IRREGULAR MIXED HODGE STRUCTURES AND MODULES

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Abstract. In the talk, I will define the (neutral Tannakian) category of irregular mixed Hodge structures, starting from the notion of mixed twistor structure (Simpson) and using the notion of mixed twistor $\mathscr{D}$-module (T. Mochizuki). Applications will be given to a Künneth formula for the irregular Hodge filtration attached to a Thom-Sebastaini sum of meromorphic functions. This is a joint work with Jeng-Daw Yu (Taipei).

1. Attempts for generalizing Hodge theory

There have been various attempts to generalizing Hodge theory, with various motivations.

(1) Deligne (1984) was considering a meromorphic bundle V on a projective curve C with connection having with possibly irregular singularities, motivated by understanding exponential periods. Meaning of "Hodge": to define a filtration F^pV by coherent $\mathscr{O}_C$-submodules satisfying $\nabla F^p \subset F^{p-1}$ such that the "E_1-degeneration property" holds:

$$H^i(C, F^p\,\mathrm{DR}(V,\nabla)) \longrightarrow H^i(C, \mathrm{DR}(V,\nabla))\text{is injective } \forall\, p.$$

The image filtration $F^pH^i(C, \mathrm{DR}(V,\nabla))$ is understood as a generalized Hodge filtration. Deligne managed to exhibit F^pV when $\nabla = \nabla^o + \mathrm{d}f$, ∇^o unitary and f meromorphic on C.

(2) Simpson (1997) introduced the notion of mixed twistor structure, as a bundle on $\mathbb{P}^1$ with an increasing filtration W opposite to its HN filtration. Meaning of "Hodge": interpretation of w-opposedness of F and $\overline{F}$ in the pure case as a bundle with pure slope w. This gives an abelian category.

(3) Barannikov (1999) introduced the notion of $\frac{\infty}{2}$ variation of Hodge structure, motivated by application understanding periods in non-commutative geometry. As with Simpson, idea that a generalized Hodge theory has objects living in dimension one, not vector spaces. Description of a $\mathbb{C}[\![z]\!]$-submodule of a $\mathbb{C}(\!(z)\!)$-vector space. The category he obtains is not abelian.

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(4) Hertling (2001), introduces the notion of TERP structure. It is similar to that of Barannikov, but with supplementary structures (connection in the z-direction, real structure, pre-polarization). Hertling emphasizes the notion of pure polarizable TERP structure, which is motivated for constructing a tt*-structure on Frobenius manifolds. Can be related to pure twistor structures: pure *integrable* twistor structure (i.e., vector bundle on $\mathbb{P}^1$ with a connection in the z-variable having a pole of order $\leqslant 2$ at $0,\infty$). Pure (or mixed) integrable twistor structures form an abelian category.

(5) Katzarkov-Kontsevich-Pantev (2008) introduced the notion of non-commutative Hodge structure defined over a subfield $\Bbbk$ of $\mathbb{R}$. If this subfield is $\mathbb{R}$, this strengthens a little the notion of pure polarized TERP structure by adding the condition that the pairing induced on the Stokes structure should be defined over $\mathbb{R}$. The motivation is similar to that of Barannikov.

(6) Kontsevich-Soibelman (2011) introduced the notion of exponential mixed Hodge structure, relying on the notion of mixed Hodge module of M. Saito. Here also, the object lives in dimension one, i.e., is a mixed Hodge module on the affine line $\mathbb{A}^1$ satisfying a supplementary property: the associated perverse sheaf should have zero global hypercohomology. This produces a neutral Tannakian category. Drawback: one does not know how to extend this notion in higher dimension.

(7) On the other hand, the notion of mixed twistor $\mathscr{D}$-module introduced by T. Mochizuki, when complemented with an integrability structure and a $\Bbbk$-structure, gives the right extension to higher dimension of the notion of non-commutative mixed Hodge structure.

2. The case of a pair (X,f)

We exemplify the various structures on the case of a pair (X,f) consisting of a smooth projective variety and a meromorphic function on it. Let U be the open set on which f is regular.

(1) $H^k_{\mathrm{DR}}(X,f):=H^k_{\mathrm{DR}}(U,f)=\boldsymbol{H}^k(U,(\Omega^\bullet_U,\mathrm{d}+\mathrm{d}f))$ and can change the compactification so that $f:X\to\mathbb{P}^1$ and $X\smallsetminus U=D$ is a ncd. Let $P=f^*(\infty)$.

- Kontsevich [**10**] defines locally free subsheaves of $\Omega^p_X(\log D)$:

$$\Omega^p_f:=\{\omega\in\Omega^p_X(\log D)\mid \mathrm{d}f\wedge\omega\in\Omega^{p+1}_X(\log D)\},$$

- This produces a complex denoted by $(\Omega^\bullet_f,\nabla)$.

• If P has multiplicities, consider also $(\Omega_f^\bullet(\alpha),\nabla)$ ($\alpha\in[0,1)\cap\mathbb{Q}$):

$$\Omega_f^p(\alpha):=\{\omega\in\Omega_X^p(\log D)([\alpha P])\mid \mathrm{d}f\wedge\omega\in\Omega_X^{p+1}(\log D)([\alpha P])\}.$$

• This complex comes equipped with the stupid filtration $\sigma^{\geqslant p}(\Omega_f^\bullet(\alpha),\nabla)$: the terms in degree $<p$ are set to zero, the other terms are unchanged.
• $\boldsymbol{H}^k(X,(\Omega_f^\bullet(\alpha),\nabla))\simeq\boldsymbol{H}^k_{\mathrm{DR}}(X,f)$.
• $\sigma^{\geqslant p}(\Omega_f^\bullet(\alpha),\nabla)$ induces

$$F^{p-\alpha}\boldsymbol{H}^k(X,(\Omega_f^\bullet(\alpha),\nabla))=:F^{p-\alpha}\boldsymbol{H}^k_{\mathrm{DR}}(X,f).$$

***Theorem* ([9], [6]).** *For each $\alpha\in[0,1)\cap\mathbb{Q}$, the spectral sequence for the stupid filtration degenerates at E_1.*

(2) For the $\frac{\infty}{2}$ Hodge structure, we consider $f:X\to\mathbb{P}^1$ and the twisted de Rham cohomology $\boldsymbol{H}^k(X,(\Omega_X^\bullet[z],z\mathrm{d}+\mathrm{d}f))$ as a $\mathbb{C}[z]$-module. It is easy to check that it has finite type over $\mathbb{C}[z]$. The Hodge property reads as follows.

***Theorem* (Barannikov-Kontsevich).** *For every k, $\boldsymbol{H}^k(X,(\Omega_X^\bullet[z],z\mathrm{d}+\mathrm{d}f))$ is $\mathbb{C}[z]$-free.*

(3) From the Higgs point of view, consider $(\mathscr{C}_U^\infty,\overline{\partial},h,\theta=\mathrm{d}f)$, where h is the trivial metric $h(1,1)=1$. If f is a map $X\to\mathbb{P}^1$, this extends as a pure twistor $\mathscr{D}$-module $\mathscr{T}^{f/z}$ (T. Mochizuki) whose components are $(\mathscr{O}_{X\times\mathbb{C}_z}(*P),z\mathrm{d}_X+\mathrm{d}f)$. (If f is not a map $X\to\mathbb{P}^1$, one defines similarly a mixed twistor $\mathscr{D}$-module).

This twistor $\mathscr{D}$-module is *integrable*, due to the natural action $z^2\partial_z\cdot 1=-f$.

Taking the push-forward of $\mathscr{T}^{f/z}$ by the constant map $X\to\mathrm{pt}$ gives the non-commutative mixed Hodge structure on $\boldsymbol{H}^k_{\mathrm{DR}}(X,f)$. Here we use the full strength of the theory of mixed twistor $\mathscr{D}$-modules.

We have an identification: integrable mixed twistor $\mathscr{D}$-modules on a point $\iff$ a non-commutative mixed Hodge structure.

(4) From the point of view of non-commutative mixed Hodge structures, integrable mixed twistor $\mathscr{D}$-modules, the E_1-degeneration property follows from the following rescaling property: if we rescale the function f as τf ($\tau\in\mathbb{C}^*$), then each $\mathscr{T}^{\tau f/z}$ is also an integrable pure or mixed twistor $\mathscr{D}$-module, and moreover, regarding τ as a variable, $\mathscr{T}^{\tau f/z}$ is itself an integrable mixed twistor $\mathscr{D}$-module.

(5) From the point of view of exponential mixed Hodge structures, assume for simplicity that f is a morphism $X\to\mathbb{P}^1$ and restrict as $f:Y\to\mathbb{A}^1$, $Y=X\smallsetminus P$. Since $\mathscr{O}_Y^{\mathrm{H}}$ is a mixed Hodge module, we have mixed Hodge modules $f_*^k\mathscr{O}_Y^{\mathrm{H}}$ on $\mathbb{A}^1$.

One can associate with them mixed Hodge modules with zero hypercohomology M_f^k. These define exponential mixed Hodge structures.

3. Irregular Hodge structures and modules

An integrable mixed twistor structure is a mixed twistor structure with meromorphic connection compatible with the W-filtration and a pole of order $\leqslant 2$ at $0, \infty$ and no other pole; we also assume that each $\mathrm{gr}_\ell^W \simeq \mathscr{O}_{\mathbb{P}^1}(\ell)^{r_\ell}$ has an integrable polarization.
One can rescale such an object, but in general the rescaled objects may not remain mixed twistor structures.

Definition. An irregular mixed Hodge structure is an integrable mixed twistor structure whose rescalings by $\tau \in \mathbb{C}^*$ form an admissible (near $\tau = 0$) variation of integrable mixed twistor structures.

Theorem.

(1) *The category* $\mathsf{IrrMHS}_{\mathbb{C}}$ *is abelian and naturally equipped with the structure of a neutral Tannakian category, with fibre functor "restriction to $z = 1$".*

(2) *There is a fully faithful functor* $\mathsf{MHS}_{\mathbb{C}} \mapsto \mathsf{IrrMHS}_{\mathbb{C}}$ *which identifies* $\mathsf{MHS}_{\mathbb{C}}$ *to a full subcategory of* $\mathsf{IrrMHS}_{\mathbb{C}}$.

(3) *There is a natural functor* "irregular Hodge filtration" *from* IrrMHS *to the category of bi-filtered vector spaces* $(H, F_\bullet^{\mathrm{irr}}, W_\bullet)$ *(where $F_\bullet^{\mathrm{irr}}$ is a filtration indexed by $\mathbb{R}$ and jumps at most at $A + \mathbb{Z}$ for some finite $A \subset [0, 1)$) which is compatible with tensor product and with taking dual. Any morphism in* $\mathsf{IrrMHS}_{\mathbb{C}}$ *gives rise to a strictly bi-filtered morphism between the corresponding bi-filtered vector spaces.*

Example. The non-commutative mixed Hodge structure on $\boldsymbol{H}^k_{\mathrm{DR}}(X, f)$ is in $\mathsf{IrrMHS}_{\mathbb{C}}$.

The theory of integrable mixed twistor $\mathscr{D}$-modules leads to various properties of the irregular Hodge filtration, in particular a Künneth formula (or Thom-Sebastiani formula).

4. Rigid irreducible $\mathscr{D}$-modules on $\mathbb{P}^1$

1. Reminder on hypergeometric equations. Consider an hypergeometric differential equation

$$\prod_{i=1}^{n}(t\partial_t-\alpha_i)-t\prod_{j=1}^{m}(t\partial_t-\beta_j),\quad \alpha_i,\beta_j\in[0,1),\quad \begin{cases}\alpha_1\leqslant\cdots\leqslant\alpha_n,\\ \beta_1\leqslant\cdots\leqslant\beta_m,\end{cases}$$

and $\alpha_i\neq\beta_j$ for all i,j. We first start with the regular case $n=m$.

- Beukers-Heckman [**2**]: the corresponding local system is irreducible and rigid,
- Simpson [**11**]: it underlies a PV$\mathbb{C}$-HS,
- Deligne [**3**]: this PV$\mathbb{C}$-HS is unique up to a shift of the Hodge filtration,
- Katz [**8**]: $\exists$ a sequence of rank-one local systems L_k on $\mathbb{P}^1\smallsetminus\{0,\infty\}$ (Kummer sheaves) whose monodromy $\lambda\in\mathbb{C}^*$ satisfies $|\lambda|=1$, such that, after a sequence of operations $L_k\star$ or $L_k\otimes$, one gets a rank-one hypergeometric local system (in particular, unitary),
- Dettweiler-CS [**5**]: computation of the behaviour of Hodge data at each step of the Katz algorithm,
- Fedorov [**7**]: explicit computation for the hypergeometric differential equation as above: set $\rho(j)=\#\{i\mid\alpha_i\leqslant\beta_j\}-j$. Then $h^p=\#\rho^{-1}(p)$.

2. What about the confluent case $n\neq m$?

- Katz [**8**]: the corresponding differential equation is irreducible and rigid,
- Arinkin [**1**], Deligne (letter to Katz, June 2006): There exists a sequence of rank-one local systems L_k on $\mathbb{P}^1\smallsetminus\{0,\infty\}$ (Kummer sheaves) whose monodromy $\lambda\in\mathbb{C}^*$ satisfies $|\lambda|=1$, such that, after a sequence of operations $L_k\otimes$, Fourier transforms and translation of the variable, one gets a rank-one regular hypergeometric local system (in particular, unitary) twisted by $\exp\varphi$ with φ meromorphic.

Question.

(1) Does there exist a kind of Hodge filtration, i.e., numbers h^p, in a canonical way up to shift?

(2) If yes, how to compute them in terms of the local exponents?

Answer to (1) known, answer to (2) still not known.

Theorem. *There exists in a canonical way, an "irregular Hodge filtration" indexed by $\mathbb{R}$ (in fact $A+\mathbb{Z}$ for some finite set $A \subset [0,1)$), unique up to a shift of the indices (by a real number).*

Why could this be true?.

(1) In rank one, any differential equation is of exponential type (i.e., twist by $\exp\varphi$ of an RS differential equation), so one can apply Deligne's construction (1984) [**4**];

(2) In view of the Arinkin-Deligne algorithm, one is reduced to prove a good behaviour of this filtration by various functors.

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