集中講義 10/6, 7, 9.

小平

S two-dim. alg. surface.

$K = \{ J_{jk} \}$: canonical bundle.
$mK = \{ J_{jk}^m \}$

問題
$$\Phi_{mK} : S \longrightarrow \Phi_{mK}(S) \subset \mathbb{P}^n$$
は birational か?

◎ 一次元のとき

C : alg. curve. $\mathfrak{k}$: canonical bundle
genus $g(C)$

Th. $g(C) \geq 2 \Rightarrow \Phi_{m\mathfrak{k}} : C \dashrightarrow \mathbb{P}^n$ is birational for $m \geq 3$.

$g(C) = 1. \Rightarrow \Phi_{m\mathfrak{k}}(C) =$ a point
$g(C) = 0 \Rightarrow \Phi_{m\mathfrak{k}}$ not exist.

Def. S is <u>of general type</u> if S is not one of the following types:

i) rational

ii) (birationally equivalent to) ruled surface $C \times \mathbb{P}^1$

iii) (—) abelian variety

iv) (—) K3 surface

v) (—) elliptic surface.

$$\begin{array}{l} \subset S \\ \quad \big\downarrow {}^{\exists}\pi \\ \quad C \end{array}$$

C の general point p にたいし

$\pi^{-1}(p)$ 、 elliptic curve.

問題 : dim ≥ 3 のときの general type は ?

Notation

$\mathcal{O}^*$: multiplicative sheaf over S of non-vanishing holomorphic functions

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0 \quad (*)$$

$$f \longmapsto e^{2\pi\sqrt{-1}f} \qquad \text{exact.}$$

$F = \{f_{jk}\} \in H^1(S, \mathcal{O}^*)$

$f_{ij} \cdot f_{jk} = f_{ik}$: 1 - cocycle

⊛ より

$$\longrightarrow H^1(S,\mathcal{O}) \longrightarrow H^1(S,\mathcal{O}^*) \longrightarrow H^2(S,\mathbb{Z}) \longrightarrow$$
$$F \longmapsto c(F)$$

Def. $F \cdot G := c(F) \cdot c(G) \in H^4(S,\mathbb{Z}) \cong \mathbb{Z}$

$F^2 := F \cdot F$

Def $[D]$ = the line bundle defined by D

i.e.

$D = \sum m_\lambda C_\lambda$

$S = \cup U_j$

各 U_j で $\exists f_j$: merom. function

$(f_j) = D$.

f_j は C_λ に $|m_j|$ 次の zero or pole をもつ.

$f_{jk} := f_j / f_k$

$[D] := \{f_{jk}\}$.

Def. $F \cdot D := F\,[D]$

Def. $D_1 \cdot D_2 := [D_1] \cdot [D_2]$ = intersection number of D_1 and D_2

Def. C is <u>exceptional</u> if $C^2 = -1$.

and if C is non-singular rational
(i.e. $\cong \mathbb{P}^1$)

いま、

$$\begin{array}{ccc} S & \supset & C \text{ exceptional} \\ \downarrow & & \downarrow \\ \tilde{S} & \supset & p \text{ (point)} \end{array}$$

とする。

$$\tilde{S} = (S - C) \cup p$$

と "reduction" でするから

仮定: S contains no exceptional curve (A)

$p_g := \dim H^0(S, \mathcal{O}(K))$ <u>geometric genus</u> of S

$P_m := \dim H^0(S, \mathcal{O}(mK))$ <u>m-genus</u> (pluri-genus)

$p^{(1)} := K^2 + 1$: linear genus

Main theorem$^{(A)}$: Assume S is of general type
$\Phi_{mK} : S \longrightarrow \mathbb{P}^n$ is holomorphic
birational for $m \geq 5$.

(Enriques)

\# Theorem. S is of general type (A)

$\Longleftrightarrow$ $P_2 > 0$ $K^2 > 0$

Main theorem (続き)

If $K^2 \geqq 2$, then Φ_{4K} is hol. birational

If. $K^2 \geqq 3$ and if $p_g \geqq 3$, then Φ_{3K} is hol. birational.

注 1. m: suff. large $\Rightarrow$ Φ_{mK} is hol. birat.

cf. Mumford Ann. of. Math 76 '62

Appendix to Zariski's paper.

∘ Šafarevič: 1965

$m \geqq 9$ $\Rightarrow$ Φ_{mK} birational

$p_g = 2$, $m \geqq 5$ $\Rightarrow$ 〃 .

$p_g = 3$, $m \geqq 4$ $\Rightarrow$ 〃 .

$p_g \geqq 4$, $m \geqq 3$ $\Rightarrow$ 〃 . } △

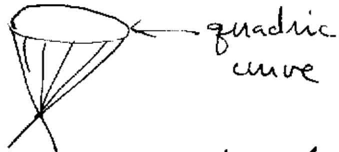
∘ $p_g \geqq 4$ $\Rightarrow$ $K^2 \geqq 3$

∘ $p_g \geqq 3$ $\Rightarrow$ $K^2 \geqq 2$) より M-T (続)

は △ の改良。

注2. この M−T は これ以上改良できない
(optimum)

∘ $K^2 = 1,\ p_g = 2 \Rightarrow \Phi_{4K}(S) = $ quadric cone



$\therefore\ \Phi_{4K}$ is not birational

∘ $K^2 = 2,\ p_g = 3 \Rightarrow \Phi_{3K}(S) = \mathbb{P}^2$
$\therefore\ \Phi_{3K}$ is not birat.

∘ $S = C_1 \times C_2$
$g(C_1) = 2 \qquad g(C_2) = k \geqq 2$ とする.
$K^2 = 8k - 8$
$p_g = 2k$
$\therefore\ \Phi_{2K}(S) = $ ruled surface
$\therefore\ \Phi_{2K}$ is not birat.

問題 : If Φ_{2K} is not birational then S contains a one-parameter family of curves of genus 2 ?

; Enriques ⇒ example.

§ Idea of the proof

F : line bundle $\qquad x$: a pt on S

$\mathcal{O}(F-x)$ = subsheaf of $\mathcal{O}(F)$ - - -

$$= \{ \varphi \in \mathcal{O}(F) \ ; \ \varphi(x) = 0 \}$$

∘ $\mathcal{O}(F-x)_z = \mathcal{O}(F)_z \qquad z \neq x.$

$$0 \to \mathcal{O}(F-x) \to \mathcal{O}(F) \longrightarrow \mathbb{C}_x \to 0.$$

exact.

ただし、$(\mathbb{C}_x)_z = \begin{cases} 0 & z \neq x \\ \mathbb{C} & z = x. \end{cases}$

∴

$$0 \longrightarrow H^0(S, \mathcal{O}(F)) \longrightarrow \mathbb{C} \longrightarrow H^1(S, \mathcal{O}(F-x)) \to$$

$$\begin{array}{ccc} \psi & & \psi \\ \varphi & \longmapsto & \varphi(x) \end{array}$$

If $H^1(S, \mathcal{O}(F-x)) = 0$ then $\exists \varphi \in H^0(F)$ with $\varphi(x) \neq 0$

∘ If $H^1(S, \mathcal{O}(F-x)) = 0$ for all $x \in S$

then $|F|$ has no base point and

Φ_F is holomorphic

$x \neq y \in S$

$\mathcal{O}(F - x - y) = \{ \varphi \in \mathcal{O}(F) \ ; \ \varphi(x) = \varphi(y) = 0 \}$

$$\begin{array}{ccccccc} \to H^0(\mathcal{O}(F)) & \longrightarrow & \mathbb{C}^2 & \longrightarrow & H^1(\mathcal{O}(F - x - y)) \to \\ \psi & & \psi & & \\ \varphi & \longmapsto & (\varphi(x), \varphi(y)) & & \end{array}$$

o If $H^1(\mathcal{O}(F - x - y)) = 0$ for all $x, y \in S$

$x \neq y$, then $\exists \varphi \in H^0(F)$ with $\varphi(x) = 0$, $\varphi(y) \neq 0$

$\exists \psi \in H^0(F)$ with $\psi(x) \neq 0$, $\psi(y) = 0$.

$\therefore \ \Phi_F(x) \neq \Phi_F(y)$

i.e.

o If $H^1(\mathcal{O}(F - x - y)) = 0$ for all $x, y \in S$, $x \neq y$

$\Longrightarrow \ \Phi_F$ is one to one.

To prove Φ_F is a hol. birat. map,

it suffices to show that

$H^1(S, \mathcal{O}(F - x)) = 0$ for all $x \in S$

$H^1(S, \mathcal{O}(F - x - y)) = 0$ for all $x, y \in S - \bigcup_{\lambda=1}^{l} C_\lambda$

$(C_\lambda$: curve$)$

約束 : C, C_i are irreducible curves

◎ $H^1(S, \mathcal{O}(F - x - y))$ の計算

suppose $\exists C \ni x, y$

def. $\mathcal{O}(F - D) = \{\varphi \in \mathcal{O}(F) \,;\, (\varphi) \geq D\}$

$\mathcal{O}(F - C) = \{\varphi \in \mathcal{O}(F) \,;\, \varphi(z) = 0$ for all $z \in C\}$

$$\mathcal{O}(F - C) \hookrightarrow \mathcal{O}(F - x - y) \to \mathcal{O}(F - x - y)_C \to 0$$

これから,

$$\to H^1(S, \mathcal{O}(F - C)) \to H^1(S, \mathcal{O}(F - x - y)) \to H^1(C, \mathcal{O}(F - x - y)_C)$$

def. $F_C = [f]_C$ ↑ divisor F の C への restriction

k : C の canonical divisor

duality th. により

$$H^1(C, \mathcal{O}(F - x - y)_C) \cong H^0(C, \mathcal{O}(k - f + x + y)_C)$$

degree $(k - f + x + y)$

$= KC + C^2 - FC + 2$

$$\mathfrak{f} = K_C + [C]_C$$

$$\deg(\mathfrak{f} - f + \sum_{\lambda=1}^{r} x_\lambda)_C = KC + C^2 - FC + r$$

Theorem 1

$x_\lambda \in \cancel{S}\ C \qquad FC - KC - C^2 \cancel{\geq}\ > r$

$$\Rightarrow H^1(C, \mathcal{O}(F - \bigcup_{\lambda=1}^{r} x_\lambda)) = 0$$

This theorem holds for any irreducible curve C on S.

（x_λ は simple pt とする必要はない）

Proof. J. of MSJ. 1968, 20, 170 ~ 192.

$x_1, \cdots, x_\ell \in C$ simple pts

$x_{\ell+1}, \cdots, x_r \in C$ singular pts のときは

$FC - KC - C^2 > \ell$ でよい。

<u>Theorem 2</u>. If $\exists$ integer $m > 0$ such that $|mF|$ has no base point, then

$$H^1(S, \mathcal{O}(K+F)) = 0$$

<u>Proof</u> $\Phi_{mK} : S \longrightarrow \mathbb{P}^N$ hol

k.f. $\longleftarrow$ kähler form

induced form Σ

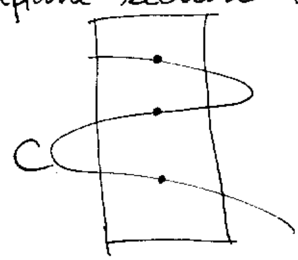
$$\gamma = \frac{\sqrt{-1}}{2\pi} \sum \gamma_{\alpha\beta}\, dz^{\alpha} \wedge d\bar{z}^{\beta}$$

∘ γ : real $\qquad d\gamma = 0$

∘ $\Phi_{mF}(S)$ is a surface in $\mathbb{P}^n$

∵) Assume $\Phi_{mF}(S) = C$ is a curve

hyperplane section ϑ in C is a divisor



$$D = \Phi_{mF}^{-1}(\vartheta) \quad : \text{ divisor on } S$$

$$[D] = mF$$

ϑ' : be a plane section

$$\vartheta \cap \vartheta' = \phi$$

$$D' = \Phi_{mF}^{-1}(\vartheta')$$

$$D' \cap D = \phi$$

$$\therefore\ m^2 F^2 = [D][D'] = D \cdot D' = 0$$

これは $F^2 > 0$ に矛盾.

q.e.d

Rk

$S - N$

$$N = \{ z \in S \ ;\ \text{Jacob}\, \Phi_{mF}(z) \leq 1 \}$$

∘ N : proper subvariety

Φ_{mF} : loc. bihol on $S-N$

∘ γ is positive definite on $S-N$

$$c(F) \sim \frac{1}{m}\gamma \qquad (c(mF) \backsim \gamma)$$

∴ by vanishing theorem

$$H^1(S, \mathcal{O}(K+F)) = 0$$

cf. Mumford : Amer. J. Math. 68 '69
94 ~ 104

Def. $\pi(D) = \frac{1}{2}(D^2 + KD) + 1$: <u>virtual genus</u>

○ 1) C : irred. non-singular $\Rightarrow \pi(C) =$ genus of C.

2) $\tilde{C}$ = non-singular model

$\Rightarrow \quad \pi(C) \geqq$ genus of $\tilde{C}$

このとき, "=" $\Leftrightarrow$ C : non-singular.

3) $\pi(C) = 0 \Leftrightarrow C$ is non-singular rational

i.e. $C \cong \mathbb{P}^1$

<u>Lemma 2</u> $\quad KC \geqslant 0$

$KC = 0 \Leftrightarrow \pi(C) = 0$ and $C^2 = -2$.

Proof. ① of general type より $\dim|2K| = P_2 - 1 \geqslant 0$

~~KC < 0 とすると~~ Hence $|2K| \ni \exists D > 0$

($D = 0$ とすると $K^2 = 0$ となり矛盾)

$KC < 0$ とすると

$2KC = DC < 0$

$\therefore \quad D = \sum n_i C_i$ とすると $n_i > 0$

$DC = \sum n_i C_i C$

$C_i C \geqslant 0$ for $C \neq C_i$ より

ある i について $C = C_i$

$\therefore \quad C^2 < 0$

$\therefore \quad \underset{\overset{\vee|}{0}}{2\pi(C)} - 2 = KC + C^2 \leqq -2$

$\therefore \pi(C) = 0, \; C^2 = KC = -1$

i.e. C : exceptional

これは 仮定に 反する。

$\therefore \quad KC \geqq 0$

② $KC = 0$ とする. $\left.\begin{array}{l} KC = 0 \\ C \not\sim 0 \end{array}\right\} \overset{\text{Lemma 1}}{\Longrightarrow} \quad C^2 < 0$

$\therefore$

$$-2 \leq 2\pi(C) - 2 = KC + C^2$$
$$= C^2 < 0$$

$\therefore \quad \pi(C) = 0 \qquad C^2 = -2$

q. e. d.

Def. $\Lambda_\ell = \{ C \subset S \,;\, KC \leq \ell,\ C^2 < 0 \}$

<u>Lemma 3</u> Λ_ℓ is a finite set

Proof. $C \sim r_0 K + \sum_{i=1}^{b} r_i B_i$

$$r_0 = \frac{KC}{K^2}, \quad r_i = \frac{B_i C}{B_i^2}$$

$$0 > C^2 = r_0^2 K^2 + \sum r_i^2 B_i^2$$
$$= r_0^2 K^2 - \sum r_i^2 |B_i^2|$$

$0 \leqq r_0 \leqq \dfrac{\ell}{K^2}$ だから

$$\sum |B_i^2| r_i^2 \leq \frac{l^2}{K^2} + l + 2$$

$\therefore$ There are only a finite number of $(r_0, r_1, \cdots, r_l)$

$\therefore$ The number of homology classes of $C \in \Lambda_l$ is finite.

一方，$C' \sim C$ とすると

$$C'C = C^2 < 0$$

If. $C' \neq C \Rightarrow C'C \geq 0$

$\therefore C' \sim C \Rightarrow C' = C$.

Hence homology class of C contains no irred. curve other than C.

$\therefore \quad \# \Lambda_l < \infty$

q.e.d

Lemma 4 (Zariski) e : integer > 0

$\dim |eK| \geq 0$.

Then there ~~are~~ exists $m_0(e)$ such that

$$\dim H^1(S, \mathcal{O}((m-e)K)) = \dim H^1(mK)$$

for $m \geq m_0(e)$

$$H^1(F) = H^1(\mathcal{O}(F)) = H^1(S, \mathcal{O}(F))$$

Proof. $|eK| \ni D > 0$

<u>Case 1) $D = C$ is an irred. curve</u>

$$0 \to \mathcal{O}(mK - C) \hookrightarrow \mathcal{O}(mK) \longrightarrow \mathcal{O}(mK)_C \to 0$$

$$\mathcal{O}(mK-C) \cong \mathcal{O}((m-e)K)$$

より

$$\to H^1((m-e)K) \to H^1(mK) \to H^1(C, \mathcal{O}(mK)_C) \to \cdots$$

By theorem 1 (p. 10)

$$mKC - KC - C^2 > 0 \Rightarrow H^1(C, \mathcal{O}(mK)_C) = 0$$

$$mKC - KC - C^2 = (m-e-1)KC = (m-e-1)eK^2$$

$\therefore$

$$m \geqq e+2 \Rightarrow H^1(C, \mathcal{O}(mK)_C) = 0$$

$\therefore$

$$m \geqq e+2 \Rightarrow \dim H^1((m-e)K) \geqq \dim H^1(mK)$$

$d(m) := \dim H^1(mK)$ とおくと

$$d(m) \geqq d(m+e) \geqq d(m+2e) \geqq \cdots \geqq 0$$
$$(m \geqq e+2)$$

$\therefore$

$\exists n_0$, $\quad d(m + n_0 e) = d(m + (n_0+1)e) = \cdots$

q. e. d.

Case 2) $|eK| \ni D = C_1 + C_2 + \cdots + C_i + \cdots + C_n$

$$D_{i-1} = C_1 + \cdots + C_{i-1}$$
$$Z_i = C_i + \cdots + C_n$$

$$0 \to \mathcal{O}(mK - C_n) \longrightarrow \mathcal{O}(mK) \to \mathcal{O}(mK)_{C_n} \to 0$$

$$0 \to \mathcal{O}(mK - C_n - C_{n-1}) \to \mathcal{O}(mK - C_n) \to \mathcal{O}(mK - C_n)_{C_{n-1}} \to 0$$

$$0 \to \mathcal{O}(mK - Z_i) \to \mathcal{O}(mK - Z_{i+1}) \to \mathcal{O}(mK - Z_{i+1})_{C_i} \to 0$$

$F_i := mK - [Z_i]$ K : canonical bdl.
とおくと．

$$0 \to \mathcal{O}(F_i) \to \mathcal{O}(F_{i+1}) \to \mathcal{O}(F_{i+1})_{C_i} \to 0$$

ただし， $\mathcal{O}(F_{n+1}) = \mathcal{O}(mK)$
$$\mathcal{O}(F_1) \cong \mathcal{O}(mK - D)$$
$$\cong \mathcal{O}((m-e)K)$$

To prove $\dim H^1((m-e)K) \geqq \dim H^1(mK)$
it suffices to show
$$\dim H^1(F_i) \geqq \dim H^1(F_{i+1})$$
$$\Uparrow$$
$$H^1(C_i, \mathcal{O}(F_{i+1})_{C_i}) = 0$$

Th 1 により,

$$F_{i+1}C_i - KC_i - C_i^2 > 0 \Longrightarrow H^1(C_i, \mathcal{O}(F_{i+1})C_i) = 0$$

$$= (mK - Z_{i+1} - K - C_i)C_i$$

$$= (mK - K - Z_i)C_i$$

$$= ((m-e-1)K + D_{i-1})C_i$$

結局,

$$(m-e-1)KC_i + D_{i-1}C_i > 0 \quad (i = 1, 2, \cdots, n)$$

$$\Longrightarrow \dim H^1((m-e)K) \geqq \dim H^1(mK)$$

$$\left(\Longrightarrow \dim H^1((m-e)K) = \dim H^1(mK) \quad \text{for } m \geqq m_0(e)\right)$$

Assume $m \geqq e+2$ として

$$KC_i + D_{i-1}C_i > 0 \quad i = 1, 2, \cdots, n$$

をいえばよい。

このために $D = C_1 + C_2 + \cdots + C_n$

の順序をうまくとることが重要

<u>Lemma 5</u> If $D \in |eK| \quad e > 0$

$D = X + Y \quad X > 0 \quad Y > 0$

then $XY \geqq 1$.

<u>Proof.</u> $X \sim r_0 K + \sum r_i B_i \qquad r_0 = \dfrac{XK}{K^2} \geq 0$

$Y \sim s_0 K + \sum s_i B_i \qquad s_0 \geq 0$

$$eK \backsim X+Y = \underbrace{(r_0+s_0)}_{\overset{\shortparallel}{e}} K + \sum \underbrace{(r_i + s_i)}_{\overset{\vee}{0}} B_i$$

$\therefore$

$$XY = r_0 s_0 K^2 + \sum r_i^2 (-B^2) \geq 0$$

もし $XY = 0$ なら

$$r_0 s_0 = r_i^2 = 0 \quad \therefore X = 0 \text{ or } Y = 0$$

となり矛盾.

$\therefore \quad XY \geq 1 \qquad (\in \mathbb{Z}$ ゆえ$)$

<u>Lemma 6</u> We can choose the order of C_i such that $KC_1 \geq 1$

$D_{i-1} C_i \geq 1 \qquad i = 2, \cdots, n.$

<u>Proof.</u> $D = \sum_{\lambda=1}^{n} C^{(\lambda)}$ とする.

$$0 < e \leq eK^2 = \sum_{\lambda=1}^{n} KC^{(\lambda)}$$

Hence some $KC^{(\alpha)} \geq 1$.

Let $C_1 = C^{(\alpha)} \qquad KC_1 \geq 1.$

$\therefore$

$$D = C_1 + Y^{(1)} \qquad Y^{(1)} = \sum_{\lambda \neq \alpha} C^{(\lambda)}$$

By Lemma 5 $\quad C_1^* Y^{(1)} \geqq 1$.

i.e. $\quad \sum_{\lambda \neq \alpha} C_1 C^{(\lambda)} \geqq 1$

We find $\beta \neq \alpha$ with $C_1 C^{(\beta)} \geqq 1$.

Let $C_2 = C^{(\beta)}$

$$D = \underbrace{C_1 + C_2}_{\substack{\shortparallel \\ D_2}} + Y^{(2)} \qquad Y^{(2)} = \sum_{\lambda \neq \alpha, \beta} C^{(\lambda)}$$

By Lemma 5

$D_2 Y^{(2)} \geqq \underset{\exists}{1}$

これから $\quad D_2 C_3 \geqq 1 \quad$ ------

q.e.d.

Lemma 6 により Lemma 4 は q.e.d.

注 We can choose any $C^{(\lambda)}$ with $KC^{(\lambda)} \geqq 1$ as C_1.

<u>Lemma 7</u> $|ek| \ni D = \sum_{\lambda=1}^{m} C^{(\lambda)}$; $KC^{(\alpha)} = 0$.

$\Rightarrow$ We can choose

$$D = C_1 + C_2 + \cdots + C_n$$

such that

① $C_n = C^{(\alpha)}$

② $KC_i + D_{i-1} C_i \geqq 1 \quad i = 1, 2, \cdots, n$

Proof. 略.

Theorem 3. Assume $\dim |eK| \geq 1$, $eK^2 \geq 2$.
(e : positive integer)
If $m \geq e+2$ and if $m \geq m_0(e)$
then $|mK|$ has no base point
and Φ_{mK} is holomorphic

Proof

$$0 \to \mathcal{O}(mK - x) \to \mathcal{O}(mK) \to \mathbb{C}_x \to 0$$

より,

$$\to H^0(mK) \to \mathbb{C} \to H^1(mK - x) \to H^1(mK) \to 0.$$

It suffices to prove

$$\begin{array}{ccccl} H^0(mK) & \longrightarrow & \mathbb{C} & \to 0 & \text{for every } x \in S \\ \Psi & & \Psi & & \\ \varphi & \longmapsto & \varphi(x) & & \end{array}$$

i.e.

It suffices to prove
$\dim H^1(mK - x) \leq \dim H^1(mK)$

$\exists$
$H^0(eK) \ni \varphi_0, \varphi_1$: lin. independent
($\dim H^0(eK) = |eK| + 1 \geq 2$)
$\varphi = a_0\varphi_0 + a_1\varphi_1$ $(a_0, a_1) \neq (0,0)$
$D = (\varphi) \ni x$.
∘ $|eK| \ni D$
$D = C_1 + C_2 + \cdots + C_n$

$KC_1 \geq 1$, $D_{i-1}C_i \geq 1$ $(i \geq 2)$ としてよい。

$$\text{def. } \underset{\Xi_i}{\overset{\parallel}{\mathcal{O}(mK-x-Z_i)}} = \mathcal{O}(mK-x) \cap \mathcal{O}(mK-Z_i)$$

$$\begin{array}{ccccccccccc}
\Xi_1 & \subset & \Xi_2 & \subset \cdots \subset & \Xi_i & \subset & \Xi_{i+1} & \subset & \cdots & \subset & \Xi_{n+1} \\
\parallel & & & & & & & & & & \parallel \\
\mathcal{O}(mK-D) & & & & & & & & & & \mathcal{O}(mK-x) \\
\parallel & & & & & & & & & & \\
\mathcal{O}((m-e)K) & & & & & & & & & &
\end{array}$$

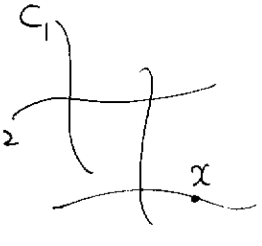
$m \geq m_0(e)$ より

$\dim H^1(\Xi_1) = \dim H^1(mK)$

∴ It suffices to show

$$\dim H^1(\Xi_i) \geq \dim H^1(\Xi_{i+1})$$

注 C_i's are not all different.



$$D = C_1 + \cdots + \underset{\substack{\cup\!\!\!\shortmid \\ x}}{C_h} + \underbrace{C_{h+1} + \cdots + C_n}_{\not\ni x}$$

とhをとる。

$$
\Xi_i = \begin{cases} \mathcal{O}(mK - Z_i) & i \leq h \\ \mathcal{O}(mK - x - Z_i) & i \geqslant h+1 \quad (x \notin Z_i) \end{cases}
$$

∴

C_{i+1}

$$
\Xi_{i+1} / \Xi_i = \mathcal{O}(mK - Z_{i+1} - \delta_{ih} x)
$$

$$
0 \to \Xi_i \to \Xi_{i+1} \to \Xi_{i+1}/\Xi_i \to 0
$$

より,

$$
H^1(C_i, \mathcal{O}(mK - Z_{i+1} - \delta_{ih} x)_{C_i}) = 0
$$

をいえばよい。 Th 1 より

$$
(mK - Z_{i+1} - K - C_i) C_i > \delta_{ih}
$$

i.e.

$$
\underbrace{(m-e-1)}_{\overset{\vee|}{1}} KC_i + D_{i-1} C_i > \delta_{ih}
$$

By hypothesis $m \geqslant e + 2$

It suffices to show

$$
KC_i + D_{i-1} C_i > \delta_{ih}.
$$

$\left(\begin{matrix} KC_1 \geqslant 1 \\ KC_i \geqslant 0 \end{matrix} \right)$

- <u>Case I $KC_h = 0$</u> ~~(will be proved 明後日~~)

- <u>Case II $KC_h \geqslant 1$</u>

<u>Case II 1) $h \geqslant 2$</u>

$KC_n = 0$, $\quad x \in C_n$.

Let $\Xi_i = \mathcal{O}(mK - Z_i)$

$$\begin{array}{ccccccccc} \Xi_1 & \subset & \cdots & \subset \Xi_i & \subset \Xi_{i+1} & \subset & \cdots & \subset & \Xi_{n+1} \\ \| & & & & & & & & \| \\ \mathcal{O}((m-e)K) & & & & & & & & \mathcal{O}(mK) \end{array}$$

$$\Xi_{i+1} / \Xi_i = \mathcal{O}(mK - Z_{i+1})_{C_i}$$

$KC_i + D_{i-1}$ $(i \geqq 1)$ より

$$H^1(C_i, \mathcal{O}(mK - Z_{i+1})_{C_i}) = 0.$$

$$\therefore \quad \dim H^1(\Xi_i) \geqq \dim H^1(\Xi_{i+1})$$

Hence

$$\begin{array}{ccc} \dim H^1((m-e)K) & \geqq & \dim H^1(\overbrace{mK - C_n}^{\Xi_n}) \\ \| & & \\ \dim H^1(mK) & & \end{array}$$

$\therefore$

$$0 \to \mathcal{O}(mK - C_n) \to \mathcal{O}(mK) \to \mathcal{O}(mK)_{C_n} \to 0.$$

$KC_n = 0$ より, $\pi(C_n) = 0$

$\therefore$ K_{C_n} は trivial bundle.

$$\therefore \quad \mathcal{O}(mK)_{C_n} \cong \mathcal{O}_{C_n}$$

$$\therefore \; \longrightarrow H^0(\mathcal{O}(mK)) \to H^0(C_n, \mathcal{O}_{C_n}) \to H^1(\mathcal{O}(mK - C_n))$$

$$\longrightarrow H^1(mK) \longrightarrow H^1(C_n, \mathcal{O}_{C_n}) = 0 \quad (C_n : \text{rational ゆえ})$$

$\dim H^1(mK) \geqq \dim H^1(mK - C_n)$ だから

$$\begin{array}{ccccc} \longrightarrow H^0(mK) & \longrightarrow & H^0(C_n, \mathcal{O}_{C_n}) & \longrightarrow & 0 \\ & & \| & & \\ \cup & & \mathbb{C} & & \\ & & \cup & & \\ \varphi & \longmapsto & \varphi_{C_n}. & & \end{array}$$

Hence there exists φ such that

$\varphi_{C_n} \neq 0$

$\therefore \; x \in C_n$ より

$\varphi(x) \neq 0.$

i.e. x は base point ではない。

Q. E. D.

Lemma 4 $e > 0$ integer

$\dim|eK| \geq 0$

$\Rightarrow$ $\exists\, m_0(e)$ such that $\dim H^1((m-e)K) = \dim H^1(mK)$ for $m \geq m_0(e)$

Theo 3 $\dim|eK| \geq 1$, $eK^2 \geq 2$, $m \geq m_0(e)$

$m \geq e + 2$

$\Rightarrow$ $|mK|$ has no base point and

$\Phi_{mK} : S \longrightarrow \mathbb{P}^n$ is holomorphic

S : general type 曲

$\dim|2K| = P_2 - 1 \geq 0$

$\therefore$ $|2K| \ni D > 0$

Hence $H^0((-nK)) = 0$ for $n > 0$

by R-R th.

$$\dim|eK| \geq \frac{1}{2}(e^2 - e)K^2 + p_g - q$$

$$\longrightarrow +\infty \quad (e \to +\infty)$$

q = irregularity of S = $\dim H^1(S, \mathcal{O})$

Hence $\dim|eK| \geq 1$ for some $e > 0$

Hence $|mK|$ has no base pt, Φ_{mK} is holomorphic for $m > \exists M_0$

By Theo 2

$\dim H^1(K+F) = 0$ if $\exists m > 0$ such that $|mF|$ has no base point,

~~therefore~~

$$\dim H^1(nK) = \dim H^1(K + \underbrace{(n-1)K}_{F})$$

if $n \geq 2$ $|m(n-1)K|$ has no base point for large m

Hence by theo 2.

<u>Theo 4</u> $\dim H^1(S, \mathcal{O}(nK)) = 0$ for $n \geq 2$.

<u>Cor 1</u> $m_0(e) = e + 2$ としてよい。

◎ For S of general type $p_g - q \geq 0$
(a classical result ?
cf. Van de Ven's paper on distributions of (c_1^2, c_2) Nat Acad Sci USA 1967 ?)
: For general S $c_1^2 = K^2 \geq 1$,

$$8c_2 \geq c_1^2 \geq 1$$

$$\therefore \quad c_2 > 0.$$

by Noether's formula

$$12 \leq c_1^2 + c_2 = 12(p_g - q + 1)$$

For $m \geq 2$

$$P_m = \dim H^0(mK) \underset{\substack{\uparrow \\ \text{R-R.}}}{=} \frac{1}{2} m(m-1)K^2 + p_g - q + 1$$

& $H^2(mK) = H^0((1-m)K) = 0$

<u>Cor 2</u> ∴ $P_m \geq \frac{1}{2} m(m-1)K^2 + 1$

とくに $P_2 \geq 2$

$P_3 \geq 4$

For $e = 2$ $\dim |2K| = P_2 - 1 \geq 1$, $2K^2 \geq 2$

$m_0(e) = 4 = e + 2$

hence by theo 3.

<u>Cor 3</u> $|mK|$ has no base point for $m \geq 4$.

(and Φ_{mK} is hol. for $m \geq 4$).

For $e = 1$. $\dim |K| = p_g - 1 \geq \overset{1}{\cancel{0}}$ $K^2 \geq 2$

$m_0(e) = 3$.

hence,

<u>Cor 4</u> if $p_g \geq 2$, $K^2 \geq 2$, $m \geq 3$, then $|mK|$ has no base point, Φ_{mK} is hol.

§ Birational maps

Theo 5. If $\dim |eK| = p_e - 1 \geq 2$, $eK^2 \geq 3$

$m \geq e + 2$,

then Φ_{mK} is a holomorphic birational map.

<u>Proof</u> $\Lambda_1 = \{ C \text{ irred curve} \,;\, KC \leq 1,\, C^2 < 0 \}$

is a finite set by Lemma 3.

$$\mathcal{C} = \bigcup_{C \in \Lambda_1} C$$

It suffices to prove that $H^1(mK - x - y)) = 0$

for $x, y \in S - \mathcal{C}$

$$\left(\text{cf.} \to H^0(mK) \longrightarrow \mathbb{C}^2 \longrightarrow H^1(mK - x - y) \to \right)$$
$$\varphi \longmapsto (\varphi(x), \varphi(y))$$

<u>Lemma 8</u> If $KC \equiv 1$, $C^2 > 0$, then $K^2 = 1, C^2 = 1$

$K \sim C$ over $\mathbb{Q}$ where K denotes a canonical

divisor on S.

<u>Proof.</u> $C \sim r_0 K + \Sigma r_i B_i \qquad r_0 = \dfrac{KC}{K^2} = \dfrac{1}{K^2}$

$$0 < C^2 = \underset{\substack{\| \\ 1}}{r_0^2 K^2} + \Sigma_i r_i^2 \underset{\substack{\wedge \\ 0}}{B_i^2} \leq \frac{1}{K^2} \leq 1.$$

$\because C^2 = 2\pi(C) - 3 = -3, -1, +1, 3, \cdots$

Hence $C^2 = 1 \quad K^2 = 1,\ r_i = 0,\ r_0 = 1,\ C \sim K.$

Proof of $H^1(mK-x-y)=0$ for $x,y\in S-\mathcal{C}$

Case I) $\exists C$ such that $x\in C$, $y\in C$, $KC=1$

$$0\to \mathcal{O}(mK-C)\to\mathcal{O}(mK-x-y)\to\mathcal{O}(mK-x-y)_C\to 0$$

より

$$\to H^1(mK-[C])\to H^1(mK-x-y))\to H^1(C,\mathcal{O}(mK-x-y)_C)\to 0$$

by th 1.

$$mK\cdot C-K\cdot C-C^2>2 \quad\Rightarrow H^1((mK-x-y)_C)=0$$

$\forall$ $C^2<0$ とあると $C\subset\mathcal{C}$ 故, $c^2>0$.

hence $C\sim K$

$$mK\cdot C-K\cdot C-C^2=(m-2)K^2\geqq eK^2\geqq 3.$$

hence $H^1((mK-x-y)_C)=0$.

by th 2. if $|mF)$ has no base point for some $m>0$ then $H^1(K+F)=0$. Let

$$mK-[C]=K+F\,,\quad F=(m-2)K+K-[C]$$

since $K\sim C$, for some positive h,

$h(K-C)\sim 0$ over $\mathbb{Z}$, hence $h\cdot c([K-C])=0$.

$$\therefore\quad c(4hF)=c((m-2)4hK)$$

$$(m-2)4h\geqq 4eh\geqq 4$$

$\therefore$ $|(m-e)4hK|$ has no base point.

recall the proof of Theo 2.: $|mF|$ has no base point $\Rightarrow$ $c(mF) \sim \gamma$ (positive definite Hermitian 2-form with $d\gamma = 0$) $\therefore$ only Chern class counts for $H^1(K+F) = 0$.

hence by theo 2 and its proof,

$$H^1(mK - C) = 0 \quad \text{for } m \geq e+2.$$

Hence. $H^1(mK - x - y) = 0$.

Case II) $x \in C, y \in C \Rightarrow KC \geq 2$.

$\exists D \in |eK|$ such that $x \in D, y \in D$ $(\dim|eK| \geq 2)$

By lemma 6.

$$D = \underbrace{C_1 + \cdots + C_{i-1}}_{D_{i-1}} + \underbrace{C_i + \cdots + C_n}_{Z_i}$$

$KC_1 \geq 1$

$D_{i-1} C_i \geq 1 \qquad 2 \leq i \leq n.$

$$\Xi_i = \mathcal{O}(mK - x - y - Z_i)$$

$$\mathcal{O}((m-e)K) \cong \mathcal{O}(mK - D) = \Xi_1 \subset \Xi_2 \subset \cdots$$
$$\subset \Xi_i \subset \Xi_{i+1} \subset \cdots \subset \Xi_{n+1} = \mathcal{O}(mK - x - y).$$

By theo 4. $H^1((m-e)K) = 0$.

$\therefore$ To prove $H^1(\mathcal{O}(mK - x - y)) = 0$ it suffices to show $\dim H^1(\Xi_i) \geq \dim H^1(\Xi_{i+1})$

i.e. to show $H^1(\Xi_{i+1}/\Xi_i) = 0$.

$$\underset{\substack{\Psi\\ x}}{C_1 + \cdots + C_h} + \cdots + \underset{\substack{\Psi\\ y}}{C_j} + \underbrace{\cdots + C_n}_{\not\ni y}$$

$$\underbrace{C_h + \cdots + C_j + \cdots + C_n}_{\not\ni x}$$

とする。

$$\Xi_i = \begin{cases} \mathcal{O}(mK - Z_i) & i \leq h \\ \mathcal{O}(mK - x - Z_i) & h+1 \leq i \leq j \\ \mathcal{O}(mK - x - y - Z_i) & i \geq j+1 \end{cases}$$

∴

$$\Xi_{i+1}/\Xi_i = \mathcal{O}(mK - \delta_{ih}x - \delta_{ij}y - Z_i)_{C_i}$$

by theo 1.

$$(mK - Z_i)C_i - KC_i - C_i^2 > \delta_{ih} + \delta_{ij}$$

$$\Rightarrow H^1(\Xi_{i+1}/\Xi_i) = 0$$

$$= (m-e-1)KC_i + D_{i-1}C_i \geq KC_i + D_{i-1}C_i$$

It suffices to verify $KC_i + D_{i-1}C_i > \delta_{ih} + \delta_{ij}$

Case II 1) $\exists C_i$ such that $x \notin C_i$, $y \notin C_i$, $KC_i \geq 1$.

We may assume $x \notin C_1$, $y \notin C_1$

cf. remark of lemma 6.

$\therefore\ h \geq 2,\ j \geq 2.$

if $h=j$ then $KC_h \geq 2$.

lemma 2: $KC = 0 \Rightarrow C^2 = -2$, hence $C \in \mathcal{C}$

より, $x \notin \mathcal{C}$, $y \notin \mathcal{C}$ だから

$KC_h \geq 1$, $KC_j \geq 1$.

<u>Case II 2). $KC_i \geq 1 \Rightarrow x \in C_i$ or $y \in C_i$</u>

<u>II 2a)</u> $h \geq 2$, $j \geq 2$ the same as II 1)

<u>II 2b)</u> $h = 1$, $j \geq 2$, $KC_1 = 1$.

<u>II 2b)</u> $h=1, j \geq 2$, $KC_1 \geq 2$ O.K.

$$3 \leq eK^2 = KD$$

$$= \underset{\substack{\Psi \\ x}}{KC_1} + \cdots + \underset{\substack{\Psi \\ y}}{KC_j} + \cdots$$

if $KC_j \geq 2$ $\quad C_j$ を C_1 におきかえる
$x \leftrightarrow y$.

if $KC_j = 1$, then $\exists\, C_\ell \ni y$ with $KC_\ell = 1$.

∘ $1 < \ell < j$

change the order

$$\underset{\substack{\Psi \\ y}}{C_1} + \cdots + \underset{\substack{\Psi \\ x}}{C_h} + \cdots + \underset{\substack{\Psi \\ y}}{C_j'}$$

then reduced to the case II 2 a).

<u>II 2d)</u> $h = j = 1$

II 2) の仮定より $KC_\ell = 0 \quad \ell \geq 2.$

$\therefore \quad 3 \leq eK^2 = KD = KC_1 \quad$ O.K.

Q.E.D. of Theo 5.

$P_3 \geqq \frac{1}{2} 3\cdot 2K^2 + 1 \geqq 4 \qquad 3\cdot K^2 \geq 3.$

① hence $m \geq 3 + 2 = 5 \quad (e = 3)$

$\Rightarrow \Phi_{mK}$ is hol. birat.

② assume $K^2 = 2$

$P_2 \geqq K^2 + 1 \geq 3 \qquad 2\cdot K^2 = 4 \geq 3.$

$\therefore \; e = 2.$

if $m \geq 4$ Φ_{mK} is hol. birat.

③ assume $K^2 \geq 3.$ $p_g = P_1 \geqq 3$

$\therefore \; e = 1$

if $m \geq 3$ Φ_{mK} is hol birat.

§ Example $\quad K_2 = 1 \Rightarrow q = 0$ (証は余り簡単ではない)

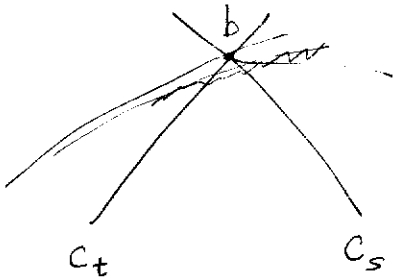
◎ $K^2 = 1,\; p_g = 2. \Rightarrow \Phi_{4K}$ is not birational

[Enriques]

$\dim H^0(K) = p_g = 2$ base $\{\varphi_0, \varphi_1\}$ of $H^0(K)$

$\therefore$

$|K| = \{C_t \mid C_t = (t_0\varphi_0, t_1\varphi_1) \;;\; t = (t_0, t_1) \in \mathbb{P}^1\}$



– base point of $|C_t|$

general member C_t is irred. non-singular.

$\dim H^0(2K) = P_2 = \frac{1}{2}\, 2\cdot 1\, K^2 + p_g + 1 = 4.$

base is $\{ \varphi_0^2, \varphi_0\varphi_1, \varphi_1^2, \psi \}$

∘ $(2K)_{C_t} = k_t$ = the canonical bundle of C_t.

∴

$$0 \longrightarrow \mathcal{O}(K) \longrightarrow \mathcal{O}(2K) \longrightarrow \mathcal{O}(k_t) \longrightarrow 0.$$

∴

$$\longrightarrow H^0(2K) \longrightarrow H^0(C_t, k_t) \longrightarrow 0. \leftarrow q = 0.$$

∴ $\{ \varphi_0^2|_{C_t}, \psi|_{C_t} \}$: basis of $H^0(C_t, k_t)$

$x = \dfrac{\varphi_1}{\varphi_0}$ $\quad y = \dfrac{\psi}{\varphi_0^2}$ are merom. functions

on S. $\quad x = \frac{t_0}{t_1}$ = const on C_t.

y = non-const. on C_t

hence x, y are alg independent.

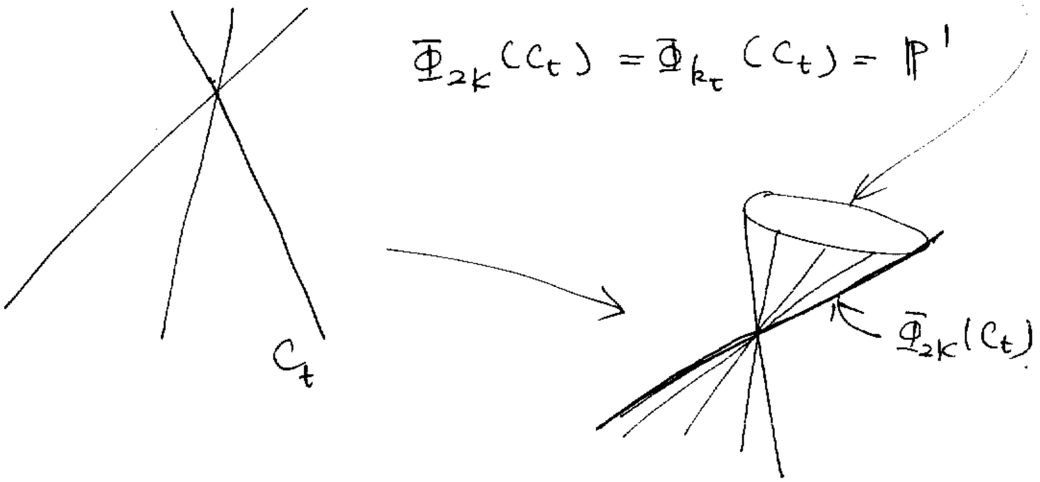
∘ $H^0(C_t, k_t)$ has no base point on C_t

∴ $2K$ has no base point on S.

∴ Φ_{2K} : is hol.

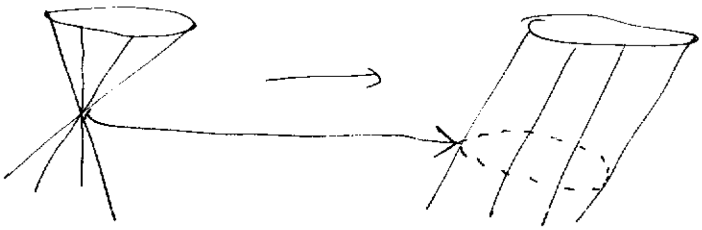
$\Phi_{2K}(S)$ is a surface in $\mathbb{P}^3$ with generic point:
$(1, x, x^2, y)$

∴ $\Phi_{2K}(S)$ is a quadric cone.



$\dim H^0(3K) = 6$.
base is $\{\varphi_0^3, \varphi_0^2\varphi_1, \varphi_0\varphi_1^2, \psi\varphi_0, \psi\varphi_1, \varphi_1^3\}$

∴ $\Phi_{3K}(S)$ is a surface in $\mathbb{P}^5$ with generic point $(1, x, x^2, x^3, y, yx)$
= ruled surface (rational

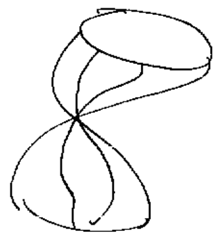


$\dim H^0(4K) = 9$

base is $\{ \varphi_0^4, \varphi_0^3\varphi_1, \cdots, \varphi_1^4, \psi\varphi_0^2, \psi\varphi_0\varphi_1, \psi\varphi_1^2, \psi^2 \}$

$\therefore$ $\Phi_{4K}(S)$ is a surface in $\mathbb{P}^8$ with generic point $(1, x, x^2, x^3, x^4, y, yx, yx^2, y^2)$

$\Phi_{4K}(S)$ is a quadric cone



$\therefore$

Conclusion : Φ_{4K} is not birational

注. ~~Φ_{2K}~~ S is a double covering of $\Phi_{2K}(S) = Q$ with branch curve of order 10.

② $K^2 = 2$, $p_g = 3$, $q = 0$. [Enriques]

$\Phi_K(S) = \mathbb{P}^2$

$\Phi_{2K}(S) = \mathbb{P}^2 \subset \mathbb{P}^6$

$\Phi_{3K}(S) = \mathbb{P}^2 \subset \mathbb{P}^{10}$

この注 Take a non-singular curve $B \subset Q$ of order 10. Then $\exists$ a double covering S of Q with branch locus B. Thus S has $K^2 = 1$, $p_g = 2$.

第2の例では.

S : double covering of $\mathbb{P}^2$ with branch locus of order 8 then $K^2 = 2$, $p_g = 3$.

—— · ——

Theo. if $m \geq 8$ then $\Phi_{mK}(S)$ is normal.

if $p_g - q \geq 3$, $m \geq 6$, $\Phi_{mK}(S)$ is normal.

$\boxed{\text{Problem}}$ Is this the best result?

T.