Classification of meromorphic differential equations

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Introduction and notation

$K = \mathbb{C}(\{z\})$ is the field of the meromorphic functions at $z = 0$, i.e., the field of the convergent Laurent series. $\widehat{K}$ is the field of the formal Laurent series. The differentiation used on these fields is $\frac{d}{dz}$.

A meromorphic (linear) differential equation is an equation with entries in K. In scalar form, this is

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

with all $a_i \in K$. In matrix form this is

$(\frac{d}{dz}+A)Y = 0$ where Y is a vector of length n and $A \in \mathrm{Matr}(n, K)$. A *differential module* over K is a K-vector space M of finite dimension provided with a $\mathbb{C}$-linear map ∂ satisfying the rule $\partial(fm) = f'm + f\partial m$ for $f \in K,\ m \in M$.

After identifying M with K^n the equation $\partial m = 0$ reduces to a differential equation in matrix form. After taking a cyclic vector $e \in M$, the monic operator $L \in K[\partial]$ of minimal degree with $Le = 0$ is a scalar equation associated to M.

The differential module M is called *regular* or *trivial* if there exists a basis $b_1, \ldots, b_m$ of M over K with all $\partial b_j = 0$. These modules are not interesting. One calls M *regular singular* if there exists a basis $b_1, \ldots, b_m$ of M over K such that $\mathbb{C}\{z\}b_1 + \cdots + \mathbb{C}\{z\}b_m$ is invariant under ∂. The structure of these modules is rather simple. They can be presented by matrix equations of the form $z\frac{d}{dz} + A$, where A is a constant matrix (say such that the eigenvalues of A do not differ by a non zero integer). They are classified by their monodromy $exp(2\pi i A)$. The remaining equations are called *irregular singular*. Their structure is highly complicated.

The aim of this talk is to identify the (Tannakian) category of all differential modules over K with a certain, more transparent, category of tuples Gr_2. For this we have to introduce *multisummation* and *Stokes maps*.

k-summation and a theorem of Sibuya

First we analyze in detail the irregular singular (inhomogeneous) equation

$$(z\frac{d}{dz}+q)y = f;\ q = q_k z^{-k} + \cdots + q_1 z^{-1} + q_0$$

and a given $f \in K$.

There is a (unique) *formal solution* $\hat{y} = \sum_{i>>-\infty} a_i z^i \in \widehat{K}$. The '*main theorem of asymptotics*' says that for a small enough sector $\mathcal{S}$, there exists a meromorphic function y on $\mathcal{S}$, solution of the equation and having $\hat{y}$ as asymptotic expansion. This means: For every $n \geq 0$ there are constants $C, \epsilon > 0$ such that $|y(z) - \sum_{i \leq n} a_i z^i| \leq C|z|^{n+1}$ for all $z \in \mathcal{S}, |z| \leq \epsilon$.

Let S denote the circle of directions at $z = 0$. The sectors at $z = 0$ are identified with intervals of S. For an interval $I \subset S$ we want to know whether there exists a solution y with $\widehat{y}$ as asymptotic expansion on the sector.

Consider two small intervals I_1, I_2 and solutions y_1, y_2 of our problem. Then $y_{12} := y_1 - y_2$ is a solution of the equation $(z\frac{d}{dz} + q)y_{12} = 0$ and y_{12} has 0 as asymptotic expansion on $I_1 \cap I_2$.

Write $\mathcal{A}_0$ for the sheaf on the circle S consisting of the meromorphic functions with 0 as asymptotic expansion. Then the *obstruction for a solution* y *on the sector* I with asymptotic expansion $\widehat{y}$ is a 1-cocycle for the sheaf $T := \ker(z\frac{d}{dz} + q, \mathcal{A}_0)$ and more precisely an element $\xi \in H^1(I, T)$.

The solution of the homogeneous equation reads

$$H(z) := \exp(\frac{q_k}{k}z^{-k} + \cdots + \frac{q_1}{1}z^{-1} - q_0 \log(z)).$$

For a direction e^{id} at $z = 0$ one considers the function $H(re^{id})$ for $r > 0$. For $r \downarrow 0$ there are three possibilities:
$Re(\frac{q_k}{k}e^{-idk}) > 0$ explosion; $Re(\frac{q_k}{k}e^{-idk}) < 0$ implosion; $Re(\frac{q_k}{k}e^{-idk}) = 0$ oscillation. In the last case, d is called a *Stokes direction*.

A *negative Stokes pair* is a pair $\{p, p + \frac{\pi}{k}\}$ of Stokes directions such that $Re(\frac{q_k}{k}e^{-idk}) < 0$ on the open interval $(p, p + \frac{\pi}{k})$.

The sheaf T of $\mathbb{C}$-linear vector spaces has therefore the property: the stalk $T_d \cong \mathbb{C}$ if d lies in the open interval $(p, p + \frac{\pi}{k})$ where $\{p, p + \frac{\pi}{k}\}$ is a negative Stokes pair. Otherwise $T_d = 0$.

An easy computation shows that: $H^1(I,T)=0$ *if and only if the interval I does not contain a negative Stokes pair*. One concludes:

(a) *For every direction d there is a solution y with asymptotic expansion $\widehat{y}$ in a sector $(d, d+\frac{\pi}{k}+\epsilon)$ for some $\epsilon>0$.*

A direction d is called *singular* if the descent of the function $|H(re^{id})|$ to 0 for $r \downarrow 0$ is maximal (i.e., $Re(\frac{q_k}{k}e^{-idk})$ is minimal). The singular directions are in fact the *midpoints of the negative Stokes pairs*.

(b) *For a non singular direction d there exists a unique solution y_d with asymptotic expansion $\widehat{y}$ on the sector $(d-\frac{\pi}{2k}-\epsilon, d+\frac{\pi}{2k}+\epsilon)$ for some $\epsilon>0$. This y_d is called the k-summation of $\widehat{y}$ in the direction d.*

Comments. There is an analytic way to obtain y_d, namely the formal k-Borel transform of $\widehat{y}$ followed by a k-Laplace integral. For a singular direction d, the y_d with the required properties does not exist in general.

The above generalizes easily to a matrix differential equation of the form $(z\frac{d}{dz}+A)(Y) = F$ where A is a $m\times m$-matrix of the form $\sum_{i\geq -k} A_i z^i$ and A_{-k} is a diagonal matrix with non zero entries, $F \in K^m$ and $\widehat{Y} \in \widehat{K}^m$ is any formal solution.

Now we consider a matrix differential equation of the form $(z\frac{d}{dz}+A)Y = 0$, where A is a $m\times m$-matrix of the form $\sum_{i\geq -k} A_i z^i$ and A_{-k} is a diagonal matrix with *distinct* non zero entries.

In order to make the procedure more transparent we use that after a suitable holomorphic transformation of the equation one may assume that A is the sum of a diagonal matrix $diag(q_1, \ldots, q_m)$ with $q_1, \ldots, q_m \in z^{-1}\mathbb{C}[z^{-1}]$, all $\deg_{z^{-1}} q_i \leq k$ and $\deg_{z^{-1}}(q_i - q_j) = k$ for $i \neq j$, and a matrix without poles. The $\{q_i\}$ are unique and will be called the *eigenvalues of the equation* or generalized local exponents. The equation is said to have *one level* k.

A *symbolic fundamental matrix* F for the equation is an invertible matrix satisfying $(z\frac{d}{dz} + A)F = 0$. The columns of F form a basis of the symbolic solutions of $(z\frac{d}{dz} + A)Y = 0$.

Classically, F is written as $\widehat{D} \cdot exp(E)$, where $\widehat{D} \in \mathrm{GL}(m, \widehat{K})$ and E is a diagonal matrix with entries $\int -q_i \frac{dz}{z},\ i = 1, \ldots, m$. The term $exp(E)$ poses no problems. The asymptotic problem is to obtain in certain sectors a meromorphic matrix B with asymptotic expansion $\widehat{D}$. The matrix $\widehat{D}$ is a formal solution of the equation

$$z\frac{d}{dz}(\widehat{D}) - \widehat{D}\cdot diag(q_1,\ldots,q_m) + A\widehat{D} = 0,$$

equivalently, $\widehat{D}$ considered as vector satisfies the differential equation

$(z\frac{d}{dz}+A)\widehat{D} = \widehat{D}(z\frac{d}{dz}+diag(q_1,\ldots,q_m))$ of size m^2. The eigenvalues of this new equation are the $q_i - q_j$. By assumption $\deg_{z^{-1}}(q_i - q_j) = k$ for $i \neq j$ and now we can apply the former result. One finds two statements *for equations with one level* k:

(a) **Theorem** (Y. Sibuya, 1968). *An asymptotic lift of* $\widehat{D}$ *exists on any sector of the form* $(d, d+\frac{\pi}{k}+\epsilon)$ *for small enough* $\epsilon > 0$.

The singular directions for $(z\frac{d}{dz}+A)Y = 0$ are defined to be the union of the singular directions for the $q_i - q_j,\ i \neq j$ (there are in total $2km(m-1)$ of them).

(b) *If* d *is not a singular direction then there exists a unique meromorphic matrix* B_d *on a sector* $(d - \frac{\pi}{2k} - \epsilon, d + \frac{\pi}{2k} + \epsilon)$ *(for small enough* $\epsilon > 0$*) with asymptotic expansion* $\widehat{D}$.
As before B_d is called the k-summation of $\widehat{D}$ in the direction d.

Comments. Sibuya's theorem is the main result on asymptotics used in the highly important paper of Jimbo–Miwa–Ueno (1981) on Painlevé equations. In Sibuya's theorem the essential assumption is that $(z\frac{d}{dz} + \sum_{i \geq -k} A_i z^i)Y = 0$ has one level k. This restriction is then also present in the work of Jimbo–Miwa–Ueno.

Stokes data for one level k equations

As before, $(z\frac{d}{dz}+A)Y=0$ is an equation with one level k and eigenvalues $q_1,\dots,q_m$.

We consider meromorphic matrices B on sectors with asymptotic expansion $\widehat{D}$ and satisfying $(z\frac{d}{dz}+A)B=B(z\frac{d}{dz}+diag(q_1,\dots,q_m))$.

For two such matrices B_1, B_2 the matrix $B_{12}:=B_2^{-1}B_1$, defined on the intersection of the two sectors, has as asymptotic expansion the identity matrix and satisfies

$$B_{12}(z\frac{d}{dz}+diag(q_1,\dots,q_m))B_{12}^{-1}=z\frac{d}{dz}+diag(q_1,\dots,q_m).$$

In other words, B_{12} is an automorphism of the 'standard' equation $z\frac{d}{dz}+diag(q_1,\dots,q_m)$ with asymptotic expansion the identity. Let T denote the sheaf (of, in general, non abelian groups) on the circle of directions S consisting of the automorphisms of this standard equation with asymptotic expansion the identity.

Then the problem of asymptotic lifts for $\widehat{D}$ defines a 1-cocycle with values in T and thus an element of $H^1(S,T)$. This 1-cocycle is called the *Stokes cocycle* and can be made explicit by choosing a covering of S by open intervals of length slightly more than $\frac{\pi}{k}$, using Sibuya's theorem.

The Stokes cocycle depends on the choice of this covering of S. In the work of Jimbo–Miwa–Ueno, a fixed covering of the circle S is taken and the matrices of the corresponding 1-cocycle are part of the analytic data in their paper.

An *intrinsic way* to represent the 1-cocycle is obtained by k-summation. Let d be a *singular direction* for $(z\frac{d}{dz}+A)Y=0$. Choose real numbers d^+, d^- with $d^- < d < d^+$ and close enough to d. Let $B_{d^-},\ B_{d^+}$ denote the k-sums of $\widehat{D}$ in the directions $d^-,\ d^+$. Then the *Stokes matrix* St_d for the singular direction is the constant matrix satisfying $B_{d^+}exp(E) = B_{d^-}exp(E)St_d$. A computation shows that St_d has a special form: If d is a singular direction for $q_i - q_j$ (with $i \neq j$), then $St_d - id$ has at most one non zero entry, namely at the place (i, j).

The *intrinsic Stokes data* for $(z\frac{d}{dz} + A)Y = 0$ is the collection of matrices with the above prescribed form $\{St_d\}_{0 \leq d < 2\pi,\ d \text{ singular}}$. It can be shown that the collection of all possible intrinsic Stokes data is in natural bijection with the set $H^1(S, T)$.

A *rather delicate fact*, for which I do not know an elementary proof and not touched upon in the Jimbo–Miwa–Ueno paper, is that for fixed eigenvalues $q_1, \ldots, q_m$, the elements of $H^1(S, T)$ (or all intrinsic Stokes data) are in bijection with the analytic equivalence classes of equations $(z\frac{d}{dz} + A)Y = 0$.

Formal classification, category Gr_1

Formal classification means the classification of linear differential equations over $\widehat{K} = \mathbb{C}((z))$. A classical way is as follows. After replacing $\widehat{K}$ by $\widehat{K_e} := \mathbb{C}((z^{1/e}))$, a given matrix differential equation can be transformed in a normal form $Y' = AY$ where $' := z\frac{d}{dz}$ and A is the direct sum of square blocks $A_{i,a}$ with $i = 1, \ldots, s$ and $1 \leq a \leq m_i$ of the form

$$\begin{pmatrix} a_i & 0 & . & . & . & 0 \\ 1 & a_i & 0 & . & . & . \\ 0 & 1 & a_i & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 1 & a_i \end{pmatrix};$$

$a_i \in \mathbb{C}[z^{-1/e}]$; $a_i - a_j \notin \mathbb{Q}$ for $i \neq j$ (and invariant under $Gal(\widehat{K_e}/\widehat{K})$).

Another classical way to normalize a formal differential equation is to write some formula for the fundamental matrix $F = \widehat{D} \cdot exp(E)$.

We prefer to work with differential modules M over $\widehat{K}$ and to use a structure on the solution space V of M for the classification.

There is an explicit universal Picard–Vessiot differential ring U_{formal}, extension of $\widehat{K}$, containing all solutions of all differential modules over $\widehat{K}$. As before, we adopt the differentiation $' := z\frac{d}{dz}$ on $\widehat{K}$.

U_{formal} contains 'symbols' $\{\underline{z}^c\}_{c\in\mathbb{C}}$, ℓ, satisfying $(\underline{z}^c)' = c \cdot \underline{z}^c$, $\underline{z}^{c_1+c_2} = \underline{z}^{c_1} \cdot \underline{z}^{c_2}$, $\underline{z}^c = z^c \in$ the algebraic closure of $\widehat{K}$ for $c \in \mathbb{Q}$. Further $\ell' = 1$.

The collection of all possible eigenvalues is $\mathcal{Q} := \bigcup_{m\geq 1} z^{-1/m}\mathbb{C}[z^{-1/m}]$. For $q \in \mathcal{Q}$ the symbols $e(q)$ satisfy $e(q)' = qe(q)$ $e(q_1 + q_2) = e(q_1) \cdot e(q_2)$.
These formulas define the universal ring $U_{formal} := \oplus_{q\in\mathcal{Q}}\widehat{K}[\{\underline{z}^c\}_{c\in\mathbb{C}}, \ell]e(q)$.

The formal monodromy γ acts as differential automorphism on U_{formal} by the formulas $\gamma \underline{z}^c = e^{2\pi i c}\underline{z}^c$, $\gamma\ell = \ell + 2\pi i$ and $\gamma e(q) = e(\gamma q)$.

The *solution space* of a differential module M of dimension m is the complex vector space $V = \ker(\partial, U_{formal} \otimes_{\widehat{K}} M)$ of dimension m.
It has an induced action of γ and a direct sum decomposition $V = \oplus_{q \in \mathcal{Q}} V_q$ given by $V_q = \ker(\partial, \widehat{K}[\{\underline{z}^c\}_{c\in\mathbb{C}}, \ell]e(q) \otimes_{\widehat{K}} M)$

and $\gamma(V_q) = V_{\gamma q}$. The q's with $V_q \neq 0$ are the *eigenvalues* (generalized exponents) of M.

Let Gr_1 denote the category with objects the tuples $(V, \{V_q\}_{q\in\mathcal{Q}}, \gamma)$ consisting of a finite dimensional vector space V, a direct sum decomposition $V = \oplus V_q$ (almost all $V_q = 0$) and a $\gamma \in \mathrm{GL}(V)$ with $\gamma(V_q) = V_{\gamma q}$ for all q.

It is obvious what the morphisms of Gr_1 are. Gr_1 is an abelian category and is in fact a neutral Tannakian category (i.e., has tensor products). The category $Diff_{\widehat{K}}$ consisting of all differential modules over $\widehat{K}$ is a Tannakian category, too. The universal ring U_{formal} provides a functor $Diff_{\widehat{K}} \to Gr_1$. We propose the classification provided by the result:

Theorem *The functor* $Diff_{\widehat{K}} \to Gr_1$ *is an equivalence of Tannakian categories.*

Meromorphic classification, category Gr_2

Let M denote a differential module over K. One associates to M the differential module $\widehat{M} = \widehat{K} \otimes_K M$ over $\widehat{K}$ and its solution space V with the additional structure $(V, \{V_q\}, \gamma)$. The eigenvalues $q_1, \ldots, q_m$ of M are the q's with $V_q \neq 0$. The *levels* of M are the degrees in the variable z^{-1} of the $q_i - q_j,\ i \neq j$. The set of the *singular directions* d for M and $\widehat{M}$ is the union of the singular directions for the $q_i - q_j,\ i \neq j$.

We make now the link with the earlier part of this lecture where we considered '*one level* $k \in \mathbb{Z}_{>0}$'. This means that all the degrees in z^{-1} of $q_i - q_j,\ i \neq j$ are k. For a non singular direction d, the k-summation method yields a $\mathbb{C}$-linear map $ksum_d : V \to Mer(d - \frac{\pi}{2k} - \epsilon, d + \frac{\pi}{2k} + \epsilon)$, where the last space is the vector space of the meromorphic functions on the indicated sector. The image of $ksum_d$ consists of the solutions of the differential module with entries in this space, i.e., $\ker(\partial, Mer(d - \frac{\pi}{2k} - \epsilon, d + \frac{\pi}{2k} + \epsilon) \otimes_K M)$.

It has the property that $ksum_d(v)$ has, for each $v \in V$, the asymptotic expansion v. For a singular direction d corresponding, say, to a single

difference $q_i - q_j$, the *Stokes map* $St_d \in \mathrm{GL}(V)$ is defined by $ksum_{d^+} \circ St_d = ksum_{d^-}$
(valid on the intersection of the sectors corresponding to d^+, d^-).
Further St_d has the form $id + L$, where L : $V \stackrel{projection}{\rightarrow} V_{q_i} \rightarrow V_{q_j} \subset V$.

Now we return to the general case. Explaining the powerful method of 'multisummation' (or even its definition) is beyond the scope of this lecture. This method generalizes k-summation to the case differential modules having any number of levels. For a non singular direction d there is a (uniquely determined) $\mathbb{C}$-linear map $multi_d : V \rightarrow Mer(d-\epsilon, d+\epsilon)$ which sends each element $v \in V$ to a meromorphic solution $multi_d(v)$ of the module on some sector around d with asymptotic expansion v. The uniqueness comes from additional, somewhat complicated, requirements on $multi_d(v)$.

As before, for a singular direction d, the Stokes map St_d is defined by
$multi_{d+} \circ St_d = multi_{d-}$;
St_d has the form $id + L$ where L is the sum, taken over all pairs (i, j) with d is singular for $q_i - q_j$, of some linear maps
$L_{i,j} : V \overset{projection}{\rightarrow} V_{q_i} \rightarrow V_{q_j} \subset V$.
It can be seen that St_d is a unipotent map (i.e., $St_d - id$ is nilpotent).

Now we define the objects of the category Gr_2. They are the tuples
$(V, \{V_q\}, \gamma, \{St_d\})$ with the requirements:
$(V, \{V_q\}, \gamma)$ is an object of Gr_1;
$St_d \in \mathrm{GL}(V)$ satisfy $\gamma^{-1} \circ St_d \circ \gamma = St_{d+2\pi}$ and St_d has the form $id + L$ where L is the sum of maps $L_{q,\tilde{q}}$, taken over all the pairs $(q, \tilde{q})$ such

that d is a singular direction for $q-\tilde{q}$, with $L_{q,\tilde{q}} : V \stackrel{projection}{\rightarrow} V_q \stackrel{linear}{\rightarrow} V_{\tilde{q}} \subset V$.
It is obvious how to define morphisms in Gr_2. The category Gr_2 is again a neutral Tannakian category. The multisummation procedure yields a functor $Diff_K \rightarrow Gr_2$. The highlight of the asymptotic theory of linear differential equations is the classification of meromorphic differential equations:

Theorem (Sibuya, Malgrange, Ramis, Balser,..., vdP) $Diff_K \rightarrow Gr_2$ *is an equivalence of categories.*

Informally, this states that a differential module over K is determined by its formal equivalence class and the Stokes maps. Moreover, for a given formal class *every* proposal for Stokes maps can be realised by a *unique* differential equation over K.

A useful formula is:
The topological monodromy is equal to (or conjugated to) the product

$$\gamma \circ St_{d_t} \circ \cdots \circ St_{d_1},$$

where $d_1 < \cdots < d_t$ *are the singular directions in* $[0, 2\pi)$.

Examples.
(1). We describe the family of the differential modules M over K having the properties: $\dim_K M = 2$, $\Lambda^2 M$ is trivial, the eigenvalues are $q_1 = z^{-2},\ q_2 = -z^{-2}$ and γ has eigenvalues α, α^{-1}.

Then $V = V_{q_1} \oplus V_{q_2} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and the matrix of γ w.r.t. the basis e_1, e_2 is $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$.

The singular directions $d \in [0, 2\pi)$ for $q_1 - q_2 = 2z^{-2}$ are defined by $|\exp(-\int 2z^{-2}\frac{dz}{z})|$ has maximal descent for $z = re^{id},\ r > 0, r \downarrow 0$. This is equivalent to $\cos(2d) = -1$ and $d = \frac{\pi}{2},\ \frac{3\pi}{2}$. The corresponding Stokes matrices have the form $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$.

The singular directions $d \in [0, 2\pi)$ for $q_2 - q_1$ are $0, \pi$ and the Stokes matrices have the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The toplogical monodromy is the product

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix}.$$

These Stokes matrices form the 'analytic part' of the family. The basis e_1, e_2 is unique up to multiplication by scalars. Thus we have to divide out the action of $\mathbb{G}_m$ given by $e_1, e_2 \mapsto te_1, t^{-1}e_2$.

This family $\mathcal{R}$ is the affine space with coordinate ring $\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{G}_m}$ which is $\mathbb{C}[x_{12}, x_{14}, x_{23}, x_{34}]$ where $x_{ij} := x_i x_j$. There is one relation, namely $x_{12}x_{34} - x_{14}x_{23} = 0$. The singular point of $\mathcal{R}$ corresponds to $x_1 = x_2 = x_3 = x_4 = 0$, i.e., the module M is the direct sum of two modules of dimension 1.

For each point in $\mathcal{R}$ there is a unique differential module M with the prescribed Stokes matrices. One can try to make this into a family $\mathcal{M}$ of differential modules. A reasonable choice for $\mathcal{M}$ is

$$z\frac{d}{dz} + \begin{pmatrix} z^{-2} & 0 \\ 0 & -z^{-2} \end{pmatrix} + \begin{pmatrix} a & b_0 + b_1 z^{-1} \\ c_0 + c_1 z^{-1} & -a \end{pmatrix},$$

with $e^{2\pi i a} = \alpha$ and divided out by the action $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ of $\mathbb{G}_m$.

$\mathcal{M}$ has dimension 3 and there is a highly transcendental map,
the *Riemann–Hilbert morphism* $RH : \mathcal{M} \to \mathcal{R}$.
It can be proven that the generic fibre of RH

is discrete and that, at least for $\alpha \neq \pm 1$, the image contains a Zariski open subset of $\mathcal{R}$.

(2). Now we consider the same type of modules but with $q_1 = z^{-5/2}$, $q_2 = -z^{-5/2}$. Write $V = V_{q_1} \oplus V_{q_2} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. The basis can be chosen such that γ has the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This determines e_1, e_2 up to multiplication by the same scalar.

The singular directions $d \in [0, 2\pi)$ for $q_1 - q_2$ are given by $\cos(5d/2) = -1$ and thus $d = 2\pi/5,\ 6\pi/5,\ (10\pi/5,\ 14\pi/5, ...)$. The Stokes matrices have the form $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$.

The singular directions $d \in [0, 2\pi)$ for $q_2 - q_1$ are given by $\cos(5d/2) = 1$ and thus $d = 0,\ 4\pi/5,\ 8\pi/5,\ (12\pi/5, ...)$. The Stokes matrices have the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The topological monodromy is the product

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ x_4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_5 \\ 0 & 1 \end{pmatrix}.$$

The space $\mathcal{R}$ of the analytic data is the affine space with coordinate ring $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$. Indeed, the only change of the basis of V is $e_1, e_2 \mapsto \lambda e_1, \lambda e_2$ with $\lambda \in \mathbb{C}^*$. This does not effect the x_i.

Finally we give the relation with the differential Galois group of the module M with tuple $(V, \{V_q\}, \gamma, \{St_d\})$:

Corollary (Ramis). *The differential Galois group is the smallest algebraic subgroup of* GL(V) *containing:*

(1). *For every homomorphism $h : \mathcal{Q} \to \mathbb{C}^*$, the map $\underline{h} \in$ GL(V), given by $\underline{h}$ is on each V_q the multiplication by $h(q)$* (the exponential torus).

(2). *The map $\gamma \in$ GL(V).*

(3). *The Stokes maps St_d.*