (2.2.1). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Assume that a $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is given by the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'))$ for some $(Q, \mathcal{O}')$ as in (2.1.1). Assume that Q is a maximal parabolic subgroup of G (i.e. $b_2(G/Q) = 1$), and this $\mathbf{Q}$-factorial terminalization is small. Then Q is a parabolic subgroup corresponding to one of the following marked Dynkin diagrams and $\mathcal{O}' = 0$:*

A_{n-1} $(k < n/2)$



D_n $(n : \text{odd} \geq 5)$





The following is the main theorem:

Theorem (2.2.2). *Let $\mathcal{O}$ be a nilpotent orbit of a classical simple Lie algebra $\mathfrak{g}$. Then $\tilde{\mathcal{O}}$ always has a $\mathbf{Q}$-factorial terminalization. If $\tilde{\mathcal{O}}$ itself does not have $\mathbf{Q}$-factorial terminal singularities, then every $\mathbf{Q}$-factorial terminalization is given by the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'))$ in (2.1.1). Moreover, any two such $\mathbf{Q}$-factorial terminalizations are connected by a sequence of Mukai flops of type A or D defined in [Na], pp. 91, 92.*

Proof. The first statement is nothing but (2.1.1). The proof of the second statement is quite similar to that of [Na], Theorem 6.1. Assume that $\tilde{\mathcal{O}}$ does not have $\mathbf{Q}$-factorial terminal singularities. Then, by (2.1.1), one can find a generalized Springer (birational) map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}.$$

Let X_Q be the normalization of $G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}')$. Then ν induces a $\mathbf{Q}$-factorial terminalization $f : X_Q \to \tilde{\mathcal{O}}$. The relative nef cone $\overline{\mathrm{Amp}}(f)$ is a rational, simplicial, polyhedral cone of dimension $b_2(G/Q)$ (cf. (1.2.2) and [Na], Lemma 6.3). Each codimension one face F of $\overline{\mathrm{Amp}}(f)$ corresponds to a birational contraction map $\phi_F : X_Q \to Y_Q$. The construction of ϕ_F is as follows. The parabolic subgroup Q corresponds to a marked Dynkin diagram D. In this diagram, there are exactly $b_2(G/Q)$ marked vertexes. Choose a marked vertex v from D. The choice of v determines a codimension one face F of $\overline{\mathrm{Amp}}(f)$. Let D_v be the maximal, connected, single marked Dynkin subdiagram of D which contains v. Let $\bar{D}$ be the marked Dynkin diagram obtained from D by erasing the marking of v. Let $\bar{Q}$ be the parabolic subgroup of G corresponding to $\bar{D}$. Then, as in (1.2.2), we have a map

$$\pi : G \times^Q (\mathfrak{n} + \bar{\mathcal{O}}') \to G \times^{\bar{Q}} (\bar{\mathfrak{n}} + \bar{\mathcal{O}}_1).$$

Let Y_Q be the normalization of $G \times^{\bar{Q}} (\bar{\mathfrak{n}} + \bar{\mathcal{O}}_1)$. Then π induces a birational map $X_Q \to Y_Q$. This is the map ϕ_F. Note that π is locally obtained by a base change of the generalized Springer map

$$L(\bar{Q}) \times^{L(\bar{Q}) \cap Q} (\mathfrak{n}(L(\bar{Q}) \cap Q) + \bar{\mathcal{O}}') \to \bar{\mathcal{O}}_1.$$

Let $Z(\mathfrak{l}(\mathfrak{q}))$ (resp. $Z(\mathfrak{l}(\bar{\mathfrak{q}}))$) be the center of $\mathfrak{l}(\mathfrak{q})$ (resp. $\mathfrak{l}(\bar{\mathfrak{q}})$). By the definition of $\bar{Q}$, the simple factors of $\mathfrak{l}(\bar{\mathfrak{q}})/Z(\mathfrak{l}(\bar{\mathfrak{q}}))$ are common to those of $\mathfrak{l}(\mathfrak{q})/Z(\mathfrak{l}(\mathfrak{q}))$

except one factor, say $\mathfrak{m}$. Put $\mathcal{O}'' := \mathcal{O}' \cap \mathfrak{m}$. By (2.2.1), π (or ϕ_F) is a small birational map if and only if $\mathcal{O}'' = 0$ and D_v is one of the single Dynkin diagrams listed in (2.2.1). In this case, one can make a new marked Dynkin diagramD' from D by replacing D_v by its dual D_v^* (cf. [Na], Definition 1). Let Q' be the parabolic subgroup of G corresponding to D'. We may assume that Q and Q' are both contained in $\bar{Q}$. The Levi part of Q' is conjugate to that of Q; hence there is a nilpotent orbit in $\mathfrak{l}(\mathfrak{q}')$ corresponding to $\mathcal{O}'$. We denote this orbit by the same $\mathcal{O}'$. Then $\mathcal{O}$ is induced from (Q', $\mathcal{O}'$). As above, let $X_{Q'}$ be the normalization of $G \times^Q (n(\mathfrak{q}') + \bar{\mathcal{O}}')$. Then we have a birational map $\phi'_F : X_{Q'} \to Y_Q$. The diagram

$$X_Q \to Y_Q \leftarrow X_{Q'}$$

is a flop. Assume that $g : X \to \tilde{\mathcal{O}}$ is a $\mathbf{Q}$-factorial terminalization. Then, the natural birational map $X - - \to X_Q$ is an isomorphism in codimension one. Let L be a g-ample line bundle on X and let $L_0 \in \mathrm{Pic}(X_Q)$ be its proper transform of L by this birational map. If L_0 is f-nef, then $X = X_Q$ and $f = g$. Assume that L_0 is not f-nef. Then one can find a codimension one face F of $\overline{\mathrm{Amp}}(f)$ which is negative with respect to L_0. Since L_0 is f-movable, the birational map $\phi_F : X_Q \to Y_Q$ is small. Then, as seen above, there is a new (small) birational map $\phi'_F : X_{Q'} \to Y_Q$. Let $f' : X_{Q'} \to \tilde{\mathcal{O}}$ be the composition of ϕ'_F with the map $Y_Q \to \tilde{\mathcal{O}}$. Then f' is a $\mathbf{Q}$-factorial terminanization of $\tilde{\mathcal{O}}$. Replace f by this f' and repeat the same procedure; but this procedure ends in finite times (cf. [Na], Proof of Theorem 6.1 on pp. 104, 105). More explicitly, there is a finite sequence of $\mathbf{Q}$-factorial terminalizations of $\tilde{\mathcal{O}}$:

$$X_0(:= X_Q) - - \to X_1(:= X_{Q'}) - - \to .... - - \to X_k(= X_{Q_k})$$

such that $L_k \in \mathrm{Pic}(X_k)$ is f_k-nef. This means that $X = X_{Q_k}$.

Example (2.3). We put $G = SP(12)$. Let $\mathcal{O}$ be the nilpotent orbit in $sp(12)$ with Jordan type $[6, 3^2]$. Let $Q_1 \subset G$ be a parabolic subgroup with flag type $(3, 6, 3)$. The Levi part $\mathfrak{l}_1$ of $\mathfrak{q}_1$ has a direct sum decomposition

$$\mathfrak{l}_1 = \mathfrak{g}l(3) \oplus sp(6).$$

Let $\mathcal{O}'$ be the nilpotent orbit in $sp(6)$ with Jordan type $[4, 1^2]$. Then $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}_1}^{sp(12)}(\mathcal{O}')$. Next consider the parabolic subgroup $Q_2 \subset SP(6)$ with flag type $(1, 4, 1)$. The Levi part $\mathfrak{l}_2$ of $\mathfrak{q}_2$ has a direct sum decomposition

$$\mathfrak{l}_2 = \mathfrak{g}l(1) \oplus sp(4).$$

Let $\mathcal{O}''$ be the nilpotent orbit in $sp(4)$ with Jordan type $[2,1^2]$. Then $\mathcal{O}' = \mathrm{Ind}_{\mathfrak{l}_2}^{sp(6)}(\mathcal{O}'')$. One can take a parabolic subgroup Q of $SP(12)$ with flag type $(3,1,4,1,3)$ in such a way that the Levi part $\mathfrak{l}$ of $\mathfrak{q}$ contains the nilpotent orbit $\mathcal{O}''$. Then $\mathcal{O}$ is the nilpotent orbit induced from $\mathcal{O}''$. We shall illustrate the induction step above by

$$([2,1^2], sp(4)) \to ([4,1^2], sp(6)) \to ([6,3^2], sp(12)).$$

Since $\tilde{O}''$ has only $\mathbf{Q}$-factorial terminal singularities, the $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$ is given by the generalized Springer map

$$\nu : G \times^Q (n(\mathfrak{q}) + \bar{\mathcal{O}}'') \to \bar{\mathcal{O}}.$$

The induction step is not unique; we have another induction step

$$([2,1^2], sp(4)) \to ([4,3^2], sp(10)) \to ([6,3^2], sp(12)).$$

By these inductions, we get another generalized Springer map

$$\nu' : G \times^{Q'} (n(\mathfrak{q}') + \bar{\mathcal{O}}'') \to \bar{\mathcal{O}},$$

where Q' is a parabolic subgroup of G with flag type $(1,3,4,3,1)$. This gives another $\mathbf{Q}$-factorial terminalization of $\tilde{\mathcal{O}}$. The two $\mathbf{Q}$-factorial terminalizations of $\tilde{\mathcal{O}}$ are connected by a Mukai flop of type A_3.

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ポスターセッション

Abelian G-Hilbert schemes via Gröbner bases

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The G-Hilbert scheme Hilb^G is defined by Ito-Nakamura. It is the irreducible component of G-fixed points of Hilbert scheme of $|G|$ points on $\mathbb{C}^n$ dominating $\mathbb{C}^n/G$ via Hilbert-Chow morphism.

A G-Hilbert scheme can be computed by using Gröbner bases.

- Fix a point p of $T = (\mathbb{C}^*)^n$.
- S : the coordinate ring of $\mathbb{C}^n$
- $I(G\cdot p) \subset S$: an ideal defining G-orbit $G\cdot p \subset \mathbb{C}^n$

Main Theorem(S—) Let G be a finite abelian subgroup of $GL(n,\mathbb{C})$. Then the following holds.

- Hilb^G is covered with affine open sets defined by reduced Gröbner bases of $I(G\cdot p)$.
- The normalization of Hilb^G is a toric variety determined by the Gröbner fan of $I(G\cdot p)$.

Notice Hilb^G is not necessary normal. Craw-Maclagan-Thomas show that Hilb^G is not normal for a subgroup $G \cong (\mathbb{Z}/5\mathbb{Z})^4$ of $GL(6,\mathbb{C})$.

For a finite small cyclic subgroup $G \subset GL(2,\mathbb{C})$, Hilb^G is the minimal resolution of $\mathbb{C}^2/G$. Hence

Corollary (Ito) For a finite small cyclic subgroup G of $GL(2,\mathbb{C})$, the toric variety determined by the Gröbner fan of an ideal $I(G\cdot p)$ is the minimal resolution of $\mathbb{C}^2/G$.

Gröbner bases can be computed by a computer, so we can examine properties of Hilb^G, for example singularity, normality and the number of torus-fixed points and so on.

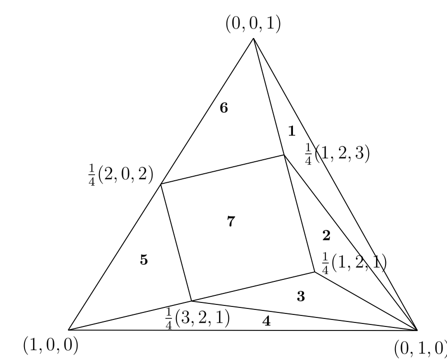
Example (Hilb^G of type $\frac{1}{4}(1,2,3)$)

- $G = \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} \middle| \varepsilon^4 = 1 \right\rangle$: a finite cyclic subgroup of $GL(3,\mathbb{C})$
- Take $p \in (\mathbb{C}^*)^3 = (1,1,1)$.

Then
$$I(G\cdot p) = \langle x^4-1, y-x^2, z-x^3\rangle.$$

The number of all reduced Gröbner bases of $I(G\cdot p)$ is seven. So Hilb^G is covered with seven affine open sets. For example the following is the reduced Gröbner basis of $I(G\cdot p)$ with respect to weight vector $(1,1,1)$;
$$\mathcal{G}_7 = \{x^2-y, y^2-1, z^2-y, xy-z, yz-x, xz-1\}.$$
The affine open set associated to $\mathcal{G}_7$ is
$$\mathrm{Spec}\,\mathbb{C}\left[\frac{x^2}{y}, y^2, \frac{z^2}{y}, \frac{xy}{z}, \frac{yz}{x}, xz\right] = \mathrm{Spec}\,\mathbb{C}\left[\frac{x^2}{y}, \frac{z^2}{y}, \frac{xy}{z}, \frac{yz}{x}\right]$$
$$\cong \mathrm{Spec}\,\mathbb{C}[X,Y,Z,W]/(XW-YZ).$$
Therefore Hilb^G is **singular**.

Moreover in this case Hilb^G is normal, so the toric variety determined by Gröbner fan of $I(G\cdot p)$ is Hilb^G. Let $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{4}(1,2,3)$. We consider Gröbner fan in $N\otimes\mathbb{R}$.



This is the cross section of Gröbner fan of $I(G\cdot p)$ and corresponding toric variety is Hilb^G.

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Non-symplectic automorphisms of prime order on $K3$ surfaces

Shingo TAKI (Graduate School of Mathematics Nagoya University)

We study non-symplectic automorphisms of prime order on algebraic $K3$ surface X which act trivially on the Néron-Severi lattice S_X i.e. $\varphi^*\omega_X = \zeta\omega_X$.
(ω_X: nowhere vanishing holomorphic 2-form, ζ: primitive p-th root of unity.).

Main Theorem

Let r be the Picard number of X. We assume that $S_X^*/S_X \simeq (\mathbb{Z}/p\mathbb{Z})^s$.

(1) If $22 - r - (p-1)s \notin 2(p-1)\mathbb{Z}_{\geq 0} \Rightarrow \nexists\ \varphi$.

(2) If $22 - r - (p-1)s \in 2(p-1)\mathbb{Z}_{\geq 0} \Rightarrow \exists\ \varphi$.

$$X^\varphi = \begin{cases} \phi & \text{if } S_X = U(2)\oplus E_8(2) \\ C^{(1)} \amalg C^{(1)} & \text{if } S_X = U \oplus E_8(2) \\ \{P_1\} \amalg \cdots \amalg \{P_M\} & \text{if } p \neq 2,\ b_1(X^\varphi) = 0 \text{ and } \chi(X^\varphi) = M \\ \{P_1\} \amalg \cdots \amalg \{P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_{N-1} & \text{otherwise} \end{cases}$$

- P_j is an isolated point.
- $C^{(g)}$ is a non-singular curve with genus g.
- E_k is a non-rational curve.

$$\chi(X^\varphi) = \frac{pr - (24-2p)}{p-1}, \qquad b_1(X^\varphi) = \frac{22 - r - (p-1)s}{p-1},$$

$$g = \frac{22 - r - (p-1)s}{2(p-1)},$$

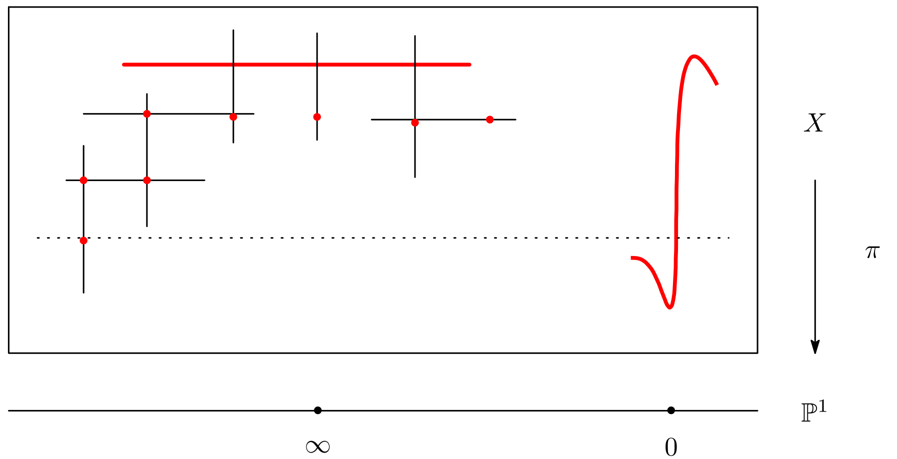
$$M = \begin{cases} 0 & p = 2 \\ \dfrac{(p-2)r + 22}{p-1} & p = 17, 19 \\ \dfrac{(p-2)r - 2}{p-1} & \text{otherwise} \end{cases}, \qquad N = \begin{cases} \dfrac{r - s + 2}{2} & p = 2 \\ \dfrac{2p + r - (p-1)s - 24}{2(p-1)} & p = 17, 19 \\ \dfrac{2p + r - (p-1)s}{2(p-1)} & \text{otherwise} \end{cases}.$$

Example

We give affine equations of elliptic $K3$ surfaces $X : y^2 = x^3 + x + t^7$. Then $S_X = U \oplus E_8$.

The elliptic fibration π has a singular fiber of type II^* over $t = \infty$ and 14 singular fibers of type I_1 over $t^{14} = -4/27$.

We put $\varphi(x, y, t) = (x, y, \zeta t)$. Then φ is a non-symplectic automorphisms of order 7.



Now $p = 7$, $r = 10$ and $s = 0 \rightsquigarrow X^\varphi = C^{(1)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 8$.

Remark

We assume $22 - r - (p-1)s \in 2(p-1)\mathbb{Z}_{\geq 0}$. For each (r, s), there exist a $K3$ surface which has a non-symplectic automorphism of order p acting trivially on S_X.

瀧　真語 (m04022x@math.nagoya-u.ac.jp)

On the arithmetical rank of squarefree monomial ideals concerned with the complete bipartite graph $K_{2,n}$

Kyouko Kimura (Nagoya University)

Joint work with Naoki Terai (Saga University) and Ken-ichi Yoshida (Nagoya University)

Introduction

S: a polynomial ring over a field K.
$I, J \subset S$: squarefree monomial ideals.

Let $X = V(I)$ be an affine algebraic set. Then how many hypersurfaces need to express X as intersection of those?

$$X = X_0 \cap X_1 \cap \cdots \cap X_{s-1}, \quad X_i = V(f_i).$$

The **arithmetical rank** of I:

$$\operatorname{ara} I := \min\{s \ : \exists f_0, f_1, \ldots, f_{s-1} \in I \text{ s. t. } \sqrt{(f_0, f_1, \ldots, f_{s-1})} = \sqrt{I}\}.$$

Fact (Lyubeznik).

$(\star)$ $\qquad$ $\operatorname{height} I \le \operatorname{pd}_S S/I \le \operatorname{ara} I \le \mu(I)$.

Problem 0.1. *When does* $\operatorname{ara} I = \operatorname{pd}_S S/I$ *hold?*

Known results:
(1) $\mu(I) - \operatorname{height} I = 0$ (clear).
(2) (KTY) $\mu(I) - \operatorname{height} I = 1$.
(3) (KTY) $\mu(I) - \operatorname{height} I = 2$.
(4) $\mu(I) - \operatorname{pd}_S S/I = 0$ (clear).
(5) (KTY) $\mu(I) - \operatorname{pd}_S S/I = 1$.
(1)* (Schenzel–Vogel, Schmitt–Vogel) $\operatorname{arithdeg} I - \operatorname{indeg} I = 0$.
(2)* (KTY) $\operatorname{arithdeg} I - \operatorname{indeg} I = 1$.
(4)* (KTY) $\operatorname{arithdeg} I - \operatorname{reg} I = 0$.

Counterexamples (char $K \ne 2$):
(6) (Yan) Stanley–Reisner ideal associated to Reisner's triangulation of the projective plane.
(7) (KTY) An ideal with $\mu(I) - \operatorname{height} I = 3$.
(7)* (KTY) An ideal with $\operatorname{arithdeg} I - \operatorname{indeg} I = 3$.

($(\cdot)^*$ is Alexander dual of $(\cdot)$.)
In this poster, we focus on the case corresponding to $(3)^*$ and $(5)^*$.

1 Alexander duality

Example 1.1 (Alexander dual ideal).

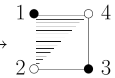
$(\clubsuit)$ $\quad I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_1, x_4) \cap (x_2, x_5) \cap (x_3, x_6)$
$\Longrightarrow I^* = (x_1x_2x_3, x_4x_5x_6, x_1x_4, x_2x_5, x_3x_6)$.

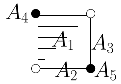
Properties:

- $I^{**} = I$.
- (Frühbis-Krüger–Terai)
  $\operatorname{indeg} I^* = \operatorname{height} I, \quad \operatorname{reg} I^* = \operatorname{pd}_S S/I, \quad \operatorname{arithdeg} I^* = \mu(I)$.
- (Hoa–Trung, Frühbis-Krüger–Terai)
  $(\star)^*$ $\qquad \operatorname{indeg} I \le \operatorname{reg} I \le \operatorname{arithdeg} I$.

2 Hypergraphs

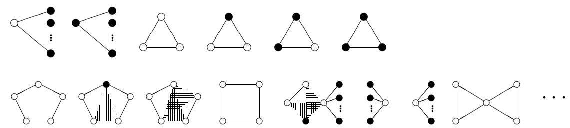
Example 2.1 (Hypergraph).

$J = (x_1x_4, x_1x_2, x_2x_3x_5, x_1x_3)$ $\longrightarrow$ 



$\longrightarrow J = (A_1A_4, A_1A_2, A_2A_3A_5, A_1A_3)$

Using hypergraph, we can classify squarefree monomial ideals with $\mu(I) - \operatorname{height} I \le 2$ (KTY).



3 The case of $(3)^*$ $\operatorname{arithdeg} I - \operatorname{indeg} I = 2$

There are 3 cases by $(\star)^*$:
(a) $\operatorname{arithdeg} I = \operatorname{reg} I = \operatorname{indeg} I + 2$.
(b) $\operatorname{arithdeg} I = \operatorname{reg} I + 1 = \operatorname{indeg} I + 2$.
(c) $\operatorname{arithdeg} I = \operatorname{reg} I + 2 = \operatorname{indeg} I + 2$.

The case (a): contained in $(4)^*$.

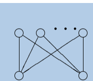
Theorem 3.1 (KTY). *For the ideal I of **the case (c)**,*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

Proof. We determine the arithmetical rank according to the classification by hypergraphs. □

The case (b) is an open problem. This case is a part of the case $(5)^*$:
$(5)^*$ $\operatorname{arithdeg} I - \operatorname{reg} I = 1$.
$\Longleftrightarrow$ $\mathcal{H}(I^*)$ **"contains" a complete bipartite graph** (Terai).

Problem 3.2.

$\mathcal{H}(J) = K_{2,n} =$ 

Determine the arithmetical rank of $I = J^*$.

Case1: assign 1 variable to each edges.

Theorem 3.3 (KTY). *For any n, the ideal I which is obtained by assigning 1 variable to each edge of hypergraphs in Problem 3.2 satisfies*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

We shall see the case $n = 3$.

$(\clubsuit)$ $\quad I = (x_1, x_2, x_3) \cap (y_1, y_2, y_3) \cap (x_1, y_1) \cap (x_2, y_2) \cap (x_3, y_3)$.

In this case, $\operatorname{pd}_S S/I = 3 = \operatorname{ara} I$:

$$\begin{cases} f_0 = x_1x_2x_3 \cdot y_1y_2y_3, \\ f_1 = x_1x_2 \cdot y_3 + x_1x_3 \cdot y_2 + x_2x_3 \cdot y_1, \\ f_2 = x_1 \cdot y_2y_3 + x_2 \cdot y_1y_3 + x_3 \cdot y_1y_2. \end{cases}$$

Case2: general cases.

Theorem 3.4 (KTY). *For $n = 2, 3, 4$, the ideal I in Problem 3.2 satisfies*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

The idea of the proof:

Example 3.5 (Known result $(4)^*$).

$$I = (y_1, x_1, x_2) \cap (y_2, x_1, x_3) \cap (y_3, x_2).$$

Then $\operatorname{ara} I = \operatorname{pd}_S S/I = 4$:

$$\begin{cases} h_0 = y_1y_2y_3, \\ h_1 = \mathbf{x_1} \cdot y_3 + \mathbf{x_2} \cdot y_2 + \mathbf{x_3} \cdot y_1y_3, \\ h_2 = \mathbf{x_1x_2} + \mathbf{x_1x_3} \cdot y_3 + \mathbf{x_2x_3}, \\ h_3 = \mathbf{x_1x_2x_3}. \end{cases}$$

Note that h_1, h_2, h_3 are **modifications of elementary symmetric functions** of x_1, x_2, x_3.

The case $n = 3$ of Theorem 3.4:

$(\clubsuit)'$ $\qquad I = P_1 \cap P_2 \cap Q_1 \cap Q_2 \cap Q_3,$

where

$$P_1 = (\underline{X_1}, \underline{X_2}, \underline{X_3}), \quad P_2 = (\underline{Y_1}, \underline{Y_2}, \underline{Y_3}), \quad Q_i = (\underline{X_i}, \underline{Y_i}),$$
$$\underline{X_i} = x_i, \underline{X_i'} = x_i, x_{i2}, \ldots, x_{i\ell_i}$$
$$\underline{Y_i} = y_i, \underline{Y_i'} = y_i, y_{i2}, \ldots, y_{im_i}, \qquad (i = 1, 2, 3).$$

Then $\operatorname{pd}_S S/I = \sum_{i=1}^{3}(\ell_i + m_i) - 3$.

$$\begin{cases} k\text{th elementary symmetric functions} \\ g_k' = S_k(\underline{X_1'} \cup \underline{X_2'} \cup \underline{X_3'} \cup \underline{Y_1'} \cup \underline{Y_2'} \cup \underline{Y_3'}) \\ \qquad (k = 1, 2, \ldots, \operatorname{pd}_S S/I - 3), \\ f_0, f_1, f_2, \qquad \operatorname{pd}_S S/I \textbf{ elements!} \text{ and modify them.} \end{cases}$$

Twisted Fourier-Mukai number of a $K3$ surface

Shouhei Ma*

In my poster, I exhibited a counting formula for the twisted Fourier-Mukai (FM) partners of a projective $K3$ surface. Let S be a projective $K3$ surface over $\mathbb{C}$. A twisted $K3$ surface (S', α') is called a *twisted FM partner* of S if there is an exact equivalence $D^b(S) \simeq D^b(S', \alpha')$ between their derived categories. Let $\mathrm{FM}^d(S)$ be the set of isomorphism classes of twisted FM partners (S', α') of S with $\mathrm{ord}(\alpha') = d$. I calculated the number $\#\mathrm{FM}^d(S)$ from severel lattice-theoretic informations about the lattice $H^2(S, \mathbb{Z})$ equipped with a natural Hodge structure. The number $\sum_d \#\mathrm{FM}^d(S)$ has the following meanings.

- The number of certain geometric origins of the category $D^b(S)$.
- The number of isomorphism classes of 2-dimensional compact moduli spaces of stable sheaves on S, considered with natural obstruction classes.
- The number of the 0-dimensional cusps of the Kahler moduli of S.

Now the formula is stated as follows.

Theorem 0.1. *Let $\varepsilon(d) = 1$ or 2 according to $d \leq 2$ or ≥ 3. For a projective $K3$ surface S the following formula holds.*

$$\begin{aligned}\#\mathrm{FM}^d(S) \quad = \quad \sum_x \Bigg\{ & \sum_M \#\Big(O_{Hodge}(T_x, \alpha_x) \backslash O(D_M) / O(M) \Big) \\ & + \varepsilon(d) \sum_{M'} \#\Big(O_{Hodge}(T_x, \alpha_x) \backslash O(D_{M'}) / O(M') \Big) \Bigg\}.\end{aligned}$$

Here x runs over the set $O_{Hodge}(T(S)) \backslash I^d(D_{NS(S)})$ and the lattices M, M' run over the sets $\mathcal{G}_1(M_x)$, $\mathcal{G}_2(M_x)$ respectively.

This formula is simplified if S satisfies either of the following conditions : (1) The Neron-Severi lattice $NS(S)$ contains the hyperbolic plane U. (2) $NS(S)$ is 2-elementary. (3) The rank of $NS(S)$ equals to 1.

As an application of the formula, I gave a set of explicit Mukai vectors for a projective $K3$ surface of Picard number 1 such that the set of the corresponding moduli spaces of stable sheaves, considered with natural obstruction classes, coincides with the set $\mathrm{FM}^d(S)$.

Finally, I would like to express my gratitude to the organizers for their efforts for the wonderful symposium.

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Enriques surfaces covered by Jacobian Kummer surfaces

October, 2008.
at Kinosaki.
H. Ohashi, RIMS, Kyoto Univ.

1 Introduction

Jacobian Kummer surface X

C: a genus 2 curve

$$\Downarrow$$

$J(C) = \mathrm{Pic}^0(C)$: Jacobian of C

$$\Downarrow$$

$\overline{X} := \overline{Km}(J(C)) := J(C)/\{\pm 1\}$

(Kummer's quartic surface)

$$\Downarrow$$

$X := Km(J(C))$: the minimal desing. of $\overline{X}$.

$$\begin{tikzcd} J(C) \arrow[d, "/\pm 1"] & \\ \overline{X} & X = Km(J(C)) \arrow[l, "\text{min.desing.}"'] \end{tikzcd}$$

Definition
X is Picard-general if $\rho(X) = 17$, which we assume in what follows.

Aut(X) has been studied by many authors.
One definitive result is the following
Theorem(S. Kondo, 1998)
Aut(X) is generated by

$$\begin{cases} 16 \times 4 & \text{Klein's involutions } (t_\alpha, \sigma_\beta, p_\alpha, p_\beta), \\ 60 & \text{Hutchinson's involutions } (\sigma_G), \\ 192 & \text{Keum's automorphisms } (\phi_{W,W'}). \end{cases}$$

Where

$$\alpha \in \{\text{2-torsion pts of } J(C)\},$$
$$\beta \in \{\text{theta characteristics of } C\}.$$

Corollary of the Main Theorem

Aut(X) is generated by

$$\begin{cases} 16 \times 4 & \text{Klein's involutions } (t_\alpha, \sigma_\beta, p_\alpha, p_\beta), \\ 60 & \text{Hutchinson's(HG) involutions } (\sigma_G), \\ 192 & \underline{\textit{Hutchinson-Weber(HW) involutions}}\ (\sigma_W). \end{cases}$$

Where did σ_W come from ?

2 Main Result

Main Theorem There are $31 = 6 + 10 + 15$ fixed-point-free involutions on X, up to the isomorphism of the quotient Enriques surfaces.
They are exactly as follows.

3 free involutions on X

Switches

$$\Theta_\beta = \{p - \beta | p \in C\}.$$

$$\text{For } p \in J(C),\ (\Theta_\beta + p) \cap (\Theta_\beta - p) = \{q, -q\}.$$

$$\sigma_\beta :\ \pm p \mapsto \pm q.$$

$$\sigma_\beta \in \mathrm{Bir}(\overline{X}) = \mathrm{Aut}(X).$$

β runs over even theta characteristics of C; we obtain 10 free switches.

HG involutions

Restriction of the Cremona involution to $\overline{X}$:

$$\sigma_G : (x, y, z, t) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}).$$

G : four points of $\overline{X}$, called Göpel subgroup.

σ_G is well-defined, because
Theorem[Hutchinson] If we choose the four points of G as the reference points of $\mathbb{P}^3$, the equation of $\overline{X}$ becomes

$$A(x^2t^2 + y^2z^2) + B(y^2t^2 + z^2x^2) + C(z^2t^2 + x^2y^2) + Dxyzt$$
$$E(yt + zx)(zt + xy) + G(zt + xy)(xt + yz) + H(xt + yz)(yt + zx)$$
$$= 0.$$

There are 15 Göpel subgroups.

HW involutions

Restriction of the Cremona involution $\sigma_W : (s_i) \mapsto (s_i^{-1})$ of $\mathbb{P}^4$ to X_W, where W: a Weber hexad (definition omitted), X_W: another quartic model of X.

$$\overline{X} \overset{|\mathcal{O}_{\mathbb{P}^3}(2) - W|}{\dashrightarrow} X_W \subset \mathbb{P}^4.$$

Theorem[Hutchinson] The equation of X_W is

$$\sum_{i=1}^{5} s_i = \sum_{i=1}^{5} \lambda_i / s_i = 0, \quad \lambda_i \in \mathbb{C}^*.$$

We obtain 6 HW involutions.

4 Sketch of the Proof

We compute certain invariant, *the patching subgroups* of free involutions. For our X, it exactly classifies the isom. classes of quotient Enriques surfaces. The definition of it uses Nikulin's lattice theory.

On The Structure of certain $K3$ surfaces

Kazuki Utsumi (Hiroshima University)

$$(\mathrm{I}_4^{*2}) := \{X \to \mathbf{P}_t^1 \mid \text{an elliptic } K3 \text{ surface } X \text{ with a section and two singular fibers both of type } \mathrm{I}_4^* \text{ at } t = 0, \infty \in \mathbf{P}_t^1 \text{ over } \mathbf{C}\}$$

E_1, E_2 : elliptic curves

$S = \mathrm{Km}(E_1 \times E_2)$ has an elliptic fibration having a section and 3 singular fibers of type I_4^*, $\mathrm{I}_{\mu_1}^*$, $\mathrm{I}_{\mu_2}^*$ at $\infty, 2, -2 \in \mathbf{P}^1$ $(0 \le \mu_1, \mu_2 \le 1)$.

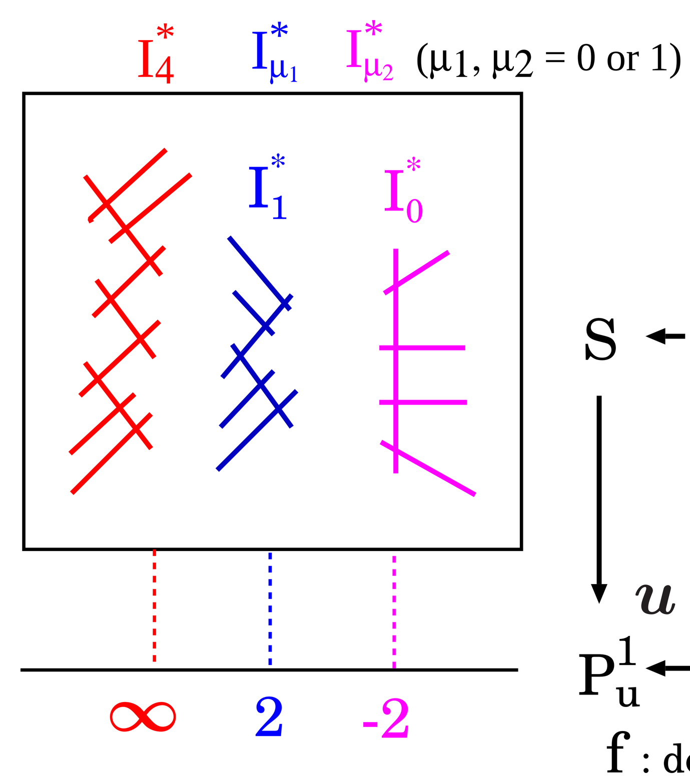
X: the double covering surface of S induced by

$$f : \mathbf{P}_t^1 \to \mathbf{P}_u^1 \; ; \; t \mapsto u = t + \frac{1}{t}$$

$$\Downarrow$$

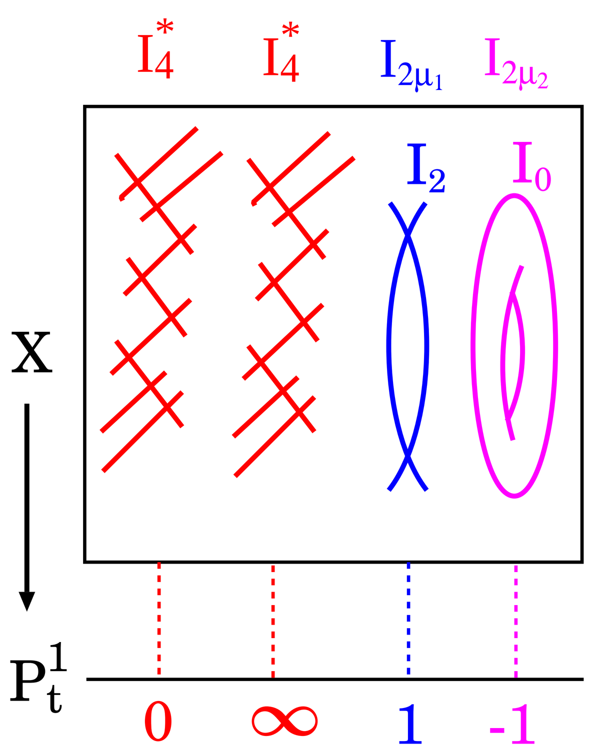
$$X \in (\mathrm{I}_4^{*2})$$

(♣) Conversely is it possible to recover such a Kummer surface S from given $X \in (\mathrm{I}_4^{*2})$?



$$\begin{tikzcd} S \arrow[d] & X \arrow[l, dashed, "\pi"'] \arrow[d] \\ \mathbf{P}_u^1 & \mathbf{P}_t^1 \arrow[l, "u = t + \frac{1}{t}"'] \end{tikzcd}$$

f : doubel covering ramified at u = 2, -2



$$\Downarrow$$

$$X \in (\mathrm{I}_4^{*2})$$

Every $X \in (\mathrm{I}_4^{*2})$ has the following Weierstrass equation

$$y^2 = x^3 - 3t^2(t^4 + a_3 t^3 + a_2 t^2 + a_1 t + 1)x + 2t^3(t^6 + b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + 1),$$

where

$$b_1 = \frac{3}{2}a_1$$
$$b_2 = \frac{3}{8}{a_1}^2 + \frac{3}{2}a_2$$
$$b_3 = -\frac{1}{16}{a_1}^3 + \frac{3}{4}a_1 a_2 + \frac{3}{2}a_3 = -\frac{1}{16}{a_3}^3 + \frac{3}{4}a_2 a_3 + \frac{3}{2}a_1$$
$$b_4 = \frac{3}{8}{a_3}^2 + \frac{3}{2}a_2$$
$$b_5 = \frac{3}{2}a_3$$
$${a_1}^4 - 8{a_1}^2 a_2 + 32 a_1 a_3 + 16 a_2^2 - 64 a_2 - 16{a_3}^2 + 64 \ne 0$$
$${a_3}^4 - 8 a_2 {a_3}^2 + 32 a_1 a_3 + 16 a_2^2 - 64 a_2 - 16{a_1}^2 + 64 \ne 0.$$

$$(\mathrm{I}_4^{*2})_2 := \{X \in (\mathrm{I}_4^{*2}) \mid a_1 = a_3\}$$

$$(\mathrm{I}_4^{*2})_* := \{X \in (\mathrm{I}_4^{*2}) \mid a_1 \ne a_3\}$$

The answer of the question (♣)

$X \in (\mathrm{I}_4^{*2})_2 \Rightarrow$ YES.

$X \in (\mathrm{I}_4^{*2})_*$ and $\rho(X) = 18 \Rightarrow$ NO.

Definition. (Shioda–Inose structure)
A $K3$ surface X admits a Shioda–Inose structure if there exists a symplectic involution ι on X with the rational quotient map $\pi : X \dashrightarrow Y$ of ι such that Y is a Kummer surface and $\pi_* : H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ induces a Hodge isometry $T_X(2) \simeq T_Y$.

If X admits a Shioda–Inose structure, the complex torus such that $Y = \mathrm{Km}(Z)$ gives a diagram

$$\begin{tikzcd} X \arrow[dr, dotted] & & Z \arrow[dl, dotted] \\ & Y & \end{tikzcd}$$

of rational maps of degree 2. This diagram induces a Hodge isometry

$$T_X \simeq T_Z.$$

Theorem 1.

If $X \in (\mathrm{I}_4^{*2})_2$, then X admits a Shioda–Inose structure.

Theorem 2.

If $X \in (\mathrm{I}_4^{*2})_*$ and Picard number of X is 18, then X does not admit a Shioda–Inose structure. However X is isomorphic to a Kummer surface of product type.

On finite group actions on an irreducible symplectic 4-fold

Kotaro Kawatani
Osaka universty

1 Introduction

In this section, we will talk about background of our study. At first, we define an irreducible symplectic manifold.

Definition 1.1. *Let X be a compact Kähler manifold. When following two conditions are satisfied, we call X* **irreducible symplcetic manifold***.*

1. X is simply connected.
2. $H^0(X, \Omega^2) = \mathbb{C}\langle\sigma_X\rangle$. where σ_X is an everywhere non-degenerate holomorphic 2-form.

In particular σ_X is said to be the symplectic form.

Remark 1.2. From existence of symplectic form, $\dim X$ is even, and a canonical bundle K_X is trivial. i.e.

$$\dim X = 2n, \ K_X \cong \mathcal{O}_X$$

We will introduce some famous examples. The easiest example is a K3 surface. Kodaira proved that a deformation equivalent class of K3 surface is unique. In higher dimensional case, there are only 4 types of deformation equivalent class which have been already known. Representative elements of each class are below.

Example

(i) n-pointed Hilbert scheme of K3 surface, $\mathrm{Hilb}^n(K3)$ ([Bea])

(ii) Generalized Kummer variety defined by Abelian surface A. We denote it by $\mathrm{Kum}^n(A)$ ([Bea]). Definition of $\mathrm{Kum}^n(A)$ is below.

$$\pi : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\mu} \mathrm{Sym}^{n+1}(A) \xrightarrow{\Sigma} A$$

Where μ is Hilbert-Chow morphism. We define $\mathrm{Kum}^n(A) := \pi^{-1}(0)$.

(iii),(iv) O'Grady's six and ten dimensional example ([Ogr2],[Ogr])

We don't know whether above classes are all or not. By the way, Beaville and Donagi found another explicit example which is different from (i)∼(iv). Let Y be a smooth cubic 4-fold, and let $F(Y)$ be all lines contained in Y. Then $F(Y)$ is an irreducible symplectic 4-fold ([B-D]). However, $F(Y)$ is deformation equivalent to a 2-pointed Hilbert scheme of a certain K3 surface $\mathrm{Hilb}^2(K3)$.

We investigated finite group actions on $F(Y)$ to make a new deformation equivalent class. We could not find it, but we met very interesting phenomena. We will introduce a part of them.

2 Preparation

In this section, we prepare some tools of our study.

Definition 2.1. *Let Y be a smooth cubic 4-fold. Let $F(Y)$ be all lines contained in Y. i.e.*

$$F(Y) := \{l \subset Y | l \cong \mathbb{P}^1, \deg l = 1\}$$

Remark 2.2. $F(Y)$ is a compact complex manifold whose dimension is 4.

Proposition 2.3 (Beauville-Donagi, [B-D]). *$F(Y)$is an irreducible symplectic manifold. In particular, $F(Y)$ is deformation equivalent to 2-pointed Hilbert scheme of a certain K3 surface $\mathrm{Hilb}^2 K3$.*

Let G be a finite group;

$$G \subset PGL(5), \ G \curvearrowright Y.$$

Since we want to make an irreducible symplectic manifold, first question is below.

Qustion 1. *When does $G \curvearrowright F(Y)$ preserve the symplectic form ?*

Let Γ be a universal family of $F(Y)$.

$$\Gamma := \{(l.y) \in F(Y) \times Y | l \ni y\}$$

There are two natural projections $p : \Gamma \to F(Y)$ and $q : \Gamma \to Y$. We define Abel-Jacobi map $\alpha : H^4(Y, \mathbb{C}) \to H^2(F(Y), \mathbb{C})$ as $\alpha(\omega) := p_* q^*(\omega)$. Abel-Jacobi map tells us whether G preserves the symplectic form or not.

$$\begin{tikzcd}
H^4(Y,\mathbb{C}) \arrow[r, "\cong", "\alpha"'] & H^2(F(Y) \\
H^{3,1}(Y) \arrow[r] \arrow[u, hook] & H^{2,0}(F(Y)) \arrow[u, hook] \\
\mathbb{C}\langle Res \frac{\Omega}{f^2}\rangle \arrow[r] \arrow[u, equal] & \mathbb{C}\langle\sigma_{F(Y)}\rangle \arrow[u, equal]
\end{tikzcd}$$

Where Ω is five form on $\mathbb{C}^6$ defined as $\Omega := \sum_{i=0}^{5}(-1)^i z_i dz_0 \wedge \cdots \check{dz_i} \cdots \wedge dz_5$. Since Abel-Jacobi map α is G-equivariant, we get a following lemma.

Answer of Question 1.

Lemma 2.4. *Notations as above.*

$$G \text{ preserves } \sigma_{F(Y)} \iff G \text{ preserves } Res\frac{\Omega}{f^2}$$

In general, $F(Y)/G$ may have singular points. So, we have to take resolution of $F(Y)/G$. We require that a resolution of $F(Y)/G$ has a symplectic form. So, second question is

Qustion 2. *When does $F(Y)/G$ have a crepant resolution $\widetilde{F(Y)/G}$?*

It is easy to find group actions $G \curvearrowright F(Y)$ which preserve the symplectic form, but it's difficult to find group actions such that $\widetilde{F(Y)/G}$ exists.

We have two examples of " good " actions. In this poster, our topic is one of them.

3 First example

First example was found by Namikawa.

Assumption

We consider special cubic 4-fold Y;

$$Y := \{f(z_0, z_1, z_2) + g(z_3, z_4, z_5) = 0\},$$

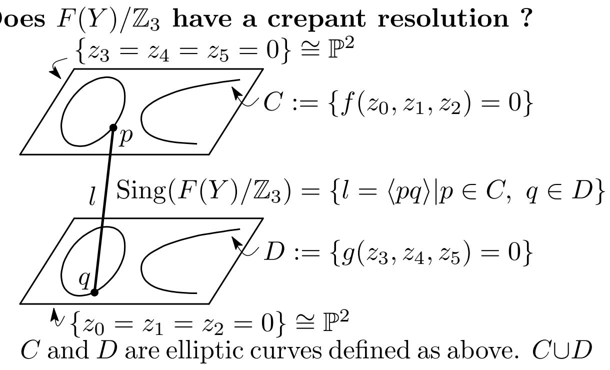
where f and g are homogeneous polynomial with degree 3.
Assume that $G = \mathbb{Z}_3$(order three cyclic group) and τ is a generator of G: $G = \langle\tau\rangle \cong \mathbb{Z}_3$. We consider following group action;

$$\tau \curvearrowright \mathbb{P}^5 \text{ as } (z_0 : z_1 : z_2 : \zeta z_3 : \zeta z_4 : \zeta z_5),$$

where $(z_0 : \cdots : z_5)$ is homogeneous coordinate of $\mathbb{P}^5$, and $\zeta = \exp(\frac{2\pi\sqrt{-1}}{3})$. In particular, G acts on Y.

From Lemma 2.4, we know that the induced action on $F(Y)$ preserves the symplectic form. Next we consider singular points of $F(Y)/\mathbb{Z}_3$.

Does $F(Y)/\mathbb{Z}_3$ have a crepant resolution ?



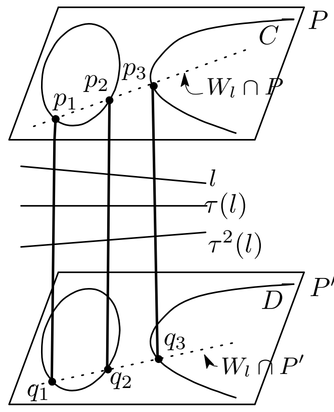
C and D are elliptic curves defined as above. $C \cup D$ is fixed locus of $\mathbb{Z}_3 \curvearrowright Y$. Singular locus of $F(Y)/\mathbb{Z}_3$ is isomorphic to $C \times D$. Since $\mathbb{Z}_3$ preserves the symplectic form, $F(Y)/\mathbb{Z}_3$ has A_2 singularities along $C \times D$. So, $\widetilde{F(Y)/\mathbb{Z}_3}$ does exist. What is $\widetilde{F(Y)/\mathbb{Z}_3}$?

Answer

Proposition 3.1 ([Nam]). *Notations as above. $\widetilde{F(Y)/\mathbb{Z}_3}$ is birational to $\mathrm{Kum}^2(C \times D)$*

Remark 3.2. If two irreducible symplectic manifold X and X' are birational, then X and X' are deformation equivalent. So, $\widetilde{F(Y)/\mathbb{Z}_3}$ is not new example.

Proof. We construct birational map $\psi : \widetilde{F(Y)/\mathbb{Z}_3} \dashrightarrow \mathrm{Kum}^2(C \times D)$. Instant picture of ψ is below.



$$\psi : \{l, \tau(l), \tau^2(l)\} \mapsto \{(p_i, q_i)\}_{i=1}^3$$

Let $\{l, \tau(l), \tau^2(l)\}$ be in $\widetilde{F(Y)/\mathbb{Z}_3}$. Let W_l be a liner space spanned by $l, \tau(l)$ and $\tau^2(l)$.

$$W_l := \langle l, \tau(l), \tau^2(l)\rangle \cong \mathbb{P}^3.$$

Suppose that $P = \{z_3 = z_4 = z_5 = 0\}, P' = \{z_0 = z_1 = z_2 = 0\}$. If we choose l in general, we may assume that $S := W_l \cap Y$ is a smooth cubic surface. There are 27 lines in S(classical results). From the configuration of 27 lines, we know that there exist three lines m_1, m_2, m_3 such that each m_i meets $l, \tau(l), \tau^2(l)$ like above picture. Each $m_i (i = 1, 2, 3)$ meets C (resp D) at one point. So we set notations as $p_i = m_i \cap C, q_i := m_i \cap D$. Since three points $\{p_1, p_2, p_3\}$(resp. $\{q_1, q_2, q_3\}$) are coliner, $p_1 + p_2 + p_3 = 0 \in C$(resp. $q_1 + q_2 + q_3 = 0 \in D$). So we have a pair of three points $\{(p_i, q_i)\}_{i=1}^3$. □

Where is the indeterminacy of ψ ?

We determine the indeterminacy of ψ and ψ^{-1}. Indeterminacy of ψ is

$$\{[l] := \{l, \tau(l), \tau^2(l)\} \in \widetilde{F(Y)/\mathbb{Z}_3} \,|[l] \text{ spans } \mathbb{P}^2\}.$$

This locus is 18 copies of $\mathbb{P}^2$. Indeterminacy of ψ^{-1} are two types. First one is

$$P_{(I)} := \{\{(p, q_1), (p, q_2), (p, q_3)\} \in \mathrm{Kum}(C \times D) | 3p = 0\}$$

Second one is

$$P_{(II)} := \{\{(p_1, q), (p_2, q), (p_3, q)\} \in \mathrm{Kum}(C \times D) | 3q = 0\}$$

$P_{(I)}$ and $P_{(II)}$ are isomorphic to 9 copies of $\mathbb{P}^2$.

Let X and X' be an irreducible symplectic **4-fold**. It is known that any birational map from X to X' is decomposed into Mukai-flop. We have a following theorem.

Theorem 3.3. *The indeterminacy of ψ can be resolved by Mukai-flop on 18 copies of $\mathbb{P}^2$.*

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Flips and variation of moduli scheme of sheaves on a surface

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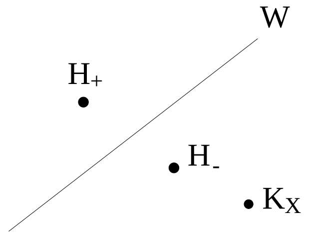
Let H be an ample line bundle on a non-singular projective surface X over $\mathbb{C}$. Denote by $M(H)$ the coarse moduli scheme of rank-two H-stable sheaves on X with Chern classes (r, c_1, c_2). We shall consider birational aspects of the problem how $\bar{M}(H)$ changes as H varies. See arXiv:0811.3522 for details.

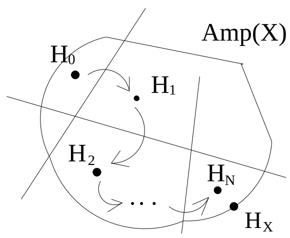
There is a union of hyperplanes $W \subset \mathrm{Amp}(X)$ called (c_1, c_2)-walls in the ample cone $\mathrm{Amp}(X)$ such that $M(H)$ changes only when H passes through walls. Let H and H_+ be ample line bundles separated by just one wall W, and $H_0 = tH \ + (1-t)H_+$ lie in W. (More exactly, we also consider parabolic stability.) For simplicity we assume that $M_\pm$ are compact, that is valid if $c_1 = 0$ and c_2 is odd for example. Denote $M_\pm = M(H_\pm)$ and $M_0 = M(H_0)$. There are natural morphisms $f \ : M \ \to M_0$ and $f_+ : M_+ \to M_0$. Let $f : X \to Y$ be a birational proper morphism such that K_X is $\mathbb{Q}$-Cartier and $-K_X$ is f-ample, and that the codimension of the exceptional set $\mathrm{Ex}(f)$ of f is more than 1. We say a birational proper morphism $f_+ : X_+ \to Y$ is a *flip* of f if (1) K_{X_+} is $\mathbb{Q}$-Cartier, (2) K_{X_+} is f_+-ample and (3) the codimension of the exceptional set $\mathrm{Ex}(f_+)$ is more than 1.

Theorem 0.1. *Assume c_2 is sufficiently large. Suppose K_X does not lies in the wall W separating H and H_+, and that K_X and H lie in the same connected components of $\mathrm{NS}(X)_{\mathbb{R}} \setminus W$. (See the left figure below.) Then the birational map*

(1)
$$\begin{tikzcd} M_+ \arrow[rr, dotted] \arrow[dr, "f_-"'] & & M \arrow[dl, "f_+"] \\ & M_0 & \end{tikzcd}$$

is a flip.





Suppose $M(H)$ is compact, and let us observe this theorem in case where X is minimal and $\kappa(X) \geq 1$. There is an ample line bundle H_X such that no wall of type (c_1, c_2) divides K_X and H_X. When $H \in \{(1-t)H_0 + tK_X | t \in [0, 1)\}$ starts from a polarization H_0 and gets closer to K_X, one gets a finite sequence of flips

$$M(H = H_0) \cdots > M(H_1) \quad \cdots > M(H_N = H_X),$$

which terminates in $M(H_X)$. (See the right figure above.) It is known that the canonical divisor of $M(H_X)$ is nef. Thus one can regard this "natural" process described in a moduli-theoretic way as an analogy of minimal model program of $M(H)$, although it is unknown whether $M(H_X)$ admits only terminal singularities.

THE REFLEXIVITY OF A SEGRE PRODUCT OF PROJECTIVE VARIETIES

(Joint work with Hajime Kaji, [Math. Ann. **342** (2008), 279–289])
Satoru Fukasawa (Waseda University/JSPS)

Main Theorem (Fukasawa-Kaji)

For a proj var $Y \subset \mathbb{P}^N$ of dim n, with Hessian rank r / char $p \geq 0$,

The Segre product $X = \mathbb{P}^m \times Y$: non-reflexive

$$\Leftrightarrow \begin{cases} (1)\ 0 < n - m < r,\ p = 2,\ m + n : \text{odd, or} \\ (2)\ n - m > r \text{ and } Y : \text{non-reflexive.} \end{cases}$$

Segre product := Product embedded by Segre embedding

Main Theorem'

$$\mathbb{P}^m \times Y: \text{reflexive} \Leftrightarrow \begin{cases} \text{(i)}\ m \geq n, \\ \text{(ii)}\ 0 \leq n - m < r,\ p \neq 2, \\ \text{(iii)}\ 0 \leq n - m < r,\ p = 2,\ m + n : \text{even} \\ \text{(iv)}\ n - m = r, \text{ or} \\ \text{(v)}\ n - m > r \text{ and } Y : \text{reflexive.} \end{cases}$$

Y	p	r	R/N	$\mathbb{P}^m \times Y$	R/N	cond.
$\mathbb{P}^n$	≥ 0	0	R	$\mathbb{P}^m \times \mathbb{P}^n$ $(m \geq n)$	R	(i)
$\mathbb{P}^1 \times \mathbb{P}^1$	$\neq 2$	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	R	(ii)
$\mathbb{P}^2 \times \mathbb{P}^2$	2	4	R	$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	R	(iii)
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	2	2	N	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	R	(iv)
$\mathbb{P}^1 \times \mathbb{P}^2$	≥ 0	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	R	
$\mathbb{P}^1 \times \mathbb{P}^3$	≥ 0	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	R	(v)
$\mathbb{P}^1 \times \mathbb{P}^1$	2	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	N	(1)
$\mathbb{P}^1 \times \mathbb{P}^2$	2	2	R	$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$	N	
F_d $(d \equiv 1 \bmod p)$	> 0	0	N	$\mathbb{P}^m \times F_d$ $(m < n)$	N	(2)

TABLE: The reflexivity of Segre products of projective varieties

R:= Reflexive / N:= Non-Reflexive
F_d: Fermat hypersurf. of deg d in $\mathbb{P}^{n+1}$

Reflexivity

$$Y \text{ is reflexive} \overset{\text{def}}{\Leftrightarrow} C(Y) = C(Y^*)$$

$\mathbb{P}^{N*} := \{H \mid \text{hyperplane in } \mathbb{P}^N\}$ dual proj. space;

$Y \subset \mathbb{P}^N$;

$C(Y) := \overline{\{(y, H) \in Y_{sm} \times \mathbb{P}^{N*} \mid \mathbb{T}_y Y \subset H\}} \subset \mathbb{P}^N \times \mathbb{P}^{N*}$

$\downarrow \pi_Y$ conormal variety

$\mathbb{P}^{N*}$

$Y^* := \pi_Y(C(Y)) \subset \mathbb{P}^{N*}$; dual variety

$C(Y^*) \subset \mathbb{P}^{N**} \times \mathbb{P}^{N*} = \mathbb{P}^N \times \mathbb{P}^{N*}$

Rem. Y: reflexive $\Rightarrow Y^{**} = Y$ (Proj. Duality)

Hessian Rank r

$$r = \text{rank} \left[\frac{\partial^2 (h|Y)}{\partial y_i \partial y_j} \right]_{i,j}$$

$(y, H) \in C(Y)$: general pt,
h: rational function defining H,
(y_i): local coord. of Y at y.

Rem. rank $d\pi_Y = N - 1 - (\dim Y - r) \leq \dim Y^*$

Monge-Segre-Wallace Criterion

Y is reflexive.

$\Leftrightarrow \pi_Y : C(Y) \to Y^*$ is generically smooth.

$\Leftrightarrow$ rank $d\pi_Y = \dim Y^*$

PROOF OF MAIN THEOREM

- Determine rank $d\pi_X$ and dim X^*.
- Use Monge-Segre-Wallace Criterion.

APPLICATION

Any Non-reflexive example of a Segre product gives a Negative Answer to Kleiman-Piene's question for Gauss maps.

Rem. The Gauss map of any Segre product is gen. smooth onto its image.

Actions of linear algebraic groups of exceptional type on projective varieties

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Main Theorem (W)

X: smooth proj. var. of dim. n / $\mathbb{C}$,

G: simple linear alg. group of exceptional type,

$G \curvearrowright X$: non-trivial, $n = r_G + 1$.

Then X is one of the following:

(1) $\mathbb{P}^6$, (2) Q^6,

(3) $E_6(\omega_1)$, (4) $G_2(\omega_1 + \omega_2)$,

(5) $Y \times C$,

where Y is $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and C is a smooth curve,

(6) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(m))$,

where Y is as in (5) and $m > 0$.

Furthermore, the action of G is unique for each case if G is simply connected.

◇ Known Results (Andreatta's Work)

$G \curvearrowright X$, G: simple alg. gp. of Dynkin type.

$$n := \dim X \geq r_G,$$

⋆ $r_G := \min\{ \dim G/P \mid P \subset G : \text{parabol. subgp.} \}$

TABLE

G	r_G	X s.t. $n = r_G$	G	r_G	X s.t. $n = r_G$
A_l	l	$\mathbb{P}^l$	E_6	16	$E_6(\omega_1)$
B_l	$2l-1$	Q^{2l-1}	E_7	27	$E_7(\omega_1)$
C_l	$l-1$	$\mathbb{P}^{l-1}$	E_8	57	$E_8(\omega_1)$
D_l	$2l-2$	Q^{2l-2}	F_4	15	$F_4(\omega_1)$
			G_2	5	$G_2(\omega_1)$ or $G_2(\omega_2)$

ω: dom. int. weight of G

V_ω: irr. rep. sp. of G with highest weight ω

$G(\omega)$: min. orbit of G in $\mathbb{P}(V_\omega)$

Question

$n = r_G + 1 \Rightarrow$ **What kinds of varieties appear?**

Theorem [A, '01]
If $n = r_G + 1$ and G is classical, then X is one of the following:
(1) $\mathbb{P}^n$, (2) Q^n,
(3) $\mathbb{P}(T_{\mathbb{P}^2})$, (4) $C_2(\omega_1 + \omega_2)$,
(5) $Y \times C$, where Y is $\mathbb{P}^{n-1}$ or Q^{n-1} and C is a smooth curve,
(6) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(m))$, where Y is as in (5) and $m > 0$.
Furthermore, the action of G is unique for each case if G is simply connected.

◇ Points of Our Argument

G-equiv. extremal contraction of X + G-orbit

$\Downarrow$

determination of the structure of X.

Different point

G: classical $\Rightarrow$ G-orbit: well-known var.

G: except. $\Rightarrow$ G-orbit: not well-known var.

Example

X: non-G-homog. var. with $\rho(X) = 1 \Rightarrow \exists G/P \subset X$: ample div.

⋆ G: classical

$\Rightarrow G/P \cong \mathbb{P}^{n-1}$ or $\mathbb{Q}^{n-1}$

$\Rightarrow X \cong \mathbb{P}^n$ or $\mathbb{Q}^n$ (well-known fact).

⋆ G: exceptional

$\Rightarrow G/P \cong E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$.

$\Rightarrow X \cong \mathbb{P}^6$, Q^6 or $E_6(\omega_1)$ by the following.

Proposition [W, '07]
(X, L): sm. polarized var. s.t. $A \in |L|$: homog. var. with $\rho(A) = 1$.
If $\dim A \geq 2$, then (X, L) is one of the following:
(1) $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(i))$, $i = 1, 2$, (2) $(Q^{n+1}, \mathcal{O}_{Q^{n+1}}(1))$,
(3) $(G(2, \mathbb{C}^{2m}), \mathcal{O}_{\text{Plücker}}(1))$, (4) $(E_6(\omega_1), \mathcal{O}_{E_6(\omega_1)}(1))$.

Rational Curves on Hypersurfaces

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We study the family of rational curves on hypersurfaces $/k = \bar{k}$, **ch $\geqslant$ 0**.

Family of Rational Curves

Let $e \geqslant 1$, $d \geqslant 1$, and $n \geqslant 3$.

Def. For $X = X_d \subset \mathbb{P}^n$: hypersurf of deg d, set

$$\boldsymbol{R_e(X)} := \left\{ \begin{array}{c} \textbf{smooth rat curves } C \textbf{ of deg } e \textbf{ in } \mathbb{P}^n \\ \textbf{lying in } X \end{array} \right\} \subset \mathrm{Hilb}^{et+1}(X/k).$$

Expected Dimension

For $\mu := (n+1-d)e + n - 4 = \chi(N_{C/X})$,

$$R_e(X) \neq \emptyset \;\Rightarrow\; \mathbf{dim}\, \boldsymbol{R_e(X)} \geqslant \boldsymbol{\mu}.$$

Starting Point (Lines, $e = 1$)

Thm A (**W. Barth-A. Van de Ven '78/79** ($\mathbb{C}$), **J. Kollár '96** (**ch** $\geqslant 0$)).

(a) $\boldsymbol{R_1(X) = \emptyset}$ **for gen X if $\mu < 0$.**

(b) $\boldsymbol{R_1(X)}$ **: smooth of dim μ for gen X if $\mu \geqslant 0$.**

(c) $\boldsymbol{R_1(X)}$ **: connected for any X if $\mu \geqslant 1$, except when $X = X_2 \subset \mathbb{P}^3$.**

What can we say about the family $R_e(X)$ for deg $e \geqslant 2$?

Thm B (**J. Harris-M. Roth-J. Starr '04** ($\mathbb{C}$)).

$$e \geqslant 1,\; d < (n+1)/2 \;\Longrightarrow\; R_e(X) : \begin{array}{l} \textbf{integral, loc comp int sch of} \\ \textbf{dim } \mu \textbf{ for gen } X. \end{array}$$

Main Thm (Furukawa '07 (ch $\geqslant$ 0)).

(a) $d \geqslant \max(e-2, 1) \;\Longrightarrow\; R_e(X) = \emptyset$ **for gen X if $\mu < 0$.**

(b) $\left. \begin{array}{l} 1 \leqslant e \leqslant 3,\; d \geqslant 1, \text{ or} \\ e \geqslant 4,\; d \geqslant 2e-3 \end{array} \right\} \Longrightarrow R_e(X)$ **: smooth of dim μ for gen X if $\mu \geqslant 0$.**

(c) $R_2(X)$ **: connected for gen X if $\mu \geqslant 1$, except when $X = X_3 \subset \mathbb{P}^3$.**

Exceptional Case of (c)

Prop. $\boldsymbol{X = X_3 \subset \mathbb{P}^3}$ **: smooth cubic**

$\Longrightarrow$ $\boldsymbol{R_2(X)}$ **has 27 connected comp.**

$\because$ For the lines $L_1, \dots, L_{27} \subset X$,

$$R_2(X) \hookrightarrow \bigcup L_i^* \subset (\mathbb{P}^3)^\vee$$
$$C \mapsto (\text{2-pl spanned by } C),$$

where $L_i^* := \{ H \subset \mathbb{P}^3 : \text{2-pl}, L_i \subset H \}$. $\square$

Key of the Proof

Fix $f : \mathbb{P}^1 \xrightarrow{\sim} C \subset \mathbb{P}^n$: deg e.

$\boldsymbol{I_C^\circ} := \{ X = X_d \subset \mathbb{P}^n : \text{smooth}, C \subset X \}$,

$\boldsymbol{Z_C^\circ} := \{ X \in \boldsymbol{I_C^\circ} \mid h^1(N_{C/X}) \neq 0 \}$.

Tools

- ♦ Castelnuovo-Mumford regularity of $\mathcal{I}_C^2$.
- ♦ Theory of catalecticant matrices.
- ♦ Bounds of a_i for $f^* N_{C/\mathbb{P}^n} \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Prop. assumption of **(b)**

$$\Longrightarrow \mathbf{codim}(\boldsymbol{Z_C^\circ}, \boldsymbol{I_C^\circ}) \geqslant \boldsymbol{\mu + 1}.$$

$\because$ $0 \to \mathcal{I}_C^2 \to \mathcal{I}_C \to N_{C/\mathbb{P}^n}^\vee \to 0$ induces

$$H^0(\mathbb{P}^n, \mathcal{I}_C(d)) \overset{\delta_C}{\twoheadrightarrow} \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f^* N_{C/\mathbb{P}^n}, f^*(\mathcal{O}_{\mathbb{P}^n}(d)))$$
$$\cup \flat \qquad\qquad\qquad \cup \sharp$$
$$\hat{Z}_C = \delta_C^{-1}(\mathsf{Z}) \twoheadrightarrow \mathsf{Z} := \{ \alpha \mid H^0(\alpha) : \text{not surj} \}.$$

(Prop) $\leftsquigarrow$ codim($\flat$) $\leftsquigarrow$ codim($\sharp$). $\square$

On the birational unboundedness of higher dimensional $\mathbb{Q}$-Fano varieties

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1 Introduction

By a $\mathbb{Q}$-**Fano variety**, we mean a normal projective $\mathbb{Q}$-factorial variety with only log terminal singularities whose anticanonical divisor is ample.

We say that a class of varieties $\mathfrak{V}$ is **birationally bounded** if there is a morphism $f\colon \mathcal{X} \to \mathcal{S}$ between algebraic schemes such that every variety in $\mathfrak{V}$ is birational to one of the geometric fibres of f. We say that $\mathfrak{V}$ is **birationally unbounded** if it is not birationally bounded.

Main theorem

Theorem 1. If $n \geq 6$ then the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally unbounded.

This result is known for 3-folds by the work of J. Lin [3].

Conjecture 2 (Borisov-Alexeev-Borisov)**.** Fix $\epsilon > 0$. Then the family of $\mathbb{Q}$-Fano varieties of a given dimension with log discrepancy greater than ϵ is bounded.

Conjecture 2 is solved in the surface case by Alexeev and in the toric case by A. Borisov and V. Borisov. Theorem 1 shows that we cannot remove the restriction on log discrepancies from the hypothesis of Conjecture 2 even if we replace "boundedness" by "birational boundedness".

Followings are some of the classes of $\mathbb{Q}$-Fano varieties which have been known to be bounded:

- Smooth Fano varieties (in arbitrary dimension) (cf. [1]).
- $\mathbb{Q}$-Fano 3-folds with canonical singularities (cf. [2]).
- Log terminal $\mathbb{Q}$-Fano pairs of bounded index (in arbitrary dimension) (cf. [4]).

2 Outline of the proof

Let a, l, m and n be positive integers, where a and l are odd. Put $b = (al-1)/2$. Let $\Bbbk$ be an algebraically closed field of char 2.

Step 1. Non-ruled $\mathbb{Q}$-Fano weighted hypersurfaces.

Let $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n, y]$ be the graded rings whose gradings are given by $\deg x_i = 1$ for $0 \leq i \leq m$, $\deg x_i = a$ for $m+1 \leq i \leq n$ and $\deg y = b$. We define weighted projective spaces as follows.

- $P_k = \mathbb{P}_k(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{a, \ldots, a}^{n\ m}, b) := \operatorname{Proj} k[x_0, \ldots, x_n, y]$.
- $Q_k = \mathbb{P}_k(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{a, \ldots, a}^{n\ m}) := \operatorname{Proj} k[x_0, \ldots, x_n]$.

For $f = f(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]_{al}$ (the degree al part), we define

- $X_f := (y^2 x_0 - f(x_0, \ldots, x_n) = 0) \subset P_k$.

Condition 3. (1) l is odd, $4 \leq n$ and $0 < m < n$.

(2) $n - m + 1 < l < 2(n - m)$.

Non-ruled weighted hypersurfaces

Theorem 4 ([5], Theorem 7.3)**.** Assume that l, m and n satisfy Condition 3. Then, the following assertions hold for every odd integer $a > (m+1)/2$.

(1) The weighted hypersurface $X_f \subset P_{\mathbb{C}}$ of degree al defined over $\mathbb{C}$ is a non-ruled $\mathbb{Q}$-Fano n-fold with Picard number one for a very general $f \in \mathbb{C}[x_0, \ldots, x_n]_{al}$.

(2) The weighted hypersurface $X_f \subset P_{\Bbbk}$ of degree al defined over $\Bbbk$ is not separably uniruled for a general $f \in \Bbbk[x_0, \ldots, x_n]_{al}$.

Step 2. Construction of "large" birationally trivial families.

For fixed l, m and n satisfying Condition 3, let $\mathcal{X}_a \to \mathcal{S}_a$ be the family of weighted hypersurfaces $X_f \subset P_{\Bbbk}$ of degree al defined over $\Bbbk$.

We say that a family of varieties is **birationally trivial** if every two members of the family are birational.

"Large" birationally trivial families

Lemma 5. Suppose that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Then, there exists a constant R such that, for every odd integer $a > m+1$ and a general point $s_a \in S_a$, there is a closed subvariety $\mathcal{B}_a$ of $\mathcal{S}_a$ with the following properties:

(1) $\mathcal{B}_a$ parametrizes a birationally trivial family.

(2) $\mathcal{B}_a$ passes through s_a.

(3) $\dim \mathcal{S}_a - \dim \mathcal{B}_a \leq R$.

Remark 6. Suppose that l, m, n satisfy Condition 3. Let $a_i > m+1$ be an odd integer and $f_i \in \mathbb{C}[x_0, \ldots, x_n]_{a_i l}$ be a very general element for $i = 1, 2$. We can prove that if X_{f_1} and X_{f_2} are birational (over $\mathbb{C}$) then their reduction mod 2 models are also birational (over $\Bbbk$). This observation is crucial in the proof of Lemma 5.

Step 3. Bounding birationally trivial families in char 2.

Let $f \in \Bbbk[x_0, \ldots, x_n]_{al}$ be a general element. We denote by $\ _f\colon X := X_f \dashrightarrow Q_{\Bbbk}$ the restriction of the natural projection $P_{\Bbbk} \dashrightarrow Q_{\Bbbk}$.

If we are over $\Bbbk$, l, m, n satisfy Condition 3 and $a > m+1$, then there is a big line bundle $\mathcal{L} \subset \Omega_Y^{n\ 1}$ on a smooth model Y of X. By analyzing the rational map associated to $\mathcal{L}$, we obtain the following.

Birational invariance of the map $\ _f$

Lemma 7. Suppose that l, m and n satisfy $l < 2(n-m) - 1$ in addition to Condition 3. Let $a > m+1$ be an odd integer and $f \in \Bbbk[x_0, \ldots, x_n]_{al}$ a general element. Then, the map $\ _f\colon X_f \dashrightarrow Q_{\Bbbk}$ is a birational invariant.

This means that, if $g \in \Bbbk[x_0, \ldots, x_n]_{a'l}$ is also general for some $a' > m+1$ and there is a birational map $\ \colon X_f \dashrightarrow X_g$, then $a = a'$, is an isomorphism and there is an automorphism of $Q_{\Bbbk}$ such that the diagram

$$\begin{tikzcd} X_f \arrow[r] \arrow[d, dotted, "\pi_f"'] & X_g \arrow[d, dotted, "\pi_g"] \\ Q_{\Bbbk} \arrow[r, "\sigma"'] & Q_{\Bbbk} \end{tikzcd}$$

commutes.

By Lemma 7, we can bound the dimension of birationally trivial subfamilies of $\mathcal{X}_a/\mathcal{S}_a$.

Bounding birationally trivial families

Lemma 8. Suppose that l, m and n satisfy $l < 2(n-m) - 1$ in addition to Condition 3. Then, for every odd integer $a > m+1$ and a general point $s_a \in \mathcal{S}_a$, there is a closed subvariety $\mathcal{W}_a$ of $\mathcal{S}_a$ with the following properties:

(1) $\mathcal{W}_a$ parametrizes the members which are birational to the member corresponds to s_a.

(2) $\dim \mathcal{S}_a - \dim \mathcal{W}_a \to \infty$ as $a \to \infty$.

If $n \geq 6$, then we can find l, m and n satisfying $l < 2(n-m) - 1$ in addition to Condition 3. Now Theorem 1 follows from Lemma 5 and 8.

Moduli of Bridgeland semistable objects on the projective plane

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Outlook

- X : a smooth projective surface
  $\mathcal{D}(X)$: the bounded derived category of $\mathrm{Coh}(X)$
  Fix $\alpha \in K(X)$ and σ : a Bridgeland stability condition on $\mathcal{D}(X)$

  Then we consider the moduli functor $\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma)$ of σ-semistable objects with class α in $K(X)$

- σ : geometric $\Longrightarrow$ $\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma) \cong \mathcal{M}_X(\alpha,\omega)$

  $\mathcal{M}_X(\alpha,\omega)$: the moduli functor of ω-semistable coherent sheaves on X,
  ω : an ample divisor class in $\mathrm{NS}(X)$

- σ : algebraic $\Longrightarrow$ $\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma) \cong \mathcal{M}_B(\alpha_B,\theta_B)$

  $\mathcal{M}_B(\alpha_B,\theta_B)$: the moduli functor of θ_B-semistable modules over a nite dimensional $\mathbb{C}$-algebra B

  where $\alpha_B \in K(\mathrm{Mod}\,B)$, $\mathrm{Mod}\,B$: an abelian category of nitely generated right B-modules

- As an application in the case of $X = \mathbb{P}^2$ we show

$$\mathcal{M}_X(\alpha,\omega) \cong \mathcal{M}_B(\alpha_B,\theta_B)$$

A Bridgeland stability condition σ

σ consists of data $(Z,\mathcal{A})$,

$$Z\colon K(X) \to \mathbb{C}, \quad \mathcal{A} \subset \mathcal{D}(X),$$

Z : a group homomorphism, $\mathcal{A}$: a full abelian subcategory
These data satisfy some axioms

(For example)
$0 \neq E \in \mathcal{A} \Longrightarrow Z(E) \in \mathbb{R}_{>0}\exp(\sqrt{-1}\pi\phi(E))$ with $0 < \phi(E) \leq 1$

De nition
A nonzero object $E \in \mathcal{A}$: semistable
$\Longleftrightarrow$ $0 \neq A \subset E \Longrightarrow \phi(A) \leq \phi(E)$

Moduli Functors $\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma)$

$\mathcal{M}_{\mathcal{D}(\mathbb{P}^2)}(\alpha,\sigma)$: (scheme/$\mathbb{C}$) $\to$ (sets)

For a $\mathbb{C}$-scheme S,

$\mathcal{M}_{\mathcal{D}(\mathbb{P}^2)}(\alpha,\sigma)(S)$

$= \{E$: a family of σ-semistable objects in $\mathcal{A}$ with class $\alpha \in K(X)\} \subset \mathcal{D}(X \times S)$

Main Theorem

In the case of $X = \mathbb{P}^2$, we nd $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}(\mathbb{P}^2)$,

which is both geometric and algebraic.

Main Theorem 0.1 (O)

For any $\alpha \in K(\mathbb{P}^2)$ with $0 < c_1(\alpha)\cdot H \leq r(\alpha)$,
$\Phi_{\mathcal{E}}\colon E \mapsto \mathrm{R\,Hom}_{\mathbb{P}^2}(\oplus E_i, E[1])$ gives the isomorphism

$$\mathcal{M}_{\mathbb{P}^2}(\alpha,H) \cong \mathcal{M}_{B_{\mathcal{E}}}(-\alpha_{B_{\mathcal{E}}},\theta_Z^{\alpha}),$$

where H is the hyperplane class on $\mathbb{P}^2$ and

(1) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(1), \Omega^1_{\mathbb{P}^2}(3), \mathcal{O}_{\mathbb{P}^2}(2)\}$

or (2) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(3)\}$ *(Le Potier)*

or (3) $\mathcal{E} = \{\mathcal{O}_{\mathbb{P}^2}(2), \Omega^1_{\mathbb{P}^2}(4), \mathcal{O}_{\mathbb{P}^2}(3)\}$.

$\Uparrow$

Geometric Stability

Theorem 0.3 (O)

$\alpha \in K(X)$ *"normalized"*, $\mu_\omega(\alpha)$ *: the slope of* α

We take $\beta \uparrow \mu_\omega(\alpha)\omega$ *in* $\mathrm{NS}(X)_{\mathbb{R}}$. *Then*

$$\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma_{(\beta,\omega)}) \cong \mathcal{M}_X(\alpha, -\frac{1}{2}K_X, \omega).$$

De nition of $\sigma_{(\beta,\omega)}$ for $\beta,\omega \in \mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X)\otimes\mathbb{R}$ with ω ample

$\exists \mathcal{A}_{(\beta,\omega)} \subset \mathcal{D}(X)$: tilting of $\mathrm{Coh}(X)$ by $\beta\cdot\omega$

$\exists Z_{(\beta,\omega)}\colon K(X) \to \mathbb{C}\colon E \mapsto -\int \exp(-\beta+\sqrt{-1}\omega)\,\mathrm{ch}(E)$

$\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$ de nes a Bridgeland stability condition on X

Algebraic Stability

If $\exists\mathcal{E} = \{E_0,\cdots,E_n\}$: a full strong exceptional collection on X

Put $B_{\mathcal{E}} = \mathrm{End}_X(\oplus_i E_i)$

Then Bondal showed that $\Phi_{\mathcal{E}} = \mathrm{R\,Hom}_X(\oplus E_i, -)$ gives the equivalence

$$\Phi_{\mathcal{E}}\colon \mathcal{D}(X) \cong \mathcal{D}(\mathrm{Mod}\,B_{\mathcal{E}}).$$

Put $\mathcal{A}_{\mathcal{E}} = \Phi_{\mathcal{E}}^{-1}(\mathrm{Mod}\,B_{\mathcal{E}}) \subset \mathcal{D}(X)$ de ned by pulling back $\mathrm{Mod}\,B_{\mathcal{E}} \subset \mathcal{D}(\mathrm{Mod}\,B)$ by $\Phi_{\mathcal{E}}$.

Proposition 0.2 (O)

For $\sigma = (Z,\mathcal{A}_{\mathcal{E}})$,

$$\mathcal{M}_{\mathcal{D}(X)}(\alpha,\sigma) \cong \mathcal{M}_{B_{\mathcal{E}}}(\alpha_{B_{\mathcal{E}}},\theta_Z^{\alpha})$$

where $\alpha_{B_{\mathcal{E}}} = \Phi_{\mathcal{E}}(\alpha) \in K(\mathrm{Mod}\,B_{\mathcal{E}})$ *and*

θ_Z^{α} *is the θ-stability of $B_{\mathcal{E}}$-modules depending on α and Z (and $\mathcal{E}$).*

$\Leftarrow$

Conclusion and Further Studies

Conclusion

Main Theorem (2) gives another proof of the result by Le Potier (1994).

(Similar results are obtained by Barth in the case of $r(\alpha) = 2$)

$\Longrightarrow$ The Bridgeland stability condition is the useful new concept to study the usual problem.

Further Studies

- Analysis of the wall-crossing phenomena of $\mathcal{M}_{B_{\mathcal{E}}}(-\alpha_{B_{\mathcal{E}}},\theta_{B_{\mathcal{E}}})$ when $\theta_{B_{\mathcal{E}}}$ varies.

  (the wall-crossing phenomena of $\mathcal{M}_{\mathbb{P}^2}(\alpha,H)$ never occur because $\mathrm{NS}(\mathbb{P}^2)_{\mathbb{R}} = \mathbb{R}H$)

- Our method is applicable for any surface X with a full strong exceptional collection (Generalization).

Homogeneous isoparametric hypersurfaces with four distinct principal curvatures and moment maps

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Main Results

Homog. Herm. isoparametric hypersurf. ($\subsetneq \mathbb{S}^n$, $g = 4$) is obtained from some moment maps!!

Our Goal

Isoparametric hypersurf. $\subsetneq \mathbb{S}^n$, $g = 4$

$\Updownarrow$ Is there any relation?

Moment maps

What is an isoparametric hypersurface?

$M \subsetneq \mathbb{S}^n$: a hypersurf. in sphere,

- M: isoparametric
  $\overset{\text{def.}}{\Longleftrightarrow}$ $^\exists\varphi : \mathbb{S}^n \to \mathbb{R}$: isoparametric & $^\exists c \in \mathbb{R}$ s.t. $M = \varphi^{-1}(c)$.
  - $\varphi : \mathbb{S}^n \to \mathbb{R}$: isoparametric
    $\overset{\text{def.}}{\Longleftrightarrow}$ $^\exists A(t), {}^\exists B(t) \in C^\infty(\mathbb{R})$ s.t. $\begin{cases} \|\operatorname{grad}\varphi\|^2 = A \circ \varphi, \\ \Delta\varphi = B \circ \varphi. \end{cases}$

Isotropy representations

G/K: Herm. symm. sp. of rank 2,

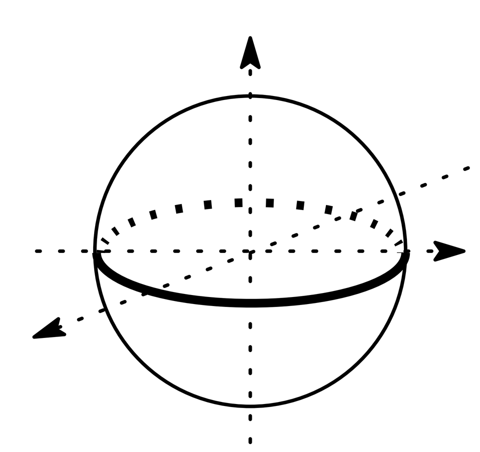
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomp. of $\mathfrak{g} = \mathrm{Lie}(G)$,

1. the principal orbit of $K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{p}$
   $=$ homog. isoparametric hypersurf.
   $\Rightarrow$ $^\exists\varphi : \mathfrak{p} \to \mathbb{R}$: isoparametric,
   $\Rightarrow$ φ: K-invariant,
2. $K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{p}$: Hamiltonian K-action,
   $\Rightarrow$ $^\exists\mu : \mathfrak{p} \to \mathfrak{k}^*$: the moment map,
   $\Rightarrow$ $\mathfrak{p} \xrightarrow[K\text{-equivariant}]{\text{moment map } \mu} \mathfrak{k}^* \xrightarrow{K\text{-invariant norm } \|\cdot\|} \mathbb{R}$

Example of isoparametric hypersurf.

$\mathbb{S}^2 \supsetneq M = \mathbb{S}^1(r)$, $0 < r \leq 1$.

$\Longrightarrow$ M: isoparametric.



$\varphi : \mathbb{S}^2 \to \mathbb{R}$,
$\varphi(x, y, z) := z$.

Facts

$M \subsetneq \mathbb{S}^n$: hypersurf.

- M: homog. $\Rightarrow$ M has constant principal curvatures,
  $\Leftrightarrow$ M: isoparametric,
- M: homog. $\Leftrightarrow$ M: the principal orbit of the isotropy rep. of $^\exists G/K$: symmetric sp. of rank 2,
- M: isoparametric,
  $g := \#\{\text{distinct principal curvatures of } M\}$,
  - g must be 1, 2, 3, 4, or 6,
  - $g \leq 3 \Rightarrow$ "M: homog. $\Longleftrightarrow$ M: isoparametric",
  - $g = 4 \Rightarrow$ $^\exists$non-homog., classification is still open.

On Mixed Plurigenera of Algebraic Plane Curves

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The purpose is to study algebraic plane curves from the viewpoint of birational geometry of pairs by making use of mixed plurigenera. To introduce the notion of mixed plurigenera, we begin by recalling birational geometry of pairs (S, D) where D is an algebraic curve on an algebraic surface S.

We shall consider algebraic varieties defined over the field of complex numbers. Here by a surface we mean a 2-dimensional projective non-singular variety.

When D is a non-singular curve on S, we consider complete linear systems $|mK_S + aD|$ where K_S is a canonical divisor on S and $m \geq a \geq 0$.

It is not hard to check that $\dim |mK_S + aD|$ is birationally invariant. Hence, $\dim |mK_S + aD| + 1$, denoted by $P_{m,a}[D]$, is called (m, a)-genus of the pair (S, D). Occasionally, it is called (m, a)-genus of D. (m, a)-genera may be called mixed plurigenera. Note that $(m, 0)$-genus becomes m-genus of S and (m, m)-genus turns out to be logarithmic m-genus of $S - D$, written simply as $P_m[D]$. From $P_m[D]$, the Kodaira dimension $\kappa[D]$ is introduced.

In what follows, suppose that S is a rational surface. Thus the study of pairs (S, D) may be understood as birational geometry of plane curves.

Let C be a curve on $\mathbf{P}^2$. Then after successive blowing ups, we obtain a non-singular curve D and a surface S which is obtained from $\mathbf{P}^2$. Then (S, D) is birationally equivalent to $(\mathbf{P}^2, C)$. By making use of (S, D), we define $P_{m,a}[C]$ to be $P_{m,a}[D]$.

Occasinally, (S, D) is said to be a non-singular model of the pair $(\mathbf{P}^2, C)$.

In 1928, Coolidge studied algebraic plane curves C and obtained the remarkable result to the effect that any rational plane curve can be transformed into a straight line on $\mathbf{P}^2$ by a birational transformation of $\mathbf{P}^2$, whenever $P_{2,1}[C] = 0$. In this case, $\kappa[C] = -\infty$.

In 1961, Nagata obtained the following result. If $g = g(D) > 0$, then $D^2 \leq 4g + 5$. Further if $D^2 = 4g + 5$, then $g = 1$ and (S, D) is birationally equivalent to $(\mathbf{P}^2, \Gamma)$, where Γ is a non-singular cubic.

Since 1983, the theory of minimal models (S, D) was introduced and has been extensively studied by Iitaka. He determined the structure of (S, D) when $\kappa[D] = 0$ or 1. Moreover he showed that, if $\kappa[D] = 2$, then any relatively minimal pair (S, D) becomes minimal. Therefore, given a plane curve C, we have a minimal pair (S, D) which is birationally equivalent to $(\mathbf{P}^2, C)$, provided that $\kappa[D] = 2$. Hereafter we suppose $\kappa[D] = 2$.

When $S \neq \mathbf{P}^2$, the minimal model (S, D) is derived from a $\sharp$-minimal pair (Σ_B, C), Σ_B being a Hirzebruch surface, which has type

$$[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$$

By Riemann-Roch theorem and vanishing theorem due to Kawamata, the following formulas are obtained by Iitaka:

$$\begin{aligned}
P_{2,1}[D] &= Z^2 + 2 - g, \\
P_{3,1}[D] &= 3Z^2 + 8 - 7g + D^2, \quad (\sigma \geq 6) \\
P_{4,2}[D] &= (2Z - D)^2 + 2(Z^2 - g + 1) + 1,
\end{aligned}$$

where $Z = K_S + D$.

Moreover,

$$P_2[D] = \begin{cases} P_{2,1}[D] = Z^2 + 2 & (g = 0), \\ P_{2,1}[D] + 1 = Z^2 + 2 & (g = 1), \\ P_{2,1}[D] + 3g - 3 = Z^2 + 2g - 1 & (g > 1). \end{cases}$$

Thus, mixed plurigenera (m, a)-genus are computed through g, Z^2 and D^2.

So far, the structures of pairs (S, D) have been studied in the following cases: (1). $P_2[D] = 2g - 1, 2g, 2g + 1$. (2). $P_{2,1}[D] = 1, 2, 3$. (3). $P_{3,1}[D] = 1, 2, 3$.

Here, we shall enumerate the types of pairs (S, D) in the following cases: $P_2[D] = 2g + 2$, $P_{2,1}[D] = 4$, $P_{3,1}[D] = 4, 5, 6$ and $P_{4,2}[D] \leq 12$.

The tables of these types will appear in the bottom of sections.

Finally, we shall give concrete examples which satisfy

$$P_2[D] \leq 2g + 2, P_{2,1}[D] \leq 4, P_{3,1}[D] \leq 6$$

.

Table 1: $P_{3,1}[D] = 4$

$k = P_{2,1}[D] - 2.$

p	α	prototype	k	g
1	2	$[7 * 7; 1]$	119	36
		$[7 * 14, 2; 1]$	119	36
	1	$[9 * 13, 1; 4^{10}]$	1	0
0	6	$[6 * 9; 1]^*$	133	40
	2	$[8 * 9; 4^8]^*$	4	8
	0	$[14 * 14; 7^7, 6, 4]^*$	1	1
		$[12 * 12; 6^6, 5^3]^*$	1	1
		$[12 * 12; 6^7, 5]^*$	3	6
		$[10 * 10; 5^7]^*$	5	11
		$[10 * 10; 5^6, 4^3]^*$	2	3
		$[8 * 8; 4^5]^*$	8	19

A classification of Lagrangian fibrations by Jacobians

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ABSTRACT

Let $\mathcal{C} \to \mathbb{P}^n$ be a family of reduced and irreducible genus n curves with 'mild singularities', such that the compactified relative Jacobian $X = \overline{\mathrm{Pic}}^1(\mathcal{C}/\mathbb{P}^n)$ is a holomorphic symplectic manifold. We prove that X is a Beauville-Mukai integrable system [1] provided the discriminant locus in $\mathbb{P}^n$ has sufficiently large degree. This work appears in preprint **arXiv:0803.1186**.

1 Introduction

An *irreducible holomorphic symplectic manifold* is a compact, simply connected, Kähler manifold X such that $\mathrm{H}^0(X,\Omega^2)$ is generated by a non-degenerate two-form σ. Matsushita [10] proved that the fibres of a non-trivial fibration on X must be Lagrangian with respect to σ, with generic fibre an abelian variety of dimension $n = \frac{1}{2}\dim X$. Hwang [7] proved that the base must be isomorphic to $\mathbb{P}^n$ if X is projective. We call $\pi : X \to \mathbb{P}^n$ a *Lagrangian fibration*.
In [15] the author described how one might use Lagrangian fibrations to classify holomorphic symplectic manifolds up to deformation. It follows from work of Beauville [1], Debarre [2], O'Grady [13], and Rapagnetta [14] that all known examples of holomorphic symplectic manifolds can be deformed to Lagrangian fibrations. The aim of this work is to classify Lagrangian fibrations whose fibres are Jacobians. We say that a family $\mathcal{C} \to \mathbb{P}^n$ of curves has *mild singularities* if the total space $\mathcal{C}$ is smooth.

Theorem: Let $\mathcal{C} \to \mathbb{P}^n$ be a family of reduced and irreducible genus n curves with mild singularities. Suppose that the compactified relative Jacobian $X = \overline{\mathrm{Pic}}^1(\mathcal{C}/\mathbb{P}^n)$ is a Lagrangian fibration, and the degree of the discriminant locus $\Delta \subset \mathbb{P}^n$ parametrizing singular curves is greater than $4n + 20$. Then X is a Beauville-Mukai integrable system [1], i.e., the family of curves $\mathcal{C}$ is a complete linear system of curves on a K3 surface S, and X can be identified with an irreducible component of the Mukai moduli space of stable sheaves on S. In particular, X is a deformation of the Hilbert scheme $\mathrm{Hilb}^n S$ of n points on S.

In dimension four, a formula of the author [16] can be combined with Guan's bounds [3] on the Chern numbers to verify the lower bound on $\deg\Delta$. We therefore recover a theorem of Markushevich [9] (the $n = 2$ case).
The proof of our theorem closely follows a construction of Hurtubise [5], which uses coisotropic reduction. The main difficulty is in extending Hurtubise's local argument to a global setting. Taking quotients of coisotropic submanifolds is a well-known idea in real symplectic geometry; the author feels it could be further exploited in holomorphic symplectic geometry. For example, Hwang and Oguiso [6] have recently used characteristic foliations to study the structure of singular fibres of Lagrangian fibrations.

2 Coisotropic reduction

Since X is the *degree one* Jacobian the relative Abel-Jacobi map $\mathcal{C} \to X$ is well-defined; the image $Y \subset X$ is smooth by the mild singularities hypothesis. Assuming that the restriction $\sigma|_Y$ of the holomorphic symplectic form to Y has rank two everywhere, Hurtubise [5] introduced the following construction. The null directions of $\sigma|_Y$ define a rank $n-1$ foliation F

$$0 \to F \to TY \to TY/F \to 0$$

known as the *characteristic foliation*. The space of leaves $Q = Y/F$ must be a holomorphic symplectic surface, since $\sigma|_Y$ descends to a non-degenerate two-form on Q. The curves in the family $\mathcal{C}$ project down to Q.
Hurtubise's argument is local: a family of curves over a small ball in $\mathbb{C}^n$ leads to an open subset Q of an algebraic surface. To obtain a nice space of leaves in a global setting we need compactness (algebraicity) of the leaves. The relevant theorems are due to Miyaoka [12], Bogomolov, and McQuillan (see Kebekus, Solá Conde, and Toma [8]). The key idea is that if Y is covered by curves on which F is ample, then by applying Mori's bend-and-break argument one can produce rational curves which must be contained in the leaves of F.

3 Rational curves in Y

Let ℓ be a generic line in the base $\mathbb{P}^n$ and let Z be the inverse image of ℓ in Y.

$$\begin{array}{ccc} Z & \hookrightarrow & Y \subset X \\ \pi\downarrow & & \downarrow\pi \\ \ell & \hookrightarrow & \mathbb{P}^n \end{array}$$

Then Z is a smooth surface fibred by genus n curves over $\ell \cong \mathbb{P}^1$, with $\deg\Delta$ singular fibres. In particular

$$c_2(Z) = 4 - 4n + \deg\Delta > 24.$$

Matsushita [11] proved that $R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^n}$ for a general Lagrangian fibration $\pi : X \to \mathbb{P}^n$. We can use this to show that $R^1\pi_*\mathcal{O}_Y$ is also isomorphic to $\Omega^1_{\mathbb{P}^n}$ and then restrict to ℓ to calculate $R^1\pi_*\mathcal{O}_Z$. Inserting this in the Leray spectral sequence yields

$$h^{0,k}(Z) = \begin{cases} 1 & k = 0, \\ 0 & k = 1, \\ 1 & k = 2. \end{cases}$$

Noether's formula now gives

$$K_Z^2 = 12\chi(\mathcal{O}_Z) - c_2(Z) < 0.$$

If Z were minimal, it would have Kodaira dimension $-\infty$ and hence p_g would vanish. Therefore Z contains at least one (-1)-curve. Varying ℓ in $\mathbb{P}^n$ gives many rational curves in Y.
Consider the following exact sequences over one of these rational curves.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & T\mathbb{P}^1 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & F|_{\mathbb{P}^1} & \to & TY|_{\mathbb{P}^1} & \to & TY/F|_{\mathbb{P}^1} & \to & 0 \\ & & & & \downarrow & & & & \\ 0 & \to & N_{\mathbb{P}^1\subset Z} & \to & N_{\mathbb{P}^1\subset Y} & \to & N_{Z\subset Y}|_{\mathbb{P}^1} & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

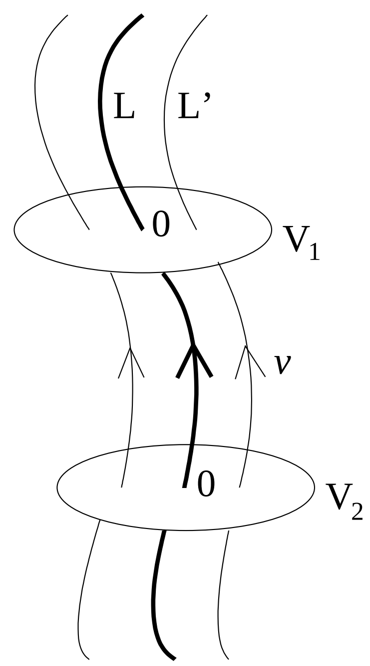
We can show that $F|_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}(1)^{\oplus(n-2)} \oplus T\mathbb{P}^1$. Thus the rational curve is contained in a leaf of F. Moreover, $F|_{\mathbb{P}^1}$ is ample which implies that all leaves of F are algebraic and rationally connected (see [8]).
Recall that we needed $\sigma|_Y$ to have rank two everywhere so that Y is coisotropic. This is proved along the way: we first define a coherent sheaf F as the kernel of the morphism $TY \to \Omega^1_Y$ given by $\sigma|_Y$. Then we show that F is locally free over a generic rational curve, and the above exact sequences imply that F has rank $n-1$. This proves that $\sigma|_Y$ has rank two at a generic point, and hence everywhere by semi-continuity.

4 The space of leaves

Algebraicity of the leaves implies that they are compact. Holmann [4] proved that a foliation on a Kähler manifold with all leaves compact must be *stable*, which implies that the space of leaves Y/F will be Hausdorff.
Let L be a leaf of the foliation F. A local model for the space of leaves is given by taking a small slice V transverse to the foliation, which meets L at $0 \in V$.
The holonomy map is a group homomorphism from $\pi_1(L)$ to the group of automorphisms of V fixing 0, and the holonomy group $H(L)$ is the image of this map. The quotient $V/H(L)$ is a local model for the space of leaves Y/F (see Holmann [4]). In our case both $\pi_1(L)$ and $H(L)$ must be trivial, because rationally connected implies simply connected. Thus the local model is simply V and Y/F is smooth.
It follows that the space of leaves is a smooth compact surface S. Given two local models V_1 and V_2 around the same leaf L, we can find a vector field v along the foliation F whose flow takes V_1 to V_2. The Lie derivative

$$\mathcal{L}_v(\sigma|_Y) = v(d\sigma|_Y) - d(i(v)\sigma|_Y)$$

vanishes, and hence the flow takes $\sigma|_{V_1}$ to $\sigma|_{V_2}$. This defines a (non-degenerate) two-form on $S = Y/F$ which is independent of the choice of local model.



The compact complex surface S therefore admits a holomorphic symplectic structure, so it must be either a K3 or abelian surface.

5 Completion of the proof

A (-1)-curve $\mathbb{P}^1 \subset Z$ will map isomorphically to the line ℓ. Comparing the normal bundle of this $\mathbb{P}^1$ inside a leaf to the normal bundle of ℓ inside $\mathbb{P}^n$ shows that each leaf maps birationally to a hyperplane in $\mathbb{P}^n$. One can show that the curves $C_t \subset Y$ in the family $\mathcal{C} \to \mathbb{P}^n$ are everywhere transverse to the leaves of the characteristic foliation, so this birational map is an isomorphism.
In particular, a leaf will intersect a curve C_t at most once, and hence each curve C_t maps isomorphically to its image in the space of leaves $S = Y/F$.
This shows that S contains an n-dimensional linear system of genus n curves, so it must be a K3 surface. The family of curves $\mathcal{C} \to \mathbb{P}^n$ is therefore a complete linear system of curves on a K3 surface S, and $X = \overline{\mathrm{Pic}}^1(\mathcal{C}/\mathbb{P}^n)$ is a Beauville-Mukai integrable system [1].

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Generalized Koszul duality and its application

(joint work with A. Takahashi)

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0. Motivation

Let k be a field and V be a finite dimensional vector space over k. Then we have the following famous diagram.

$$\begin{tikzcd} \mathcal{D}^b(\mathrm{coh}\,\mathbb{P}(V)) \arrow[r, "\text{Beilinson}", "\sim"'] \arrow[d, dash, "\text{BGG corresp.}"', "\wr"] & \mathcal{D}^b(\text{mod-}B_n) \\ \underline{\mathrm{grmod}}\bigwedge(V^*) & \end{tikzcd}$$

where

$$\begin{cases} B_n \ : \ \text{(the Beilinson algebra)} \\ \bigwedge(V^**) \ : \ \text{(the exterior algebra of } V^*) \\ \underline{\mathrm{grmod}} \ : \ \text{(the stable category)} \end{cases}$$

The BGG correspondence was developed into the theory of Koszul duality.

Let Q be a quiver whose path algebra kQ is finite dimensional and has infinite representation type.

Then we have the following diagram similar to the above diagram.

$$\begin{tikzcd} \mathcal{D}^b(\mathrm{qcoh}\,\Pi(Q)) & \mathcal{D}^b(\text{mod-}kQ) \arrow[l, "M."', "\sim"] \arrow[dl, "\sim", "\text{Happel}"'] \\ \underline{\mathrm{grmod}}\ T(Q) & \end{tikzcd}$$

where

$$\begin{cases} D(kQ) := \mathrm{Hom}_{k\text{-vect}}(kQ, k) \\ \rho := \mathbb{R}\,\mathrm{Hom}_{kQ}(D(kQ), kQ)[1] \\ \Pi(Q) := \bigoplus_{n\geq 0} \rho^{\otimes n} \quad \text{(the preprojective algebra)} \\ T(Q) := kQ \oplus D(kQ) \quad \text{(the trivial extension algebra)} \end{cases}$$

Comparing these diagrams, it is natural to consider that $T(Q)$ and $\Pi(Q)$ are "Koszul dual" to each other. But they are not Koszul algebra in the classical sense. To get a framework including this case, we work with graded dg-algebras.

1. Generalized Koszul duality

Let $\mathcal{A}$ be a dg-algebra over k and

$$\mathcal{R} = \mathcal{A} \oplus P_1\langle -1\rangle \oplus P_2\langle -2\rangle \oplus P_3\langle -3\rangle \oplus \cdots$$

be a connected positively graded dg-algebra over $\mathcal{A}$. (where $\langle n\rangle$ is the graded-degree shift operator by n.) We assume that each term P_i is obtained from $\mathcal{A}\otimes_k \mathcal{A}^{op}$ by taking finite number of cones,shifts and direct summand.

We construct its "Koszul dual" $\mathcal{R}^!$.

Definition 1.

$$Q_n := \left(\bigoplus_{\lambda\models n} \bigotimes_{\mathcal{A}} (P_{\lambda_i}[-\lambda_i + 1]),\ \text{"one-sided-twist"} \right)$$

where the "one-sided-twist" is obtained from the totalization of the bar resolution of $\mathcal{R}$ over $\mathcal{A}$.

$$\mathcal{R}^! := \mathcal{A} \oplus Q_1^\vee[-1]\langle 1\rangle \oplus Q_2^\vee[-2]\langle 2\rangle \oplus Q_3^\vee[-3]\langle 3\rangle \oplus \cdots$$

where we set $(-)^\vee := \mathrm{Hom}_{\mathcal{A}\text{-}bimod}(-, \mathcal{A})$.

This $\mathcal{R}^!$ is a connected negatively graded dg-algebra over $\mathcal{A}$. $\mathcal{R}$ and $\mathcal{R}^!$ satisfy the following properties similar to the classical case

Theorem 2.

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{R}\text{-}\mathrm{gr}}(\mathcal{A},\mathcal{A}) \simeq \mathcal{R}^!$$

$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{R}^!\text{-}\mathrm{gr}}(\mathcal{A},\mathcal{A}) \simeq \mathcal{R}.$$

We can also construct the "Koszul complex " $\mathcal{K}$ which is graded $\mathcal{R}\otimes(\mathcal{R}^!)^{op}$ module. For any graded $\mathcal{R}$ module $\mathcal{M}$, we have

$$\mathrm{Hom}_{\mathrm{gr}\text{-}\mathcal{R}}(\mathcal{K},\mathcal{M}) \simeq \mathbb{R}\,\mathrm{Hom}_{\mathrm{gr}\text{-}\mathcal{R}}(\mathcal{A},\mathcal{M}).$$

From the functor

$$F := \mathrm{Hom}_{\mathrm{gr}\text{-}\mathcal{R}}(\mathcal{K}, -) \ : \ \mathcal{D}(\mathrm{gr}\text{-}\mathcal{R}) \longrightarrow \mathcal{D}(\mathrm{gr}\text{-}\mathcal{R}^!)$$

we obtain the following equivalence

Theorem 3.

$$\mathrm{fg}_{/\mathcal{A}}(\mathrm{gr}\text{-}\mathcal{R}) \simeq \mathrm{Perf}(\mathrm{gr}\text{-}\mathcal{R}^!) \tag{1}$$

where $\mathrm{fg}_{/\mathcal{A}}(\mathrm{gr}\text{-}\mathcal{R})$ (resp. $\mathrm{Perf}(\mathrm{gr}\text{-}\mathcal{R}^!)$) is the full sub triangulated category of $\mathcal{D}(\mathrm{gr}\text{-}\mathcal{R})$ (resp. $\mathcal{D}(\mathrm{gr}\text{-}\mathcal{R}^!)$) generated by $\{\mathcal{A}\langle n\rangle \mid n\in\mathbb{Z}\}$ (resp. $\{\mathcal{R}^!\langle n\rangle \mid n\in\mathbb{Z}\}$).

2. Application

2-1 Let $\mathcal{A}$ be a homologically smooth dg-algebra and P be an invertible $\mathcal{A}$ bi-module and $R = T_{\mathcal{A}}(P)$ be the tensor algebra of P over $\mathcal{A}$.

Theorem 4. *we have the following equivalences by the functor F*

$$\mathrm{Perf}(\mathrm{gr}\text{-}\mathcal{R}) \simeq \mathrm{fg}_{/\mathcal{A}}(\mathrm{gr}\text{-}\mathcal{R}^!) \tag{2}$$

From the equivalences 1 and 2, we get the following corollary which generalize the Happel's theorem.

Corollary 5.

$$\mathrm{Perf}(\mathrm{gr}\text{-}\mathcal{R})/\,\mathrm{fg}_{/\mathcal{A}}(\mathrm{gr}\text{-}\mathcal{R}) \simeq \mathrm{fg}_{/\mathcal{A}}(\mathrm{gr}\text{-}\mathcal{R}^!)/\,\mathrm{Perf}(\mathrm{gr}\text{-}\mathcal{R}^!)$$
$$\simeq \mathrm{Perf}(\mathcal{A})$$

2-2 Let A be a finite dimensional algebra of finite global dimension and P be a tilting A bi-module.

Reversing Serre's vanishing theorem, we define a pair $(\mathcal{D}^{P,\geq 0}, \mathcal{D}^{P,\leq 0})$ of full sub categories of $\mathcal{D}^b(\text{mod-}A)$ as follows.

Definition 6.

$$M^\cdot \in \mathcal{D}^{P,\geq 0} \Longleftrightarrow M \otimes^L P^{\otimes^L n} \in \mathcal{D}^{\geq 0}(\text{mod-}A)$$

$$\textit{resp. } M^\cdot \in \mathcal{D}^{P,\leq 0} \Longleftrightarrow M \otimes^L P^{\otimes^L n} \in \mathcal{D}^{\leq 0}(\text{mod-}A)$$

$$\textit{for } n >> 0$$

Corollary 7. *Assume that $P^{\otimes^L n}$ is pure for every $n \geq 0$, then the following conditions are equivalent*

(i) $(\mathcal{D}^{P,\geq 0}, \mathcal{D}^{P,\leq 0})$ *is a t-structure in* $\mathcal{D}^b(\text{mod-}A)$

(ii) the tensor algebra $T_A(P)$ is a graded coherent ring.