Rewiev of Some Results on PVI

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Today we review results about **asymptotic analytic properties** of solutions of the Sixth Painlevé equation (denoted PVI).

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 + \\ - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx} + \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]$$

Complex constant coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$,

Singular points $x = 0,\ 1,\ \infty.$

Results reviewed in this talk come from the **references** below:

– M.Jimbo: Publ. RIMS, Kyoto Univ.,**18** (1982).

– S.Shimomura: J.Math. Soc. Japan, **39**, (1987)

– P.Boalch: Proc. London Math. Soc. (2005) 90(1)

– K.Kaneko: Proceedings of the Japan Academy, Vol 82 (2006).

– D. Guzzetti: Comm. Pure Appl. Math., (2002).

: J. Phys. A: Math. Gen. (2006) and (2008)

: International Mathematics Research Notices (2011)

: Physica D (2012)

: Nonlinearity (2012)

: arXiv:1210.0311 (2012)

Recall well known facts:

The Painlevé equations are **non linear ODE's**

$\longrightarrow$ a non-linear dynamical system.

In general, non-linear dynamical systems have **chaotic behavior**.

Painlevé equations are the basic equations of **integrable systems**

Why?

Integrable $\longleftrightarrow$ **Painlevé Property**.

Painlevé Property

Consider a general solution depending on two integration constants c_1, c_2:

$$y(x) = y(x; c_1, c_2).$$

Painlevé property = the critical points of the general solution (= **essential singularities, branch points**) are only the singular points of the equation, $x = 0, 1, \infty$.

Remark: The **poles** depend on the integration constants c_1, c_2. Namely, the poles are movable.

What does it mean that we know a solution of PVI?

A generic solution of PVI (Painlevé Transcendent) is not a classical function.

Classical function = Rational, Algebraic, Contour integral of rational and algebraic functions, Solution of linear homogeneous ODE with rational coefficients, Solution of algebraic ODE of I order (rational coeff), Elliptic functions, etc.

[Umemura 1987-'90]

So, what does it mean that we know a solution of PVI?

Painlevé property $\longrightarrow$ "Bad" singular points (critical points) depend only on the equation, as in the case of classical special functions.

What do we know of classical (linear) special functions?

i) Asymptotic behavior at the singular points.

ii) Connection formulae.

i) Distribution of the poles (e.g. Poles of Elliptic functions, Γ functions, etc).

Then, we require the same knowledge i), ii), iii) for Painlevé transcendents.

What does it mean that we konw a solution?

i) Explicit critical behaviors, namely the behaviors of a transcendent $y(x)$ at singular points $x = 0$, 1 and ∞

$$y(x) \sim \begin{cases} y_0(x, c_1^{(0)}, c_2^{(0)}), & x \to 0 \\ y_1(x, c_1^{(1)}, c_2^{(1)}), & x \to 1 \\ y_\infty(x, c_1^{(\infty)}, c_2^{(\infty)}), & x \to \infty \end{cases}$$

$y_u(x, c_1^{(u)}, c_2^{(u)})$ = an **expansion** (convergent or asymptotic), or the **leading term**, for $x \to u \in \{0, 1, \infty\}$.

Explicit = each term of the expansion is a classical function of $(x, c_1^{(u)}, c_2^{(u)})$.

What does it mean that we konw a solution?

ii) Explicit connection formulae.

Two critical points: $u, \ v \in \{0, 1, \infty\}, \qquad u \neq v,$

and corresponding critical behaviors.

$$y(x) \sim y_u(x, c_1^{(u)}, c_2^{(u)}), \quad x \to u$$
$$y(x) \sim y_v(x, c_1^{(v)}, c_2^{(v)}), \quad x \to v$$

Then:

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(c_1^{(u)}, c_2^{(u)}) \\ c_2^{(v)} = c_2^{(v)}(c_1^{(u)}, c_2^{(u)}) \end{cases}$$

Connection Formulae

iii) The distribution of the movable poles.

The tools to acheive the knowledge of i), ii), iii)

i) Finding **critical behavior** is a **local problem** $\longrightarrow$ Local Analysis:

- Shimomura 1987
- Elliptic representation [D.G. 2001-2]]
- Power Geometry [Bruno- Goryuchkina (2010)]
- Method of Monodromy Preserving Deformations [Jimbo-Miwa-Ueno '81] $\rightarrow$ **Global analysis**

ii) Method of Monodromy Preserving Deformations provides **connection formulae**.

iii) No global results about poles, except for special valuse of $\alpha = \beta = \gamma = \delta - 1/2 = 0$ [Brezhnev 2009].
But we can find the **asymptotic distribution of the poles close to** $x = 0, 1, \infty$ [D.G. 2012].

Monodromy Preserving Deformations

PVI is equivalent to the **Schlesinger equations** (1912) for 2×2 matrices $A_0(x)$, $A_x(x)$, $A_1(x)$

$$\frac{dA_0}{dx} = \frac{[A_x, A_0]}{x},$$

$$\frac{dA_1}{dx} = \frac{[A_1, A_x]}{1-x},$$

$$\frac{dA_x}{dx} = -\frac{[A_1, A_x]}{x} - \frac{[A_1, A_x]}{1-x},$$

[,] = commutator of matrices.

$x = 0, 1, \infty$ singular (critical) points.

Monodromy Preserving Deformations

Suppose that $A_0(x)$, $A_x(x)$, $A_1(x)$ satisfy conditions on eigenvalues:

$$\text{Eigen}(A_0) = \pm\theta_0/2, \quad \text{Eigen}(A_1) = \pm\theta_1/2, \quad \text{Eigen}(A_\mathbf{x}) = \pm\theta_x/2 \tag{1}$$

$$A_0 + A_1 + A_x = \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{pmatrix}, \quad \theta_\infty \neq 0. \tag{2}$$

Einegvalues are ralated to the coefficients of PVI:

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_2^2.$$

Monodromy Preserving Deformations

Then, the entries of the $A_i(x)$'s are simple algebraic functions of $y(x)$, derivative $y'(x)$ and primitive $\int^x y(s)ds$ [Jimbo-Miwa (1981)].

Viceversa: $$y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_0)_{12}}$$

◇ PVI is the isomonodromy deformation equation of a 2×2 Fuchsian system of ODE [Jimbo, Miwa, Ueno (1981)]:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0(x)}{\lambda} + \frac{A_1(x)}{\lambda - 1} + \frac{A_x(x)}{\lambda - x} \right] \Psi.$$

The monodromy of $\Psi(x, \lambda)$ is independent of small deformations of x.

◇ **Complex power and oscillatory**. Here $\sigma, a, \phi \in \mathbb{C}$, $\nu \in \mathbb{R}$ are integration constants:

$$y(x) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + O(x^{2-2\sigma}), \qquad 0 < |\Re\sigma| < 1$$

$$y(x) = x \left\{ A \sin(i\sigma \ln x + \phi) + B \right\} + O(x^2), \qquad \Re\sigma = 0, \quad \sigma \neq 0$$

$$y(x) = \frac{1}{\mathcal{A} \sin(\nu \ln x + \phi) + \mathcal{B} + O(x)}, \qquad \nu \neq 0$$

◇ **Logarithm and Taylor**:

$$y(x) = x \left[\frac{\theta_x^2 - \theta_0^2}{4} (\ln x + a)^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2} \right] + O(x^2 \ln^3 x),$$

$$y(x) = \frac{4}{(\theta_1^2 - (\theta_\infty - 1)^2) \ln^2 x} \left[1 - \frac{2a}{\ln x} + O\left(\frac{1}{\ln^2 x} \right) \right],$$

$$y(x) = \sum_{n=0}^{\infty} c_n(a) x^n.$$

a) **Lemma** [Sato-Miwa-Jimbo '79]: Let A_0^0, A_x^0 and A_1^0 be constant matrices satisfying

$$\text{Eigen}(A_0^0) = \pm\theta_0/2, \quad \text{Eigen}(A_1^0) = \pm\theta_1/2, \quad \text{Eigen}(A_{\mathbf{x}}) = \pm\theta_x/2$$

$$A_0^0 + A_1^0 + A_x^0 = \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{pmatrix}, \quad \theta_\infty \neq 0.$$

Let

$$\Lambda := A_0^0 + A_x^0, \quad \text{with eigenvalues } \pm\frac{\sigma}{2}$$

Suppose $0 \leq \Re\sigma < 1$ and $\Lambda \neq 0$.

Then, the Schlesinger equations have a *unique* solution with critical behavior for $x \to 0$:

$$A_1(x) = A_1^0 + O(|x|^\delta)$$
$$x^{-\Lambda} A_0(x) x^{\Lambda} = A_0^0 + O(|x|^\delta), \qquad \delta \leq 1 - \Re\sigma$$
$$x^{-\Lambda} A_x(x) x^{\Lambda} = A_x^0 + O(|x|^\delta)$$

Observation: By elementary algebra we find the explicit matrices

$$A_0^0(\theta_0, \theta_x, \sigma, a), \quad A_x^0(\theta_0, \theta_x, \sigma, a),$$

$$A_1^0(\theta_1, \theta_\infty, \sigma, a), \qquad a = \text{ additional parameter.}$$

i) Critical behaviors

Substitute the result into $y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12}+(A_1)_{12}]-(A_0)_{12}}$.

$$y(x,\sigma,a) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + O(x^{2-2\sigma}),$$

$$\sim a x^{1-\sigma} \qquad\qquad 0 < \Re\sigma < 1$$

$$y(x,\sigma,a) = x\left\{ A\sin(i\sigma \ln x + \phi) + B \right\} + O(x^2), \qquad \Re\sigma = 0, \quad \sigma \neq 0$$

where $e^{i\phi} = 2ia/A$.

$A^2 = \frac{(\sigma^2-(\theta_0+\theta_x)^2)(\sigma^2-(\theta_0-\theta_x)^2}{4\sigma^2}$, $B = \frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2}$,

i) Critical behaviors

Remark: In Lemma of Sato-Miwa-Jimbo, we have

$$A_0(x) + A_x(x) \to \Lambda \neq 0 \quad \text{and} \quad xA_0(x) \to 0, \; xA_x(x) \to 0$$

When $x \to 0$, we need to consider also cases

$$A_0(x) + A_x(x) \to 0, \quad \text{or} \quad xA_0(x) \not\to 0, \; xA_x(x) \not\to 0.$$

We need a different method than the Lemma of Sato-Miwa-Jimbo.

b) Matching method [D.G. (2006)].
Consider

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_x}{\lambda - x}\right]\Psi.$$

Divide the λ plane into two domains ("outside a circle"):

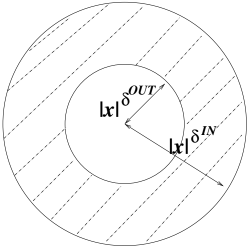
$$\{\lambda \in \mathbb{C} \mid \ |\lambda| \geq |x|^{\delta_{OUT}}\}, \qquad \delta_{OUT} > 0,$$

and ("inside a circle")

$$\{\lambda \in \mathbb{C} \mid \ |\lambda| \leq |x|^{\delta_{IN}}\}, \qquad \delta_{IN} > 0.$$

Matching method

We require that the domains overlap ($0 < \delta_{IN} < \delta_{OUT} < 1$):



λ *plane*

Matching method

For $|\lambda| \geq |x|^{\delta_{OUT}}$ we do an approximation of

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{A_1}{\lambda - 1}\right]\Psi$$

$$\frac{x}{\lambda} \to 0$$

$$\Downarrow$$

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{x\hat{A}_x}{\lambda^2}\left(1 + \frac{x}{\lambda} + \ldots + \frac{x^{N_{OUT}}}{\lambda^{N_{OUT}}}\right) + \frac{\hat{A}_1}{\lambda - 1}\right]\Psi_{OUT}$$

$N_{OUT} \in \mathbb{N}$ is a **truncation** of the full expansion.

Here $\hat{A}_0(x)$, $\hat{A}_0(x)$, $\hat{A}_0(x)$ are the critical behaviors (or leading terms) of $A_0(x)$, $A_x(x)$, $A_1(x)$ for $x \to 0$.

Matching method

For $|\lambda| \leq |x|^{\delta_{IN}}$ we approximate

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_{\mathbf{x}}}{\lambda - x} + \frac{A_1}{\lambda - 1}\right]\Psi$$

$$\lambda \to 0,$$

$$\Downarrow$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} - \hat{A}_1 \sum_{n=0}^{N_{IN}} \lambda^n\right]\Psi_{IN}$$

$N_{IN} \in \mathbb{N}$ is a **truncation** of the full expansion.

Then, **impose the matching condition** for $x \to 0$:

$$\Psi_{IN}(x,\lambda) \sim \Psi_{OUT}(x,\lambda), \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}} \to 0.$$

Matching method

- The matching condition determines the explicit asymptotic behavior of $\Psi_{IN}(x,\lambda)$ and $\Psi_{OUT}(x,\lambda)$ for $x \to 0$.
- The entries of $\Psi_{IN}(x,\lambda)$ and $\Psi_{OUT}(x,\lambda)$ are expressed in a unique way in terms of the entries of $\hat{A}_0(x), \hat{A}_x(x), \hat{A}_1(x)$.
- Thus, the matching condition determines the explicit asymptotic behavior of $\hat{A}_0(x), \hat{A}_x(x), \hat{A}_1(x)$ for $x \to 0$. Note: $\hat{A}_0, \hat{A}_x, \hat{A}_1$ also depen on two constants. Let us call them $c_1^{(0)}, c_2^{(0)}$.
- Finally, we find the explicit critical behavior of $y(x)$, for $x \to 0$, if we substitute into

$$y(x) \sim \frac{x(\hat{A}_0)_{12}}{x[(\hat{A}_0)_{12} + (\hat{A}_1)_{12}] - (\hat{A}_0)_{12}} \equiv y_0(x; c_1^{(0)}, c_2^{(0)}).$$

Matching method

Example 1: $N_{IN} = N_{OUT} = 0$. The original system is approximated by two Fuchsian systems

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + 0 + \frac{\hat{A}_1}{\lambda - 1}\right]\Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} + 0\right]\Psi_{IN}$$

The approximation is possible iff $\hat{A}_1$ is constant, $\hat{A}_0(x) + \hat{A}_x(x) = \Lambda \neq 0$ constant.

$\Longrightarrow$ So, there exists a 2×2 matrix $K(x)$ s.t.

$$K(x)^{-1}\left[\hat{A}_0(x) + \hat{A}_x(x)\right]K(x) = \begin{pmatrix} \sigma/2 & 0 \\ 0 & -\sigma/2 \end{pmatrix}$$

Matching method

Standard theory of Fuchsian systems gives:

$$\Psi_{OUT} = \sum_{n=0}^{\infty} G_n \lambda^n \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix}, \qquad \lambda \to 0,$$

$$\Psi_{IN} = \left[K(x) + \sum_{n=1}^{\infty} K_n(x) \frac{x^n}{\lambda^n} \right] \begin{pmatrix} (\lambda/x)^{\frac{\sigma}{2}} & 0 \\ 0 & (\lambda/x)^{-\frac{\sigma}{2}} \end{pmatrix}, \quad \frac{\lambda}{x} \to \infty.$$

<u>We obtain $K(x)$ by matching $\Psi_{OUT} \sim \Psi_{IN}$:</u>

$$G_0 \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \sim K(x) \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \begin{pmatrix} x^{-\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}.$$

$$\Downarrow$$

$$K(x) \sim G_0 \begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{-\frac{\sigma}{2}} \end{pmatrix}, \qquad x \to 0$$

Matching method

The final result is

$$\hat{A}_1(x) = A_1^0$$
$$\hat{A}_0(x) = x^{\Lambda} A_0^0 \, x^{-\Lambda},$$
$$\hat{A}_x(x) = x^{\Lambda} A_x^0 \, x^{-\Lambda}.$$

This is the result of Sato-Miwa-Jimbo!
Hence, we obtain the critical behavior as before:

$$y(x) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + O(x^{2-2\sigma}), \qquad 0 < |\Re\sigma| < 1$$

$$y(x) = x \left\{ A \sin(i\sigma \ln x + \phi) + B \right\} + O(x^2), \qquad \Re\sigma = 0, \quad \sigma \neq 0$$

Example 2: Matching in the case of Example 1 with eigenvalue $\underline{\sigma = 0}$ of $\Lambda \neq 0$.

We obtain Logarithmic behaviors for $x \to 0$:

$$y(x) = x\left[\frac{\theta_x^2 - \theta_0^2}{4}(\ln x + a)^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2}\right] + O(x^2 \ln^3 x), \quad \theta_0^2 \neq \theta_x^2$$

$$y(x) = x\Big(a \pm \theta_0 \ln x\Big) + O(x^2 \ln^2 x), \qquad \theta_0^2 = \theta_x^2$$

a is the integration constant.

Example 3: Case $\lim_{x\to 0} A_0(x) + A_x(x) = 0$,

$A_x(x) \to A$ constant matrix, with eigenvalues $\pm\theta_x/2$.

Write:

$$A_0 + A_x = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix},$$

$$\lim_{x\to 0} a(x) = \lim_{x\to 0} b(x) = \lim_{x\to 0} c(x) = 0.$$

and

$$A = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)\, s}{r} & -s - \frac{\theta_x}{2} \end{pmatrix}, \quad r, s \in \mathbb{C}, \quad r \neq 0.$$

$\hat{A}_1 \sim -\theta_\infty/2\ \sigma_3, \qquad \hat{A}_0 \sim -A.$

Matching method

The functions $a(x)$, $b(x)$ and $c(x)$ can be asymptotically determined by the matching condition for approximated systems with $N_{OUT} = N_{IN} = 1$:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{xA}{\lambda^2} + \frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1}{\lambda - 1}\right]\Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} - \hat{A}_1\right]\Psi_{IN}$$

The final result is

$$y(x) = \frac{1}{1-\theta_\infty} + ax + O(x^2), \qquad a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}.$$

Matching method

- The matching procedure for cases

$$A_0(x) + A_1(x) \to 0,$$

or

$$xA_0(x) \not\to 0, \; xA_x(x) \not\to 0,$$

provides all the Taylor expansions:

$$y(x) = \sum_{n=0}^{\infty} c_n(a) x^n, \qquad a \in \mathbb{C}$$

Critical behaviors – Summary

So far we have computed the following behaviors for $x \to 0$

◇ Complex power:

$$y(x) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + O(x^{2-2\sigma}), \qquad 0 < |\Re\sigma| < 1$$

$$y(x) = x\left\{ A \sin(i\sigma \ln x + \phi) + B \right\} + O(x^2), \qquad \Re\sigma = 0, \quad \sigma \neq 0$$

◇ Logarithm and Taylor:

$$y(x) = x\left[\frac{\theta_x^2 - \theta_0^2}{4} (\ln x + a)^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2} \right] + O(x^2 \ln^3 x), \quad \theta_0^2 \neq \theta_x^2$$

$$y(x) = x\left(a \pm \theta_0 \ln x \right) + O(x^2 \ln^2 x), \qquad \theta_0^2 = \theta_x^2$$

$$y(x) = \sum_{n=0}^{\infty} c_n(a) x^n.$$

c) We find other critical behavior (from the above) making use of the **symmetries** of PVI [K. Okamoto (1987)]:

- Consider

$$y'(x') = \frac{x}{y(x)}, \quad x' = x,$$
$$\theta_0' = \theta_\infty - 1, \quad \theta_x' = \theta_1, \quad \theta_1' = \theta_x, \quad \theta_\infty' = \theta_0 + 1$$

Symmetry means that $y'(x')$ solves PVI with coefficients $\theta_0', \theta_x', \theta_1', \theta_\infty'$ if and only if $y(x)$ solves PVI with $\theta_0, \theta_x, \theta_1, \theta_\infty$.

From the above, we compute (just as examples):

$$y(x) = \frac{1}{\mathcal{A}\sin(\nu \ln x + \phi) + \mathcal{B} + O(x)}, \qquad \nu \in \mathbb{R}\backslash\{0\}$$

$$y(x) = \frac{4}{(\theta_1^2 - (\theta_\infty - 1)^2)\ln^2 x}\left[1 - \frac{2a}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \theta_1^2 \neq (\theta_\infty - 1)^2$$

$$y(x) = \frac{1}{(\theta_\infty - 1)\ln x}\left[1 - \frac{a}{(\theta_\infty - 1)\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \theta_1^2 = (\theta_\infty - 1)^2$$

$x \to 0$.

- We obtain critical behaviors at $x = 1$ from the behaviors at $x = 0$. Use the symmetry

$$y'(x') = 1 - y(x), \quad x' = 1 - x, \quad \theta'_0 = \theta_1, \quad \theta'_1 = \theta_0.$$

- We obtain critical behaviors at $x = \infty$ from the behaviors at $x = 0$. Use the symmetry

$$y'(x') = \frac{1}{x} y(x), \quad x' = \frac{1}{x}, \quad \theta'_x = \theta_1, \quad \theta'_1 = \theta_x.$$

- There are other symmetries (a representation of the affine D_4) which generate other critical behaviors $\longrightarrow$ Table of Critical Behaviors. Please see D.G. Nonlinearity 25 (2012) 3235-3276.

i) Critical behaviors – Full expansions

So far, we have seen the leading term of critical behavior.

$$y(x) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + O(x^{2-2\sigma}), \quad 0 < |\Re\sigma| < 1$$

Can we compute the full expansion of $O(x^{2-2\sigma})$? Yes:

$$O(x^{2-2\sigma}) = \sum_{n=2}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}(a)(x^{\sigma})^m.$$

This is convergent in a suitable domain contained in *the universal covering* $\tilde{\mathcal{U}}$ of

$$\mathcal{U} := \{x \in \mathbb{C} \mid 0 < |x| < 1\}.$$

Theorem:[Shimomura (1987)]. $\forall a \neq 0$ and $\sigma \notin (-\infty, 0] \cup [1, \infty)$ there exists $r < 1$ such that PVI has a holomorphic solution in

$$\mathcal{D}(r; \sigma, a) := \{x \in \tilde{\mathcal{U}} \mid |a x^{1-\sigma}| < 4r, \ |x^{\sigma}/a| < r/4\}$$

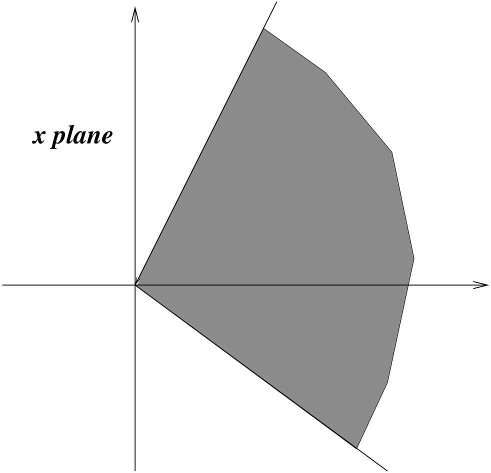
with convergent expansion

$$y(x, \sigma, a) = \left\{ a x^{1-\sigma} + \frac{4A^2}{a} x^{1+\sigma} + Bx \right\} + \sum_{n=2}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}(a)(x^{\sigma})^m.$$

Remark: The same results can be obtained by the elliptic representation of PVI [D.G. (2001-2)].

We represent $\mathcal{D}(r;\sigma,a)$ in the plane $(\ln|x|, \Im\sigma \arg x)$:

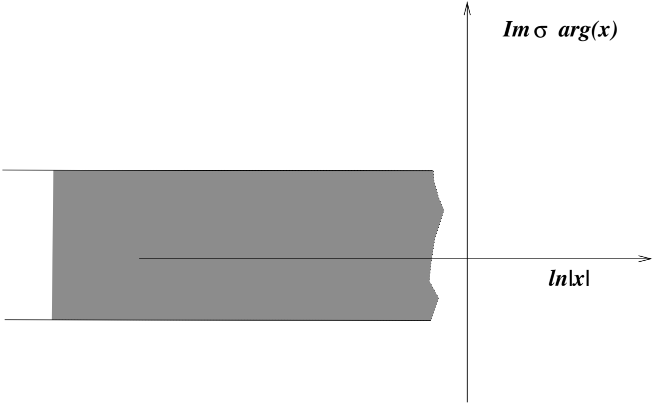
Consider a sector in the x plane.



How is the sector represented in the $(\ln |x|, \arg(x))$ plane?

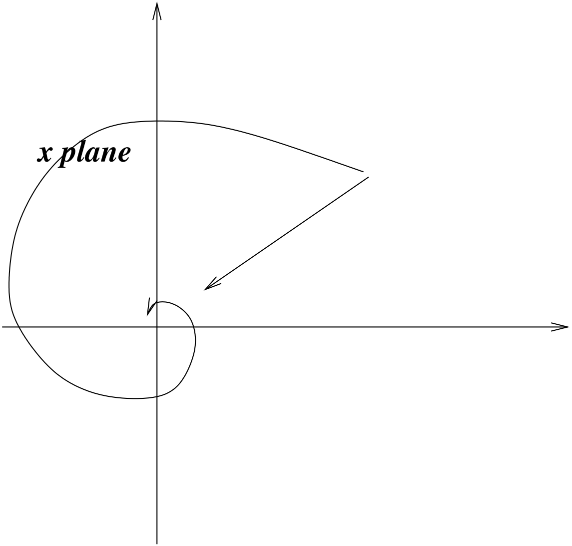
Representation in the $(\ln|x|, \arg(x))$ plane

The sector becomes a strip:

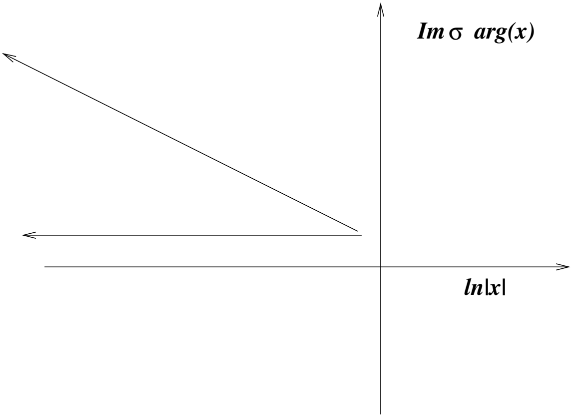


Representation in the $(\ln|x|, \arg(x))$ plane

Consider paths (curves) converging to $x = 0$:



The curves are:



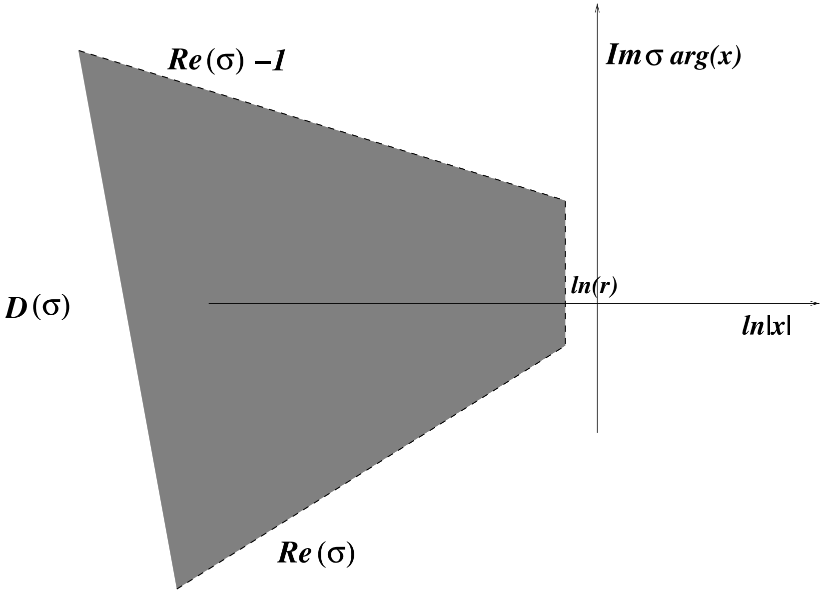
$\mathcal{D}(r;\sigma,a)$ in the $(\ln|x|, \arg(x))$ plane

$\mathcal{D}(r;\sigma,a)$ in the plane $(\ln|x|, \Im\sigma \arg x)$ is

$$\Re\sigma \ln|x| - \ln\frac{r|a|}{4} < \Im\sigma \arg x < (\Re\sigma - 1)\ln|x| + \ln(4r/|a|),$$

$$\ln|x| < \ln(r).$$

$\mathcal{D}(r;\sigma,a)$ in the $(\ln|x|, \arg(x))$ plane



Recall that in $\mathcal{D}(r;\sigma,a)$ the behavior is

$$y(x,\sigma,a)=\left\{ax^{1-\sigma}+\frac{4A^2}{a}x^{1+\sigma}+Bx\right\}+\sum_{n=2}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}(a)(x^{\sigma})^m.$$

To be precise, the behavior along paths

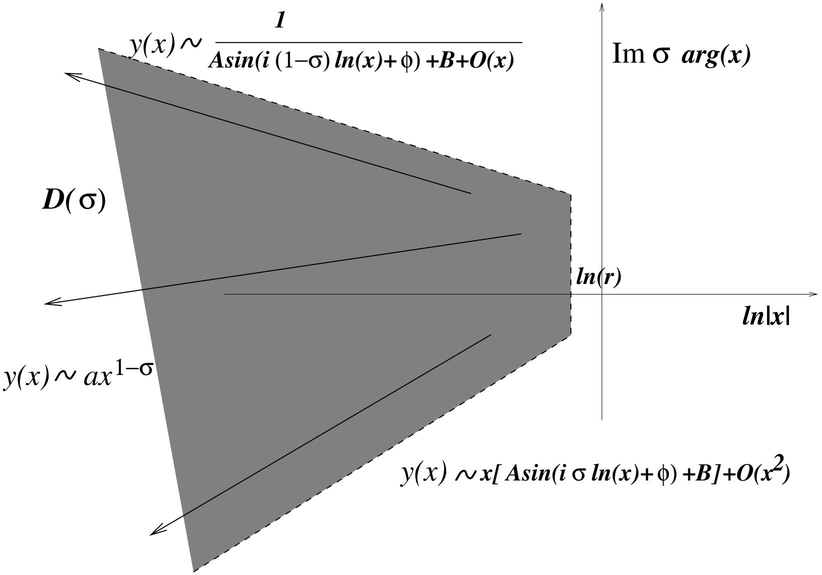
$$\Im\sigma\arg x=(\Re\sigma-1)\ln|x|+\Im\sigma\arg x_0$$

is

$$y(x,\sigma,\phi(a))=\frac{1}{\mathcal{A}\sin(i(1-\sigma)\ln x+\phi)+\mathcal{B}+O(x)},$$

$$O(x)=\sum_{n=2}^{\infty}x^{n-1}\sum_{m=-n}^{n}A_{nm}(\nu)e^{im\phi}x^{(\sigma-1)m}.$$

Critical behaviors in $\mathcal{D}(r;\sigma,a)$



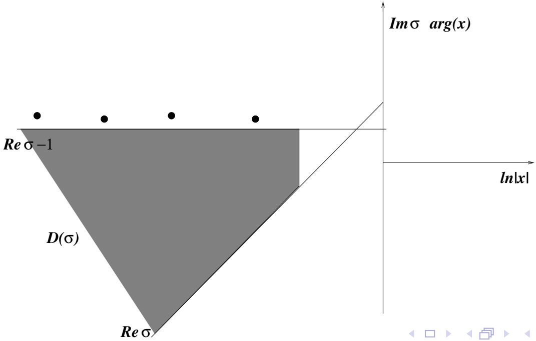
iii) Poles

Case $\sigma = 1 + i\nu$, and $x \to 0$ along a radius. Then:

$$y(x) = \frac{1}{\mathcal{A}\sin(\nu \ln x + \phi) + \mathcal{B} + O(x)} \qquad \nu \in \mathbb{R}\backslash\{0\}$$

$$= \frac{1}{A_{11} e^{i\phi} x^{i\nu} + A_{10} + A_{1,-1} e^{-i\phi} x^{-i\nu} + O(x)}$$

- The poles are outside $\mathcal{D}(r; \sigma, \nu)$.



- The poles are approximated for $x \to 0$ by the roots of

$$A_{11} e^{i\phi} x^{i\nu} + A_{10} + A_{1,-1} e^{-i\phi} x^{-i\nu} = 0$$

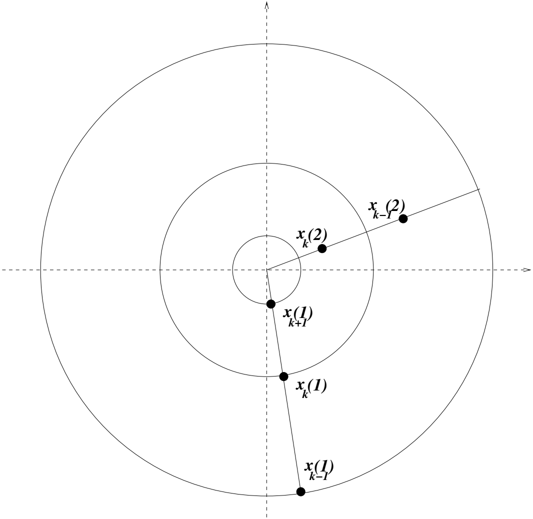
- There are two sequences of roots $\{x_k(1)\}_{k\in\mathbb{Z}}$ and $\{x_k(2)\}_{k\in\mathbb{Z}}$

$$x_k(j) = \exp\left\{ -\frac{\phi}{\nu} - \frac{i}{\nu} \ln\left[(-)^j \sqrt{\frac{A_{10}^2}{4A_{11}^2} - \frac{A_{1,-1}}{A_{11}}} - \frac{A_{10}}{2A_{11}} \right] - \frac{2k\pi}{\nu} \right\}$$

$$k \in \mathbb{Z}.$$

iii) Poles

These zeros accumulate at $x = 0$ when $k \to +\infty$:



iii) Poles

Consider the full espansion of

$$y(x) = \frac{1}{A_{11}e^{i\phi}x^{i\nu} + A_{10} + A_{1,-1}e^{-i\phi}x^{-i\nu} + O(x)}$$

$$O(x) = \sum_{n=2}^{\infty} x^{n-1} \sum_{m=-n}^{n} A_{nm}(\nu)e^{im\phi}x^{(\sigma-1)m}$$

We obtain the asymptotic expansion of the poles for $k \to +\infty \Longleftrightarrow x_k(j) \to 0$:

$$\xi_k(j) = x_k(j) + \sum_{N=2}^{\infty} \Delta_N(j;\nu)x_k(j)^N,$$

$\Delta_N(j;\nu) \in \mathbb{C}$.

We find the connection formulae with the **method of monodromy preserving deformations**. Recall that PVI is the isomonodromy deformation equation of

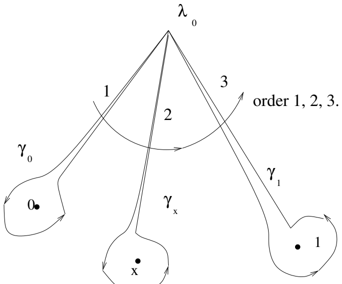
$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_{\mathbf{x}}}{\lambda - x}\right]\Psi$$

The **monodromy matrices** M_0, M_x, M_1 of $\Psi(x,\lambda)$ are independent of small deformations of x

$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_0,$$
$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_x,$$
$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_1,$$

$$M_\infty := M_1 M_x M_0$$

Note: $|\arg x| < \pi$, $|\arg(1-x)| < \pi$.



Monodromy data:

$$\Theta := \{(\theta_0, \theta_1, \theta_x, \theta_\infty) \in \mathbf{C}^4 \mid \theta_\infty \neq 0\}/\sim .$$

Equivalence $\sim$ is $\theta_k \mapsto -\theta_k$, $\theta_\infty \mapsto 2 - \theta_\infty$.

$M :=$

$$\{(M_0, M_x, M_1) \mid \mathrm{Tr} M_\mu = 2\cos\pi\theta_\mu,\ \ \mu = 0, 1, x, \infty\}/\text{conjugation}$$

Note: Conjugation is: $M_i \mapsto C M_i C^{-1}$, $\mathrm{Det}\, C \neq 0$

Definition: The monodromy data of the class of Fuchsian systems, with the basis of loops ordered as in figure, are elements of the set

$$\mathcal{M} := \Theta \cup M.$$

$y(x) \longrightarrow$ system $\frac{d\Psi}{d\lambda} = A(x,\lambda)\Psi \longrightarrow$ monodromy data $\mathcal{M}$.

$f : \{y(x)$ branch of solution of PVI$\} \to \mathcal{M}$,   Monodromy Map

- 1) f is **one-to-one** if restricted to f^{-1}(subspace of $\mathcal{M}$ where $M_0, M_x, M_1, M_\infty \neq I$).
- 2) If the group generated by M_0, M_x, M_1 is irreducible then good **coordinates** on $\mathcal{M}$ are

$$\begin{cases} p_{ij} = p_{ji} := \mathrm{Tr} M_i M_j & i \neq j \in \{0, x, 1\} \\ p_\mu := \mathrm{Tr} M_\mu = 2\cos \pi\theta_\mu & \mu \in \{0, x, 1, \infty\} \end{cases}$$

Note: Only two of p_{0x}, p_{x1}, p_{01} are independent (cubic relation) [Jimbo (1982), K.Iwasaki (2003)].

ii) Connection Problem

Let us consider a critical behavior

$$y(x) = y_u(x, c_1^{(u)}, c_2^{(u)}), \quad x \to u \in \{0, 1, \infty\}$$

If 1) and 2) hold, then we **parametrize the integration constants**:

$$\begin{cases} c_1^{(u)} = c_1^{(u)}(p_0, p_x, p_1, p_\infty, p_{0x}, p_{x1}, p_{01}) \\ c_2^{(u)} = c_2^{(u)}(p_0, p_x, p_1, p_\infty, p_{0x}, p_{x1}, p_{01}) \end{cases}$$

$$\text{and} \quad \begin{cases} p_{0x} = p_{0x}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{x1} = p_{x1}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{01} = p_{01}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \end{cases}$$

These formulae are computed **explicitly** by *asymptotic computation of monodromy matrices* M_0, M_x, M_1.

ii) Connection Problem

At the critical point $x = u$ we have:

$$\begin{cases} p_{0x} = p_{0x}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{x1} = p_{x1}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{01} = p_{01}(p_0, p_x, p_1, p_\infty, c_1^{(u)}, c_2^{(u)}) \end{cases}$$

and, for anather critical point $v \in \{0, x, 1\}$, $v \neq u$ we have,

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(p_0, p_x, p_1, p_\infty, p_{0x}, p_{x1}, p_{01}) \\ c_2^{(v)} = c_2^{(v)}(p_0, p_x, p_1, p_\infty, p_{0x}, p_{x1}, p_{01}) \end{cases}$$

Combine the above: $\longrightarrow$ We find **connection formulae** between integration constants at u and v

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(c_1^{(u)}, c_2^{(u)}) \\ c_2^{(v)} = c_2^{(v)}(c_1^{(u)}, c_2^{(u)}) \end{cases}$$

ii) Connection Problem – Asymptotic computation of monodromy data

Asymptotic computation of monodromy data. The matching method is useful when we are able to compute exactly the monodromy of $\Psi_{IN}(x,\lambda)$ and $\Psi_{OUT}(x,\lambda)$ for

$$\frac{d\Psi_{OUT}}{d\lambda}=\left[\frac{\hat{A}_0+\hat{A}_x}{\lambda}+\frac{x\hat{A}_x}{\lambda^2}\sum_{n=0}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n+\frac{\hat{A}_1}{\lambda-1}\right]\Psi_{OUT}$$

$$\frac{d\Psi_{IN}}{d\lambda}=\left[\frac{\hat{A}_0}{\lambda}+\frac{\hat{A}_x}{\lambda-x}-\hat{A}_1\sum_{n=0}^{N_{IN}}\lambda^n\right]\Psi_{IN}$$

The monodrmy of $\Psi_{IN}(x,\lambda)$ and $\Psi_{OUT}(x,\lambda)$ provides the monodromy of $\Psi(x,\lambda)$ for

$$\frac{d\Psi}{d\lambda}=\left[\frac{A_0}{\lambda}+\frac{A_x}{\lambda-x}+\frac{A_1}{\lambda-1}\right]\Psi$$

Asymptotic computation of monodromy data

- Recall the matching $(x \to 0)$

$$\Psi_{IN}(x,\lambda) \sim \Psi_{OUT}(x,\lambda), \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}} \to 0.$$

- We need to check also the matching with $\Psi(x,\lambda)$:

$$\Psi(x,\lambda) \sim \Psi_{OUT}(x,\lambda) C_{OUT}, \qquad \text{for } |\lambda| >> 1$$

$$\Psi(x,\lambda) \sim \Psi_{IN}(x,\lambda) C_{IN}, \qquad \text{for } \begin{cases} \lambda \to x \\ \lambda \to 0 \end{cases}$$

when x is fixed and small.
We also compute the *connection matrices* C_{IN} and C_{OUT}.

Suppose the above matchings are verified:

We consider the approximated system:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{x\hat{A}_x}{\lambda^2}\sum_{n=0}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n + \frac{\hat{A}_1}{\lambda - 1}\right]\Psi_{OUT}$$

and **compute exactly** $M_1^{OUT}(c_1^{(0)}, c_2^{(0)}, p_1, p_\infty)$. This is possible in the examples considered before.

Then (isomonodromy property and matching $\Psi \sim \Psi_{OUT}$)

$$M_1 = C_{OUT}^{-1} M_1^{OUT}(c_1^{(0)}, c_2^{(0)}, p_1, p_\infty) C_{OUT}.$$

ii) Connection Problem

Then, consider the approximated system

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} - \hat{A}_1 \sum_{n=0}^{N_{IN}} \lambda^n \right] \Psi_{IN}$$

and **compute exactly**

$$M_0^{IN}(c_1^{(0)}, c_2^{(0)}, p_0, p_x) \quad \text{and} \quad M_x^{IN}(c_1^{(0)}, c_2^{(0)}, p_0, p_x).$$

This is possible in the examples considered before.

Then (isomonodromy property and matching $\Psi \sim \Psi_{IN}$):

$$M_0 = C_{IN}^{-1} M_0^{IN}(c_1^{(0)}, c_2^{(0)}, p_0, p_x) C_{IN},$$

$$M_x = C_{IN}^{-1} M_x^{IN}(c_1^{(0)}, c_2^{(0)}, p_0, p_x) C_{IN}.$$

◇ Finally, compute $p_{ij} = \mathrm{Tr}(M_i M_j)$.

Example [Jimbo 1982, Boalch 2005] Consider the critical behavior we have seen before

$$y(x,\sigma,a)=\sum_{n=1}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}(a)x^{m\sigma}\sim ax^{1-\sigma},\quad 0\le\Re\sigma<1$$

We compute with the above procedure:

$$2\cos(\pi\sigma)=p_{0x},\qquad a=\frac{(\theta_x-\theta_0-\sigma)(\theta_x+\theta_0-\sigma)(\theta_\infty+\theta_1-\sigma)}{4\sigma^2(\theta_\infty+\theta_1+\sigma)}\,\frac{1}{\mathbf{F}}$$

where $\mathbf{F}=\mathbf{F}(p_{01},p_{x1})$:

$$\mathbf{F}:=\frac{\Gamma(1+\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma)+1\right)}{\Gamma(1-\sigma)^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma)+1\right)}\times$$

$$\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma)+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma)+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma)+1\right)}\,\frac{\mathbf{V}}{\mathbf{U}},$$

$$\mathbf{U} := \left[\frac{i}{2}\sin(\pi\sigma)p_{x1} - \cos(\pi\theta_x)\cos(\pi\theta_\infty)\right.$$

$$\left. - \cos(\pi\theta_0)\cos(\pi\theta_1)\right] e^{i\pi\sigma} +$$

$$+\frac{i}{2}\sin(\pi\sigma)p_{01} + \cos(\pi\theta_x)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_0)$$

$$\mathbf{V} := 4\sin\frac{\pi}{2}(\theta_0 + \theta_x - \sigma)\sin\frac{\pi}{2}(\theta_0 - \theta_x + \sigma)$$

$$\sin\frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma)\sin\frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma).$$

– For other examples see K.Kaneko (2006) and D.G. (2006∼2012) (in particular Nonlinearity (2012)).

Behavior on the Universal Covering of $x = 0$

A PVI transcendent $y(x, \sigma, a) \sim a x^{1-\sigma}$ is in one to one correspondence with p_{0x}, p_{x1}, p_{01}.

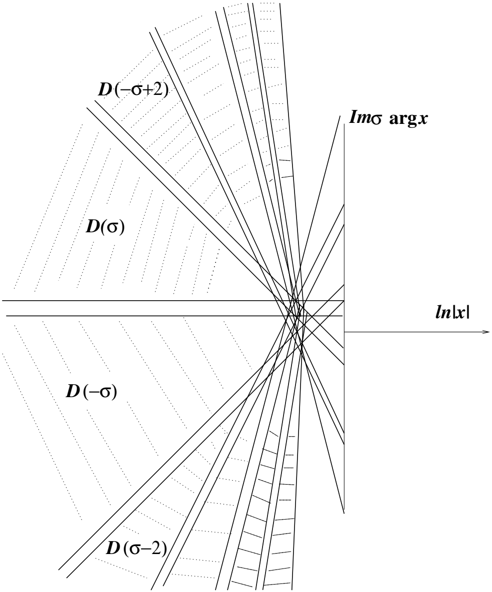
Observe that the integration constant $a = a(\sigma, p_{01}, p_{x1})$.

Let us observe the equation which gives σ:

$$2 \cos \pi\sigma = p_{0x} \tag{3}$$

- For given p_{0x}, p_{x1}, p_{01}, there is a solution σ of (3) such that $0 \leq \Re\sigma < 1$.
  This determines the behavior of Jimbo solutions.
- There are also soultions $\sigma_N^{\pm} := \pm\sigma + 2N$, for any $N \in \mathbb{N}$.
- Consider also $a_N^{\pm} := a(\sigma_N^{\pm}, p_{01}, p_{x1})$
  and the domains $\mathcal{D}(r_N^{\pm}, \sigma_N^{\pm}, a_N^{\pm})$.

The domains $\mathcal{D}(r_N^{\pm}, \sigma_N^{\pm}, a_N^{\pm})$



We conclude (see D.G. Comm. Pure Apl. Math (2002)) that a given transcendent

$$y(x, \sigma, a) \sim a x^{1-\sigma}$$

associated to p_{0x}, p_{x1}, p_{01} has also several other different behaviors on each $\mathcal{D}(r_N^{\pm}, \sigma_N^{\pm}, a_N^{\pm})$:

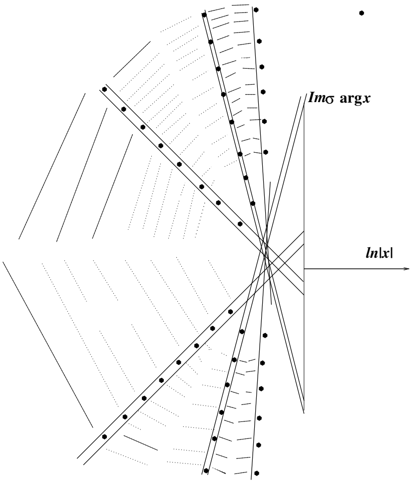
$$y(x, \sigma_N^{\pm}, a_N^{\pm})) \sim a_N^{\pm} x^{1-\sigma_N^{\pm}}$$

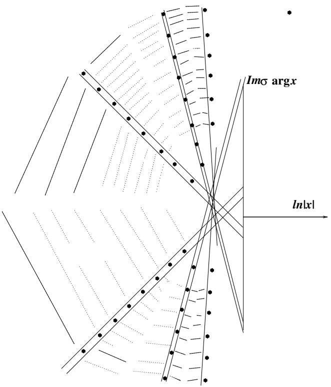
Remark: the behaviors along boundary lines of $\mathcal{D}(r_N^{\pm}, \sigma_N^{\pm}, a_N^{\pm})$:

$$y(x, \sigma_N^{\pm}, a_N^{\pm})) \sim x \Big\{ A_N^{\pm} \sin(i\sigma_N^{\pm} \ln x + \phi(a_N^{\pm})) + B_N^{\pm} \Big\}$$

$$y(x, \sigma_N^{\pm}, a_N^{\pm})) = \frac{1}{\mathcal{A}_N^{\pm} \sin(i(1-\sigma_N^{\pm}) \ln x + \phi(a_N^{\pm})) + \mathcal{B}_N^{\pm} + O(x)}$$

It can be proved that the critical behavior extends to the separating region of $\mathcal{D}(r;\sigma,a(\sigma))$ and $\mathcal{D}(r;-\sigma,a(-\sigma))$.



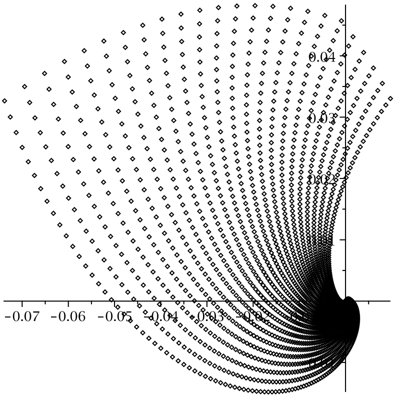


The behaviors along directions of boundaries are:

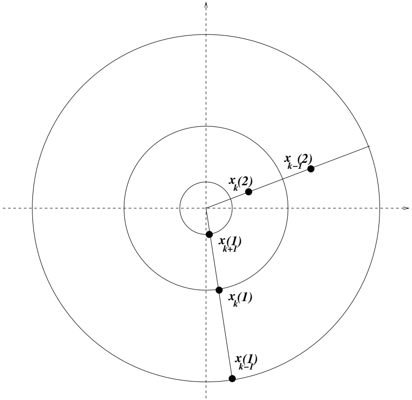
$$y(x) = \frac{1}{\mathcal{A}_N^{\pm} \sin(i(1-\sigma_N^{\pm}) \ln x + \phi_N^{\pm}) + \mathcal{B}_N^{\pm} + O(x)}$$

The poles lie outside the union of the (extended) domains.

x plane:
The poles lie alog spirals in the universal covering of $\mathcal{U} = \{x \in \mathbb{C} \mid 0 < |x| < \max_N r_N^{\pm}\}$ (up to a fixed "big" N).



For $\Re\sigma = 1$ we recover the two sequences of poles accumulating at $x = 0$ when we project to the x plane.



$$y(x) = \frac{1}{\mathcal{A}\sin(\nu \ln x + \phi) + \mathcal{B} + O(x)}, \quad \nu \in \mathbb{R}, \quad \sigma = 1 + i\nu.$$

The same analysis at $x = 1$ and $x = \infty$ is acheived by making use of the symmetries of PVI.

Summary: We have:

- The critical behaviors and their complete tabulation.
- The corresponding connection formulae.
- The asymptotic distribution of the poles.

The study of the poles deserves more investigations.