Solving linear differential equations

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1. Introduction to differential Galois theory

A *differential field* is a field K provided with a differentiation $'$. We suppose that the field of constants $C := \{a \in K|\ a' = 0\}$ is algebraically closed, has characteristic zero and $C \neq K$. The basic example is $K = \mathbb{C}(z)$ and $' = \frac{d}{dz}$. A homogeneous linear differential equation over K has the form

$$Ly = 0;\ L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0,$$

$$\partial = {}' = \frac{d}{dz}; \quad \text{all } a_i \in K.$$

The solutions $y \in K$ form a C-vector space. One expects that this space has dimension n. However, in general, there are not enough solutions in K itself. There exists a minimal differential ring $R \supset K$ such that the solutions $y \in R$ form indeed a C-vector space of dimension n. This R is unique, up to non unique isomorphism, and is called the *Picard-Vessiot ring* of the equation L over K.

One builds R as follows. Consider the differential ring

$$R_0 = K[X_1, \ldots, X_n, X_1^{(1)}, \cdots, X_{n-1}^{(n-1)}, X_n^{(n-1)}, \frac{1}{D}],$$

where $'$ on K is the given one; $X_i' := X_i^{(1)}$; $(X_i^{(1)})' = X_i^{(2)}$ etc.,
$(X_i^{(n-1)})' + a_{n-1}X_i^{(n-1)} + \cdots + a_1 X_i^{(1)} + a_0 X_i = 0$ and $D = \det(X_i^{(j)})_{i,j=0}^{n-1}$.

Let $I \subset R_0$ be an ideal, maximal among the differential ideals of R_0. Then I is a prime ideal and $R = R_0/I$ is the Picard-Vessiot ring. Its field of fractions is the *Picard-Vessiot field* of L over K.

The *differential Galois group* G of L/K is the group of the K-linear differential automorphisms of $R = K[x_1, \ldots, x_n, \ldots]$.
An element $g \in G$ acts linearly on *the solution space* $V := Cx_1 + \cdots + Cx_n$.
This induces a homomorphism $G \to \mathsf{GL}(V)$, which is injective and has as image a linear algebraic group. Thus the differential Galois group has a natural structure as linear algebraic group and V is a faithful representation of G. The Lie algebra $\mathfrak{g}$ of G can be identified with the K-linear derivations d on R commuting with $'$. Further V is a faithful representation of $\mathfrak{g}$.

Examples.

(1). $y' = ay$ has differential Galois group $G \subset C^*$. Let $m \geq 1$ be minimal such that $f' = maf$ has a non zero solution t in K, then $R = K(\sqrt[m]{t})$ and $G = \mu_m$. If m does not exists, then $G = C^*$.

(2). If $y' = a$ has no solution in K, then $R = K[t]$ with $t' = a$ and $G = C$.
For $K = \mathbb{C}(z)$ it is easy to provide an algorithm for these examples.

(3). A breakthrough is Jerry Kovacic's algorithm which computes the Picard-Vessiot ring and the differential Galois group for the order two linear equations $y'' + a_1 y' + a_0 y = 0$. It is based upon the following classification of (the conjugacy classes) of the algebraic subgroups $H \subset \mathrm{SL}(2, \mathbb{C})$

(a). $H = \mathsf{SL}_2(\mathbb{C})$;
(b). H is a subgroup of the Borel group B;
(c). H is not contained in the Borel group B and lies in the infinite dihedral group D_∞;
(d). H is one of the groups $A_4^{\mathsf{SL}_2}, S_4^{\mathsf{SL}_2}, A_5^{\mathsf{SL}_2}$, i.e., the preimages of the groups $A_4, S_4, A_5 \subset$ PSL$(2, \mathbb{C})$ under SL$(2, \mathbb{C}) \to$ PSL$(2, \mathbb{C})$.

With a small trick one reduces to $a_1 = 0$ and then the differential Galois group lies in SL$(2, \mathbb{C})$. One considers the Riccati equation $u' + u^2 + a_0 = 0$ satisfied by $u = \frac{y'}{y}$ if $y'' + a_0 y = 0$ and $y \neq 0$. In case (a) the Riccati equation has no algebraic solution. In the next cases the Riccati equation has an algebraic solution of degrees 1, 2, 4, 6, 12. For the differential field $K = \mathbb{C}(z)$ there is an algorithm computing the algebraic solutions of this Riccati equation. This algorithm first computes truncated local formal solutions at the singular points of the equation (i.e., the poles of a_0) and then tries to match these data globally.

Example: $y'' = ry$, $u' + u^2 = r := \frac{5}{16}z^{-2} + z$. The truncated local solutions at $z = 0$ are $u = (\frac{1}{2} \pm \frac{3}{4})z^{-1} + *z^{>0}$ and at $z = \infty$ they are $\pm z^{1/2} + *z^{\leq 0}$. Inspection of the 4 combinations yield the solutions $u = -\frac{1}{4}z^{-1} \pm z^{1/2}$. A basis of solutions for $y'' = ry$ is then $z^{-1/4}e^{\pm\frac{2}{3}z^{3/2}}$. The differential Galois group is D_∞.

(4). There is a vast literature on extending Kovacic's algorithm to higher order equations. Everything is based upon the classification of the algebraic subgroups of $\mathrm{SL}(n, \mathbb{C})$ and representations of linear algebraic groups. A basic ingredient is the (local formal, global) factorization of differential operators by Mark van Hoeij and others.

(5). A result of C. Tretkoff and M. Tretkoff is: *any linear algebraic group* $G \subset \mathrm{GL}(n, \mathbb{C})$ *can be realized as differential Galois group of a regular singular differential equation of order* n *over the field* $\mathbb{C}(z)$.

More sophisticated solutions of the '*inverse problem*', involving irregular singularities, have been developed by J.-P. Ramis. Explicit equations have been found by M.F. Singer and C. Mitschi. As an example: any connected semi-simple group G can be obtained as the differential Galois group of a matrix differential equation of the form $y' = (A_0 + A_1 z)y$ with $A_0, A_1 \in \mathrm{Matr}(n, \mathbb{C})$. More recent work of Julia Hartmann covers the general case explicitly.

2. The work of Fano

Gino Fano published in 1900 an extremely interesting paper (of almost 100 pages) on linear differential equations. His aim was to answer a question posed by L. Fuchs:

Is it possible to express the n independent solutions of a scalar linear differential equation of order n over $K = \mathbb{C}(z)$, *under the assumption that these solutions satisfy a non trivial homogeneous equation over* $\mathbb{C}$, in terms of solutions of scalar linear differential equations of lower order ?

Fano gave a positive answer for $n \leq 6$ and many examples for larger n. An *interesting example* is: A basis of solutions for $L_5 :=$

$\partial^5 + 2p\partial^3 + 3p'\partial^2 + (3p'' + p^2 - 4q)\partial + (p''' + pp' - 2q')$

with $p, q \in K$ can be written as $u_1 u_2' - u_1' u_2$ where u_1, u_2 are solutions of $L_4 := \partial^4 + \partial p \partial + q$.

A modern interpretation of Fano's method.

Let L_n be again a scalar equation of order n. Consider the subring $\mathbb{C}[X_1,\ldots,X_n]$ of the ring

$$R_0 = K[X_1,\ldots,X_n,X_1^{(1)},..,X_{n-1}^{(n-1)},X_n^{(n-1)},\frac{1}{D}],$$

that we had before. The ideal $H \subset \mathbb{C}[X_1,..,X_n]$, generated by the homogeneous polynomials in the variables $X_1,\ldots,X_n$ that belong to the ideal I. Since I is a prime ideal, also H is a (homogeneous) prime ideal and defines an irreducible projective variety $S \subset \mathbb{P}(V) \cong \mathbb{P}^{n-1}$. Fano formulates this by: "*the solutions of L lie on S*". The interpretation of this terminology seems to be the following.

Take a point z_0 in the complex plane where the equation L_n has n independent local, meromorphic solutions $f_1,\ldots,f_n$. For z in a neighbourhood D of z_0, there is a well defined analytic map $m : D \to \mathbb{P}^{n-1}$, given by the formula

$z \mapsto (f_1(z) : f_2(z) : \cdots : f_n(z))$. The smallest projective subspace of $\mathbb{P}^{n-1}$, containing the image of m, is easily seen to be S.

The group F that Fano considers is the algebraic subgroup of PGL(V) consisting of the elements A with $A(S) = S$. This group is a rather coarse approximation of the differential Galois group $G \subset$ GL$(n, \mathbb{C})$. In fact, F contains the image of G in PGL$(n, \mathbb{C})$ and is, in general, much larger.

In Fuchs' question it is given that $S \neq \mathbb{P}^{n-1}$. Fano considered especially the cases $\dim S = 1, 2$ and S is a hypersurface. Using his extensive knowledge of low dimensional varieties he was able to prove the conjecture for $n \leq 6$.
Example: Let a scalar equation L_4 of order 4 give rise to a non singular quadric surface $S \subset \mathbb{P}^3$. Now S is the image of $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.

This implies that there are two scalar equations L_2, L_2' of order 2 such that a basis of the solutions of L_4 has the form $\{fg\}$ where f and g are solutions of L_2 and L_2'.

A modern solution of Fuchs' question.
M.F. Singer (1988) produced a counterexample for $n = 7$. The complete answer to this question was given by A.K. Nguyen and MvdP. The important step is the **Observation**:

Suppose that the solutions of a linear differential equation cannot be expressed in terms of equations of lower order and algebraic extensions of K.
Then the Lie algebra $\mathfrak{g}$ of the differential Galois group is simple and its action on the solution space V is a faithful representation of smallest dimension.

The question now translates into: Let $\mathfrak{g}$ be simple and let V be a faithful representation of smallest dimension. Does there exists a non trivial, $\mathfrak{g}$-invariant homogeneous form F on V? The next table is the answer.

Simple Lie algebras, smallest dimension, degree of F

symbol		Lie algebra	smallest	deg F
A_n	$n \geq 1$	$\mathfrak{sl}_{n+1}$	$n+1$	NO
B_n	$n \geq 3$	$\mathfrak{so}_{2n+1}$	$2n+1$	2
C_n	$n \geq 2$	$\mathfrak{sp}_{2n}$	$2n$	NO
D_n	$n \geq 4$	$\mathfrak{so}_{2n}$	$2n$	2
E_6		$\mathfrak{e}_6$	27	3
E_7		$\mathfrak{e}_7$	56	4
E_8		$\mathfrak{e}_8$	248	2
F_4		$\mathfrak{f}_4$	26	2
G_2		$\mathfrak{g}_2$	7	2

We note that : $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$, $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ and $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. This confirms Fano's result for $n \leq 6$ and Singer's counterexample is the case $\mathfrak{so}_7$.

3. Differential modules, Tannakian categories

In the above example of Fano, the operator L_5 corresponds to the Lie algebra $\mathfrak{g} = \mathfrak{sp}_4$ and V_5 is its 5-dimensional representation. The operator L_4 corresponds to $\mathfrak{sp}_4$ with the standard 4-dimensional representation. The formula for the solutions reflects that $\Lambda^2 V_4 = V_5 \oplus \mathbf{1}$.

In the sequel we show how this ad hoc case of reducing a differential equation to one of lower degree can be done systematically. First we have to introduce *differential modules*, replacing scalar linear differential equations.

A differential module M is a vector space over K provided with a C-linear map $\partial : M \to M$ satisfying $\partial(fm) = f'm + f\partial m$ for $f \in K,\ m \in M$. In other words M is a left $K[\partial]$-module which has finite dimension over K. It is known that M has a *cyclic vector* e, i.e., $K[\partial]e = M$. Since $K[\partial]$ is a left principal ideal domain, one has $M \cong K[\partial]/K[\partial]L$ for some monic operator $L \in K[\partial]$. Operations of linear algebra with differential modules are the obvious ones. The translations into operations with operators are somewhat unnatural.

Let Diff_K denote the *category of all differential modules over* K. This category has the following features:
(1) For two objects M, N, there is a *tensor product* $M \otimes N$. It is the K-vector space $M \otimes_K N$ equipped with $\partial(m \otimes n) := \partial(m) \otimes n + m \otimes \partial n$. It has a *unit object* $\mathbf{1} := K$ equipped with $\partial ='$.

(2) For each object M there is a *dual* $M^* := \mathrm{Hom}_K(M,K)$, equipped with $(\partial\ell)(m) = \ell(\partial m) - (\ell(m))'$ for $m \in M,\ \ell \in M^*$.
(3) It is an *abelian category*, i.e., a category of (left) modules over some ring, closed under taking kernels, cokernels and finite direct sums.
(4) There is an isomorphism $\mathrm{End}(\mathbf{1}) \cong C$.
(5) There is a *fibre functor* $\omega : \mathrm{Diff}_K \to \mathrm{Vect}_C$ (meaning C-linear, faithful, exact, commuting with tensor products). Here Vect_C denotes the category of the finite dimensional vector spaces over C. Indeed, there exists a universal Picard-Vessiot ring $Univ$, the direct limit of the Picard–Vessiot rings for all objects M of Diff_K. The fibre functor is defined by $M \mapsto \omega(M) = \ker(\partial, Univ \otimes_K M)$.

A category with these 5 features (including a lot of rules and commutative diagrams) is called a *C-linear, neutral Tannakian category*.

For any linear algebraic group G over C (or more generally an affine group scheme over C) the objects of the category Repr_G are the representations of G on finite dimensional C-vector spaces and the morphisms are the usual homomorphisms of representations. This is a C-linear neutral Tannakian category. The fibre functor $\omega : \mathrm{Repr}_G \to \mathrm{Vect}_C$ is the forgetful functor which associates to a representation V of G the C-vector space V.

For any (C-linear, neutral) Tannakian category A and any object M of A one forms the full subcategory $\{\{M\}\}$ whose objects are the subquotients of finite sums of objects of the form $M^* \otimes \cdots \otimes M^* \otimes M \otimes \cdots \otimes M$. Then $\{\{M\}\}$ is also a neutral Tannakian category. For a set of objects $\{M_i\}$ one defines the full subcategory $\{\{\{M_i\}\}\}$ of A in a similar way.

A useful result, valid for linear algebraic groups (but not for Lie algebra's!), is:
Let V be a faithful representation of the linear algebraic group G over C, then $\{\{V\}\} = \text{Repr}_G$.

A **main result** on C-linear, neutral Tannakian categories is:

For every C-linear, neutral Tannakian category A, there exists a unique affine group scheme G over C (i.e., a projective limit of linear algebraic groups over C) such that A with all its structure is equivalent to the category Repr_G.

This result applied to Diff_K yields an affine group scheme which is, generally, too big and too complicated to be useful. However, for any object M of Diff_K, the linear algebraic group over C corresponding to the C-linear, neutral Tannakian category $\{\{M\}\}$ is the *differential Galois group of* M.

Explicitly: Let $L \supset K$ denote the Picard–Vessiot field of M and let G be the group of the K-linear differential automorphisms of L/K. Then the equivalence $\{\{M\}\} \to \text{Repr}_G$ is given by $N \mapsto \ker(\partial, L \otimes_K N)$. Moreover, the objects N of $\{\{M\}\}$ are the differential modules such that the solutions of N can be expressed in the solutions of M. Indeed, the coordinates of the solutions of M generate the Picard–Vessiot field extension L of K and the solution space of N is $\ker(\partial, L \otimes_K N)$.

A useful example:
Let $\mathfrak{g}$ be a *semi-simple Lie algebra over* $\mathbb{C}$. The category $\mathsf{Repr}_{\mathfrak{g}}$ is also a neutral Tannakian category. The corresponding group is in fact the simply connected, semi-simple group G^+ with Lie algebra $\mathfrak{g}$.

The functor $\mathsf{Repr}_{G^+} \to \mathsf{Repr}_{\mathfrak{g}}$, which associates to each (complex) representation of G^+ the representation of its Lie algebra $\mathfrak{g}$, is in fact an equivalence of Tannakian categories.
A faithful representation W of $\mathfrak{g}$ of minimal dimension can be seen to come from a faithful representation of G^+ of minimal dimension. In particular $\{\{W\}\} = \mathsf{Repr}_{\mathfrak{g}}$. We give a special case.

For $\mathfrak{g} = \mathfrak{sl}_2$, one has $G^+ = \mathsf{SL}_2$. The standard 2-dimensional representation W of $\mathfrak{sl}_2$ comes from the standard representation of SL_2 and thus $\{\{W\}\} = \text{Repr}_{\mathfrak{sl}_2}$. We note that the second symmetric power $V = sym^2W$ is a faithful representation of $\mathfrak{sl}_2$ but $\{\{V\}\} \neq \text{Repr}_{\mathfrak{sl}_2}$.

An *essential aspect of Fano's work* consists of reducing (with respect to solutions), if possible, a differential equation over $K = \mathbb{C}(z)$, to equations of lower order and finite extensions of K. This leads to replacing K by its algebraic closure $\overline{K}$ and to the following translation:

Let M be a differential module over $\overline{K}$. Produce differential modules $\{N_i\}$ with $\max\{\dim N_i\}$ *minimal and such that M belongs to $\{\{\{N_i\}\}\}$.*

There are some obvious cases where M can be solved by modules of lower dimension, e.g., M reducible or $M = A \otimes B$ with $\dim A, \dim B <$

dim M. We now sketch the proof of the '**Observation**' of Section 2:

The differential module M over $\overline{K}$ cannot be reduced to modules of lower dimension if and only if the Lie algebra $\mathfrak{g}$ of the differential Galois group is simple and its representation on the solution space has smallest dimension.

Proof. M is irreducible and one may suppose that det $M = \mathbf{1}$ (by solving an equation of order 1). The differential Galois group $G \subset \mathrm{SL}(V)$ is connected (because $\overline{K}$ is algebraically closed) and semi-simple because the representation is irreducible.

Let $G^+ \to G$ denote the universal covering of G. Now there is a subtle point, namely the existence of a differential module M^+ over $\overline{K}$ with differential Galois group G^+ such that M

belongs to $\{\{M^+\}\}$ and thus $\{\{M\}\} \subset \{\{M^+\}\}$. The equivalences $\{\{M^+\}\} \to \mathsf{Repr}_{G^+} \to \mathsf{Repr}_{\mathfrak{g}}$ permit us to work with representations of $\mathfrak{g}$. Let M correspond to the irreducible representation V of $\mathfrak{g}$. If $\mathfrak{g}$ is semi-simple but not simple, then it is known that V is a tensor product. Thus $\mathfrak{g}$ is simple. Let W be a faithful representation of $\mathfrak{g}$ of smallest dimension, then $V \in \{\{W\}\} = \mathsf{Repr}_{\mathfrak{g}}$. Thus V has smallest dimension. $\square$

4. Representations of semi-simple Lie algebras

For an irreducible differential module M, with a corresponding representation V of a semi-simple Lie algebra $\mathfrak{g}$, there are two cases of *reduction to lower dimension*:
(a) $\mathfrak{g}$ is semi-simple but not simple.
(b) $\mathfrak{g}$ is simple but V does not have smallest

dimension.
We give explicit data.

A table of irreducible representations

We present here a list of irreducible representations V, $\dim V = d$, of semi-simple Lie algebras, including the decomposition of $\Lambda^2 V$ and $sym^2 V$.
We adopt here and in the sequel the efficient notation of the online program [LiE] *for irreducible representations*.
This is the following. After a choice of simple roots $\alpha_1, \ldots, \alpha_d$, the Dynkin diagram (with standard numbering of the vertices by the roots) and the fundamental weights $\omega_1, \ldots, \omega_d$ are well defined. The irreducible representation with weight $n_1\omega_1 + \cdots + n_d\omega_d$ is denoted by $[n_1, \ldots, n_d]$.

In particular, $[0,\ldots,0]$ is the trivial representation of dimension 1.

Table of the irreducible representations of dimension $d \leq 6$.

d	Lie alg	repr	Λ^2	sym^2
2	$\mathfrak{sl}_2$	$[1]$	$[0]$	$[2]$
3	$\mathfrak{sl}_2$	$[2]$ $*$	$[2]$	$[4],[0]$
3	$\mathfrak{sl}_3$	$[1,0]$	$[0,1]$	$[2,0]$
4	$\mathfrak{sl}_2$	$[3]$ $*$	$[4],[0]$	$[6],[2]$
4	$\mathfrak{sl}_4$	$[1,0,0]$	$[0,1,0]$	$[2,0,0]$
4	$\mathfrak{sp}_4$	$[1,0]$	$[0,1],[0,0]$	$[2,0]$
4	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$	$[1]\otimes[1]$ $*$	$[0]\otimes[2],[2]\otimes[0]$	$[0]\otimes[0],[2]$
5	$\mathfrak{sl}_2$	$[4]$ $*$	$[6],[2]$	$[8],[4],[0]$
5	$\mathfrak{sp}_4$	$[0,1]$ $*$	$[2,0]$	$[0,2],[0,0]$
5	$\mathfrak{sl}_5$	$[1,0,0,0]$	$[0,1,0,0]$	$[2,0,0,0]$
6	$\mathfrak{sl}_2$	$[5]$ $*$	$[8],[4],[0]$	$[10],[6],[2]$
6	$\mathfrak{sl}_3$	$[2,0]$ $*$	$[2,1]$	$[4,0],[0,2]$
6	$\mathfrak{sl}_4$	$[0,1,0]$ $*$	$[1,0,1]$	$[0,2,0],[0,0$
6	$\mathfrak{sl}_6$	$[1,0,0,0,0]$	$[0,1,0,0,0]$	$[2,0,0,0,0]$
6	$\mathfrak{sp}_6$	$[1,0,0]$	$[0,1,0],[0,0,0]$	$[2,0,0]$
6	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$	$[1]\otimes[2]$ $*$	$[0],[0]\otimes[4],[2]\otimes[2]$	$[0]\otimes[2],[2]$
6	$\mathfrak{sl}_2 \times \mathfrak{sl}_3$	$[1]\otimes[1,0]$ $*$	$[0]\otimes[2,0],[2]\otimes[0,1]$	$[0]\otimes[0,1],[$

For the $\mathfrak{sl}_n$ with $n > 2$ we have omitted duals of representations. Further we have left out symmetric cases. The $*$ indicates that there is a reduction to lower dimension and all these cases are present in Fano's work.

The decompositions of the second symmetric power and the second exterior power are useful to distinguish the various cases. We are here especially interested in those representations which can be expressed in terms of representations of lower dimension. In dimensions $7-11$, one finds for the new items of this sort (here we omit the case $\mathfrak{sl}_2$ and duals and symmetric situations) the list:
$\mathfrak{sl}_3$ with $[1,1]$ (dim 8), $[3,0]$ (dim 10);
$\mathfrak{sl}_4$ with $[2,0,0]$ (dim 10);
$\mathfrak{sl}_5$ with $[0,1,0,0]$(dim 10);
$\mathfrak{so}_7$ with $[0,0,1]$(dim 8);
$\mathfrak{sp}_4$ with $[2,0]$(dim 10);
$\mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [3]$ (dim 8), with $[2] \otimes [2]$

(dim 9), with $[1] \otimes [4]$ (dim 10);
$\mathfrak{sl}_2 \times \mathfrak{sl}_3$ with $[2] \otimes [1,0]$ (dim 9); $\mathfrak{sl}_2 \times \mathfrak{sl}_4$ with $[1] \otimes [1,0,0]$ (dim 8);
$\mathfrak{sl}_2 \times \mathfrak{sp}_4$ with $[1] \otimes [1,0]$ (dim 8), with $[1] \otimes [0,1]$ (dim 10);
$\mathfrak{sl}_2 \times \mathfrak{sl}_5$ with $[1] \otimes [1,0,0,0]$ (dim 10); $\mathfrak{sl}_3 \times \mathfrak{sl}_3$ with $[1,0] \otimes [1,0]$ (dim 9);
$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $[1] \otimes [1] \otimes [1]$ (dim 8).

Subcategories of $\mathrm{Repr}_{\mathfrak{g}}$ **for a semi-simple** $\mathfrak{g}$
Let G^+ denote the simply connected, linear algebraic group with Lie algebra $\mathfrak{g}$. Its center Z is a finite group. Any connected group with Lie algebra $\mathfrak{g}$ has the form G^+/Z' where Z' is a subgroup of Z. The list of the groups Z that occur for the simple Lie algebra's is:
$\mathbb{Z}/(n+1)\mathbb{Z}$ for A_n; $\mathbb{Z}/2\mathbb{Z}$ for B_ℓ, C_ℓ, E_7; $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for D_ℓ with ℓ even; $\mathbb{Z}/4\mathbb{Z}$ for D_ℓ with ℓ odd; $\mathbb{Z}/3\mathbb{Z}$ for E_6; 0 for E_8, F_4, G_2.

As remarked before, the obvious functor T : $\mathrm{Repr}_{G+} \to \mathrm{Repr}_{\mathfrak{g}}$ is an equivalence of Tannakian categories. For any subgroup Z' of Z, the restriction of the functor T to $\mathrm{Repr}_{G+/Z'}$ induces an equivalence with a full Tannakian subcategory of $\mathrm{Repr}_{\mathfrak{g}}$. And this describes all these subcategories. The largest one $\mathrm{Repr}_{\mathfrak{g}}$, corresponding to G^+, equals $\{\{V\}\}$, where V is the standard faithful $\mathfrak{g}$-module. The smallest one, corresponding to G^+/Z, equals $\{\{W\}\}$, where W is the adjoint representation.

Example: $G^+ = \mathrm{SL}_3$ is simply connected and $\mathfrak{g} = \mathfrak{sl}_3$. There is only one other connected group with Lie algebra $\mathfrak{sl}_3$, namely $\mathrm{PSL}_3 = \mathrm{SL}_3/\mu_3$. Let V be the standard representation of SL_3 with T-image $(\mathfrak{sl}_3, [1,0])$. Then sym^3V is a faithful representation for PSL_3 and its T-image is $W := (\mathfrak{sl}_3, [3,0])$. Then $\{\{V\}\} = Repr_{\mathfrak{sl}_3}$ and $\{\{W\}\}$ is the full subcategory of $Repr_{\mathfrak{sl}_3}$ for which the irreducible objects are the $[a, b]$ with $a \equiv b \mod 3$. The adjoint representation of PSL_3 on $\mathfrak{sl}_3$ has T-image $(\mathfrak{sl}_3, [1, 1])$ which also generates this full subcategory.

5. Strategy for reduction to lower dimension

The *essential case* to consider is an irreducible differential module N over K with connected differential Galois group and such that the action of its semi-simple Lie algebra $\mathfrak{g}$ on the solution space is the adjoint representation. We will call N an *adjoint differential module*.

An *explicit standard differential module* (M, ∂_S) for $\mathfrak{g}$ is defined by:

(a) Let, as before, G^+ be the simply connected group with Lie algebra $\mathfrak{g}$. Write $G^+ = G_1 \times \cdots \times G_s$ with each G_i simple and let V_i be a faithful representation of smallest dimension of G_i. Define the (standard) G^+-module $V := \oplus_{i=1}^s V_i$.
(b) Define $M := K \otimes_{\mathbb{C}} V$ and define ∂_S on M by $\partial_S = \partial_0 + S$. Here ∂_0 is the derivation on M given by $\partial_0(f \otimes v) = f' \otimes v$ for $f \in K,\ v \in V$. Further $\mathfrak{g} \subset \mathrm{End}(V)$ and $S \in \mathfrak{g}(K)$.
(c) S should be such that the differential Galois group of M is G^+.

The step from an explicit standard differential module (M, ∂_S) to an adjoint module N, is just a construction of linear algebra. One considers the direct summand $N := K \otimes_{\mathbb{C}} \mathfrak{g}$ of $\mathrm{End}_K(M) = K \otimes_{\mathbb{C}} \mathrm{End}(V)$. Let ∂_0 be the derivation on N, given by $\partial_0(f \otimes g) = f' \otimes g$ for $f \in K,\ g \in \mathfrak{g}$. One easily verifies that (M, ∂_S)

induces on N the derivation $A \mapsto \partial_0(A) + [A, S]$. In this way (M, ∂_S) induces the adjoint differential module N.

The other direction: from an adjoint differential module N for $\mathfrak{g}$ to an explicit standard differential module is the *strategy for reduction to lower dimension*.

Theorem
Let N be an adjoint differential module for G. The $\mathbb{C}$-Lie algebra structure of on the solution space $\mathfrak{g}$ of N induces a K-Lie algebra structure $[\ ,\ ]$ on N satisfying $\partial[a, b] = [\partial a, b] + [a, \partial b]$ for all $a, b \in N$. This structure is unique up to multiplication by an element in $\mathbb{C}^*$.

The assumption that K is a C_1-field implies that there exists an isomorphism of K-Lie algebras $\phi : N \to K \otimes_{\mathbb{C}} \mathfrak{g}$. After choosing ϕ, there *exists a unique $S \in \mathfrak{g}(K)$ such that N is isomorphic to the adjoint module induced by the explicit standard module (M, ∂_S).*

Comments. The computation of the Lie algebra structure on N amounts to computing a rational solution (i.e., with coordinates in K) of the differential module $\mathrm{Hom}(\Lambda^2 N, N)$. The computation of $S \in \mathfrak{g}(K)$ is an easy exercise on Lie algebras. The computation of an isomorphism ϕ leads to

The problem:
Let $K \supset \mathbb{C}$ be a C_1-field and N a semi-simple Lie algebra over K. Compute an isomorphism $\phi : K \otimes_{\mathbb{C}} \mathfrak{g} \to N$ for some $\mathbb{C}$-Lie algebra $\mathfrak{g}$. This amounts to finding a Cartan subalgebra of N, defined over $\mathbb{C}$.

Find an algorithm, based on the C_1-property of K, that produces a Cartan subalgebra defined over $\mathbb{C}$.

The case $\overline{K} \otimes_K N = \overline{K} \otimes_{\mathbb{C}} \mathfrak{sl}_2$.
Take a basis n_1, n_2, n_3 of N. An element $n := \sum x_i n_i$ is mapped to the characteristic polynomial of $[\ ,n]$ acting on N, which is $T^3 - q(x_1, x_2, x_3)T$ and q is homogeneous of degree 2. By the C_1-property of K there exists a non trivial solution of $q(x_1, x_2, x_3) = 0$ in K. The resulting element e has the property $[\ ,e]$ is nilpotent. Inspection of $\overline{K} \otimes_K N$ yields that Ke lies in $[N, e]$. Thus there exists an element $h \in N$ with $[h, e] = 2e$. The eigenvalues of $[h,\ ]$ are $0, 2, -2$ and has eigenvectors $h,\ e_1 = e,\ e_2$. After multiplying e_2 by an element in K^* one has $[e_1, e_2] = h$. Thus $\mathbb{C}h + \mathbb{C}e_1 + \mathbb{C}e_2 = \mathfrak{sl}_2$ and we have an explicit isomorphism $N \to K \otimes_{\mathbb{C}} \mathfrak{sl}_2$.

6. Examples

For all the cases with $\dim M \leq 11$, where reduction to lower dimension is possible, explicit algorithm are given. In many cases, a shortcut is chosen instead of going to an adjoint differential module. Instead of working with the algebraic closure $\overline{K}$ of $\mathbb{C}(z)$ one makes the more natural choice: 'K is a finite extension of $\mathbb{C}(z)$'. This introduces a new technical problem (with technical solutions for $\dim M \leq 11$) of computing the minimal finite extension $K^+ \supset K$ such that the differential Galois group of $K^+ \otimes_K M$ is connected.
Notation: $\overline{M} = \overline{K} \otimes_K M$.

These technicalities are reading material but come along with interesting statements, like:

Theorem *The* $(\mathfrak{sl}_4, [0,1,0])$ *case.* Let M be a differential module of dimension 6. The following properties of M are equivalent (no conditions on M and K).
(1) $M \cong \Lambda^2 N$ for some module of dimension 4 with $\det N = \mathbf{1}$.
(2) There exists $F \in sym^2 M$ with $\partial F = 0$ such that F is non degenerate and M has a totally isotropic subspace of dimension 3.
Using F one finds an explicit formula for N.

Theorem *The* $(\mathfrak{sp}_4, [0,1])$ *case.* Let M be absolutely irreducible of dimension 5 and $\det M = 1$. The representation of the Lie algebra of the differential Galois group is $(\mathfrak{sp}_4, [0,1])$ if and only if $sym^2 M$ is a direct sum of two irreducible spaces of dimensions $1, 14$.
Working with $M \oplus \mathbf{1}$ and using the previous result one finds a standard differential module for $(\mathfrak{sp}_4, [1,0])$.

Theorem *The* $(\mathfrak{sl}_2 \times \mathfrak{sl}_2, [1] \otimes [1])$ *case.* Let M be a differential module over K of dimension 4 with $\det M = \mathbf{1}$ (no further conditions on M and K). Then $M \cong A \otimes B$ for modules A, B of dimension 2 and with $\det A = \det B = \mathbf{1}$ if and only if there exists $F \in sym^2 M$, $\partial F = 0$, F is non degenerate and has an isotropic subspace of dimension 2.
Using F one obtains explicit formulas for A, B.

The $(\mathfrak{sl}_2 \times \mathfrak{sl}_3, [1] \otimes [1, 0])$ *case.*
Let M with $\det M = \{1\}$ be an absolutely irreducible differential module correspond to these data. The *problem* is to decompose M as $N_2 \otimes N_3$ with $\dim N_i = i$, $\det N_i = \mathbf{1}$, corresponding to and $(\mathfrak{sl}_2, [1])$ and $(\mathfrak{sl}_3, [1, 0])$.

The construction follows from the observation that $[1] \otimes [1, 0]$ 'generates' the Tannakian category $Repr_{\mathfrak{sl}_2 \times \mathfrak{sl}_3}$.

Explicitely, $([1] \otimes [1,0]) \otimes ([1] \otimes [1,0])$ is the direct sum of the irreducible objects
$[0] \otimes [2,0]$, $[0] \otimes [0,1]$, $[2] \otimes [2,0]$, $[2] \otimes [0,1]$
of dimensions $6, 3, 18, 9$. The corresponding direct sum decomposition of $\overline{M} \otimes \overline{M}$ is already present over K, since the Galois group $Gal(\overline{K}/K)$ cannot permute subspaces of distinct dimensions. Choose N_3 to be the dual of the factor of $M \otimes M$ of dimension 3.

Next, consider $([1] \otimes [1,0]) \otimes ([0] \otimes [0,1])$ which is the direct sum of the irreducible objects
$[1] \otimes [0,0]$, $[1] \otimes [1,1]$ of dimensions $2, 16$.
As before, $M \otimes N_3^*$, has a direct summand N_2 of dimension 2 and it can be proven that
$M \cong N_2 \otimes N_3$.