Higher direct images in nonabelian Hodge theory

Dedicated to **Masa-Hiko Saito**
on the occasion of his 60^{th} birthday

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This talk is about joint work with Ron Donagi and Tony Pantev.

Many people have constructed and studied moduli spaces of parabolic connections and local systems.

I learned a lot about these moduli spaces from Masa-Hiko, in particular they can have a very nice geometric structure.

An interesting next question will be to understand natural maps between the moduli spaces.

These can come from various operations : tensor products and other maps between structure groups, pullbacks, ...

A particular kind of map comes from taking higher direct images.

Suppose $f : X \to Y$ is a smooth map between smooth projective varieties. Let $M_{dR}(X, GL(r))$ denote the moduli space of $GL(r)$-bundles with flat connection on X, and the same on Y. If $(E, \nabla) \in M_{dR}(X, GL(r))$ then

$$DR(X/Y, E, \nabla) := \left(E \to E \otimes \Omega^1_{X/Y} \xrightarrow{d_\nabla} E \otimes \Omega^2_{X/Y} \to \dots \right)$$

is the *relative de Rham complex*.

The map

$$(E,\nabla)\mapsto \mathbb{R}^i f_* DR(X/Y,E,\nabla)$$

gives a rational map

$$M_{dR}(X,GL(r)) \; - - \rightarrow M_{dR}(Y,GL(r'))$$

defined on the open subset $M_{dR}(X,GL(r))'$ where the dimensions of the cohomology groups are constant equal to r'.

The direct image bundle depends only on the relative connection $\nabla_{X/Y}: E\rightarrow E\otimes\Omega^1_{X/Y}$ but to get the *Gauss-Manin connection* ∇_{GM} downstairs, we should use the full connection upstairs.

By the Riemann-Hilbert correspondence the direct image construction may also be viewed in the Betti world :

$$M_B(X, GL(r)) - - \to M_B(Y, GL(r'))$$

given by

$$L \mapsto \mathbb{R}^i f_*(L),$$

again a rational map defined on the open subset where the dimension is constant.

Question : What is the Higgs bundle version of this construction ?

Recall that the Hitchin equations give the nonabelian Hodge isomorphisms of M_{dR} and M_B with the moduli space of Higgs bundles

$$M_H(X, GL(r)) = \left\{ \begin{array}{c} (E, \varphi) \text{ where } \ \varphi : E \to E \otimes \Omega^1_X, \\ \varphi \wedge \varphi = 0, \text{ polystable, } c_i = 0 \end{array} \right\}.$$

We can also define a rational map

$$M_H(X, GL(r)) - - \to M_H(Y, GL(r')).$$

Given (E, φ) define the *Dolbeault complex*

$$DOL(X/Y, (E, \varphi)) := \left(E \to E \otimes \Omega^1_{X/Y} \stackrel{\wedge \varphi_{X/Y}}{\longrightarrow} E \otimes \Omega^2_{X/Y} \to \ldots \right)$$

depending on the relative Higgs field.

The higher direct image vector bundle is

$$(E, \varphi) \mapsto F := \mathbb{R}^i f_* DOL(X/Y, (E, \varphi)),$$

and the global Higgs field upstairs leads to a *Gauss-Manin Higgs field* φ_{GM} on this bundle.

The Dolbeault construction is interesting because it can sometimes be calculated more explicitly.

If $\varphi_{X/Y}$ has isolated zeros in each fiber (generically the case), then the subscheme of zeros

$$\begin{array}{ccc} Z & \subset & X \\ & \searrow & \downarrow \\ & & Y \end{array}$$

becomes the *spectral covering* for the Higgs bundle (F, φ_{GM}) over Y, with the global Higgs field providing the required differential form.

Work of Donagi and Pantev (see Ron's talk) gives an explicit construction of certain Higgs bundles over $\mathbf{Bun}_G$, and they would like to understand the images by the *Hecke correspondences*.

Hecke correspondences are nice canonical examples of the direct image construction, crucial for the Geometric Langlands program.

Therefore, Donagi and Pantev were interested in understanding the Dolbeault higher direct image construction. This was the motivation for our project.

The above discussion has skipped an important aspect : most maps between algebraic varieties are *not* smooth maps of smooth projective varieties.

At the very least, there will be singular fibers, and often the natural local systems we would like to consider come from $\mathcal{D}$-modules with singularities. They are local systems only over some quasiprojective open subsets.

The theory of Higgs bundles over quasiprojective varieties brings in the notion of parabolic structure.

We would therefore like to understand the Dolbeault higher direct images for parabolic Higgs bundles.

Suppose $D = D_1 + \ldots + D_k \subset X$ is a simple normal crossings divisor. A *parabolic bundle* on (X, D) is the structure of a collection of locally free sheaves E_β on X, indexed by *parabolic levels*

$$\beta = (\beta_1, \ldots, \beta_k)$$

with inclusions $E_\alpha \subset E_\beta$ if $\alpha_i \leq \beta_i$, such that $E_{\beta+\delta_j} = E_\beta(D_j)$, satisfying a semicontinuity property.

Another popular viewpoint considers the vector bundle E_0 on X together with filtrations of $E_0|_{D_j}$ that are provided with parabolic weights. These viewpoints are equivalent.

We denote the parabolic bundle, that is the whole collection of E_β, by $\underline{E}$.

A *logarithmic Higgs field* on $\underline{E}$ is a compatible collection of maps

$$\varphi : E_\beta \to E_\beta \otimes \Omega^1_X(\log D).$$

Takuro Mochizuki's *Kobayashi-Hitchin correspondence* provides an equivalence between :
—logarithmic parabolic Higgs bundles,
—filtered local systems, and
—parabolic vector bundles with logarithmic connection ;
where one requires polystability and vanishing of the parabolic Chern classes in all cases.

Over a divisor component D_j, for any parabolic weight b, we obtain the bundle

$$\mathrm{gr}_{j,b}(\underline{E}) := E_{\ldots,b,\ldots}/E_{\ldots,b-\epsilon,\ldots}.$$

It actually has a parabolic structure with respect to the intersection $D_{\cap j} := D_j \cap (D - D_j)$.

On these graded pieces $\mathrm{gr}_{j,b}(\underline{E})$ the logarithmic Higgs field induces a *residue*

$$N_{j,b} = \mathrm{res}_{j,b}(\varphi) : \mathrm{gr}_{j,b}(\underline{E}) \to \mathrm{gr}_{j,b}(\underline{E}).$$

Nilpotence hypothesis : *The Higgs field residues $N_{j,b}$ are nilpotent.*

In the Betti picture, this corresponds to looking at local systems with trivial filtrations whose monodromy eigenvalues are in $S^1 \subset \mathbb{C}$.

On the de Rham side, it corresponds to looking at parabolic connections such that the eigenvalue of $\mathrm{res}_{j,b}(\nabla)$ is real, equal to the parabolic weight b.

Consider now our map $f : X \to Y$. Assume :
–Y is a curve and f is a family of semistable marked curves ;
–the divisor decomposes as

$$D = D_H + D_V$$

where the components of D_H are etale over Y and $D_V = f^{-1}(Q)$ for a divisor $Q \in Y$;
—the vertical divisor D_V contains all the singular fibers.

Assume we start with a parabolic logarithmic Higgs bundle $(\underline{E}, \varphi)$ on (X, D), polystable with vanishing parabolic Chern classes.

We would like to construct the higher direct image parabolic logarithmic Higgs bundle $(\underline{F}, \varphi_{GM})$ on (Y, Q).

It should be compatible with the Kobayashi-Hitchin correspondence with the higher direct images of the corresponding local system and vector bundle with connection.

Consider first the smooth fibers $X_y = f^{-1}(y)$, $y \in Y - Q$. Steve Zucker and Claude Sabbah's construction says that the intersection cohomology of the local system on the open curve $X_y - D_{H,y}$ is calculated by an L^2 *holomorphic Dolbeault complex*.

We can define the relative version. The main ingredient is the *weight filtration* $W_\bullet$ for the 0-th graded piece $\mathrm{gr}_{j,0}$ along a horizontal divisor component. That is the weight filtration of the nilpotent endomorphism $N_{j,0}$.

Suppose α is a parabolic level whose horizontal components are equal to 0. Then the weight filtration leads to subbundles we denote $W_m E_\alpha$.

Zucker's Dolbeault calculation tells us to look at the L^2 Dolbeault complex

$$DOL_{L^2}(X/Y, (\underline{E}, \varphi); \alpha) := \left(W_0 E_\alpha \xrightarrow{\wedge\varphi} W_{-2} E_\alpha \otimes \Omega^1_{X/Y}(\log D) \right).$$

As said above, the parabolic levels associated to horizontal divisor components D_j should be $\alpha_j = 0$.

We may say, more briefly, that $DOL_{L^2}(X/Y, (\underline{E}, \varphi); \alpha)$ is the complex of holomorphic forms that are L^2 with respect to the Poincaré metric near the horizontal singularities.

The question is what to do for the vertical components $D_V \subset D$.

For simplicity of notation let us suppose that Q has only one point q. A parabolic level for Q is therefore a real number a. Define the parabolic level $\alpha(a)$ by setting $\alpha(a)_j = 0$ for horizontal divisor components D_j, and $\alpha(a)_j = a$ for vertical components D_j.

Theorem

Suppose $(\underline{E}, \varphi)$ is a parabolic logarithmic Higgs bundle on (X, D) corresponding to a local system L on $X - D$ and assume the Nilpotence Hypothesis. Define

$$F_a := R^i f_* \left(DOL_{L^2}(X/Y, (\underline{E}, \varphi); \alpha(a))\right).$$

Then $\underline{F} = \{F_a\}$ is a parabolic bundle on (Y, Q) with logarithmic Higgs field φ_{GM}.
These form the parabolic logarithmic Higgs bundle that corresponds to the higher direct image of the local system $R^i f_*(L)$ on $Y - Q$.

The strategy of proof is as follows. Sabbah and Mochizuki have created a theory of twistor $\mathcal{D}$-modules, generalizing Morihiko Saito's theory of Hodge modules.

The basic underlying objects are $\mathscr{R}_{\mathscr{X}}$-modules. Here we consider the λ-line $\mathscr{A} := \mathbb{A}^1$, one of the coordinate patches of the twistor $\mathbb{P}^1$. We put $\mathscr{X} := X \times \mathscr{A}$ etc., and $\mathscr{R}_{\mathscr{X}}$ is the sheaf of rings on $\mathscr{X}$ obtained by applying the Rees construction to the standard filtration of $\mathcal{D}_X$.

Given a harmonic bundle there is a naturally associated family of λ-connections, hence an $\mathscr{R}_{\mathscr{X}-\mathscr{D}}$-module.

One looks at the *minimal extension* to an $\mathscr{R}_{\mathscr{X}}$-module, it corresponds to taking the intersection cohomology middle extension of the local system.

This plays the role of the $\mathcal{D}_X$-module plus Hodge filtration, in the theory of Hodge modules.

Following Kashiwara-Malgrange, in the $\mathscr{R}$-module context Sabbah shows that taking the V-filtration commutes with higher direct image.

This gives a result close to the one we are looking for. Indeed, the V-filtration (for negative values of the index) corresponds to the parabolic filtration, on the λ-connections for all values of λ.

The answer we are interested in is the V-filtration on the Dolbeault fiber at $\lambda = 0$.

The axiomatic characterization of the V-filtration depends on the $\mathscr{R}_{\mathscr{X}}$-module structure over the full λ-line, it is not *a priori* well-defined just within the fiber at $\lambda = 0$.

This question is solved by the fact that Sabbah and Mochizuki provide an explicit generation expression for the V-filtration that can be interpreted just within the Dolbeault fiber.

That leads directly to our theorem, in the case when f is smooth.

Basic difficulty : At a normal crossing point of D_V, the map f is not smooth so, in order to define the V-filtration with respect to f one should use the graph embedding $g : X \to X \times Y$.

In local coordinates $f = xy$ and the graph is

$$g(X) = \{(x, y, t) : \ \ t = xy\} \subset X \times Y.$$

Given our $\mathscr{R}_{\mathscr{X}}$-module $\mathscr{E}$ we may now take the V-filtration of $g_+(\mathscr{E})$ along the coordinate plane $t = 0$.

Formation of this V-filtration is compatible with formation of the relative de Rham complex, which over $\lambda = 0$ is a Dolbeault complex for the $\mathscr{R}_{\mathscr{X}\times\mathscr{Y}}$-module $g_+(\mathscr{E})$.

We obtain a collection of bundles

$$V_{a-1}R^i f_* DOL(X \times Y/Y, g_+(E))$$

$$= R^i f_* DOL(X \times Y/Y, V_{a-1}(g_+(E)))$$

and by Sabbah's theory this collection indeed constitutes the parabolic Higgs bundle corresponding to the higher direct image.

In order to obtain the formula of our theorem, the $\mathscr{R}$-module Dolbeault complex needs to be compared with the collection of bundles given by the parabolic structure, which may be considered as a sort of *multi-V-filtration* with respect to the normal crossings divisor.

Theorem

At a normal crossings point, the natural map

$$DOL_{L^2}(X/Y, (\underline{E}, \varphi); \alpha(a)) \to g^{-1} DOL(X \times Y/Y, V_{a-1}(g_+(E)))$$

is a quasiisomorphism.

Corollary

This yields the proof of the main theorem.

We know the quasiisomorphism away from the normal crossing points. Recall that Sabbah has treated the L^2 aspect along the horizontal divisor.

The normal crossing points form a scheme that is finite over Y.

Therefore, it suffices to prove the quasiisomorphism after applying $\mathbb{R}f_*$.

But that makes the left hand side into a perfect complex over Y, by usual semicontinuity theory. The right side becomes a perfect complex by the decomposition theorem for twistor $\mathcal{D}$-modules.

In order to prove that a map between perfect complexes is a quasiisomorphism it suffices to prove it after restriction to closed points.

It therefore suffices to show that the restriction to $q \in Q$ is a quasiisomorphism. This is now filtered by the parabolic weights between say $a-1$ and a.

It therefore suffices to prove the analogous result for the associated-graded $\mathrm{gr}_a = ()_a/()_{a-\epsilon}$. We now go back upstairs to do that.

Let us write down explicitly the map. Denote by $E_{a,a}$ the component of the parabolic bundle with respect to our two crossing divisors $D_1 = (x = 0)$, $D_2 = (y = 0)$.

Then the generation expression for $a < 1$ says that $V_{a-1}(g_+(E))$ is the $V_0 R_{X\times Y,0}$-submodule of $g_+(E)$ generated by $E_{a,a}$. Here $R_{X\times Y,0} = \mathrm{Sym}^\bullet T(X \times Y)$ is the Dolbeault fiber at $\lambda = 0$ of $\mathscr{R}_{\mathscr{X}\times\mathscr{Y}}$.

The ring $V_0 R_{X\times Y,0}$ contains a subring $R_X[s]$ where s corresponds to $t\partial_t$, and R_X is the symmetric algebra on $u := \partial_x$ and $v := \partial_y$.

The generation statement can be improved : $V_{a-1}(g_+(E))$ may be characterized as the submodule of $g_+(E)$ generated by $E_{a,a}$ over the ring $R_X[s]$, but then even better, over just $R_X = \mathcal{O}_X[u, v]$.

Put $w := xu - yv$; it generates the logarithmic relative tangent bundle of X/Y which is locally free.

Let

$$M = \psi_{a,a} := E_{a,a}/E_{a-\epsilon,a-\epsilon}$$

viewed as a $\mathcal{O}_D[w]$-module, and for $a < 1$ let

$$N = \Psi_{a-1} := g^{-1}V_{a-1}(g_+(E))/V_{a-\epsilon-1}(g_+(E)),$$

viewed as a $\mathcal{O}_D[u, v]$-module.

Lemma (Tensor product formula)

The map

$$\psi_{a,a} \otimes_{\mathcal{O}_D[w]} \mathcal{O}_D[u, v] \to \Psi_{a-1}.$$

is an isomorphism.

On the gr_a's, our map of Dolbeault complexes is

$$\begin{array}{ccccccc} \mathbf{Kosz}(V_1, M) & = & & & M & \rightarrow & M \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbf{Kosz}(V_2, N) & = & N & \rightarrow & N^{\oplus 2} & \rightarrow & N \end{array}$$

with $V_1 = \Omega^1_{X/Y}(\log D)$ (acting by w) and $V_2 = \Omega^1_X$ (acting by u, v), and we calculate that the vertical map of complexes is a quasiisomorphism.

For this calculation we may reduce to the cases of $\varphi = 0$ with rank 1 modules supported on D_1, D_2 or $D_1 \cup D_2$, then it can be done explicitly by hand. This completes the proof.

Corollary

The generalization holds for a family of semistable curves over a higher-dimensional base.

This allows, in principle, to treat arbitrary maps by devissage.

It seems reasonable also to formulate a direct statement for higher-dimensional fibers over a base curve, although the definition of the Dolbeault complex, and the proof, will be more complicated.

The case of higher-dimensional base and fibers looks less clear, one could try using Abramovich-Karu toric reduction.

Happy Birthday, Masa-Hiko !